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NOTE ON FERMAT'S NUMBERS.

BY DR. J. C. MOREHEAD AND MR. A. E. WESTERN.

(Read before the Chicago Section of the American Mathematical Society, April 9, 1909.)

In June, 1658, Fermat wrote to Sir Kenelm Digby a letter,* in which, after referring to certain theorems proved by him, which he might propose to Viscount Brouncker and John Wallis, in order to give them something to do, he said that, instead of these theorems, he would submit to them, as problems, some theorems which he admitted he could not prove, though he was convinced of their truth. The first of these problems was to prove that $2^{2^n} + 1$ is a prime, and he gave as examples the numbers corresponding to $n = 1, 2, 3, 4$, which in fact are primes. Fermat challenged his English friends to furnish a proof of this proposition, which was certainly very beautiful, and which he believed was true. He added that perhaps the proof would give the key to penetrate all the mystery of prime numbers.†

As is well known, the theorem is untrue for many values of n.‡ In 1905 Dr. Morehead read before the AMERICAN MATHEMATICAL SOCIETY a "Note on Fermat's numbers,"‡ which stated the result of a calculation proving that $F_7 = 2^{128} + 1$ is composite. Mr. Western had independently performed the same calculation, and communicated the result almost simul-

* Pierre de Fermat, Œuvres, Paris, 1896, vol. 2, p. 405 (in Latin); vol. 3, p. 315 (French translation).

† That Fermat attached great importance to this theorem is further evidenced by the fact that he referred to it in six other letters and papers written between 1640 and 1659. Cf. Œuvres, vol. 1, p. 131; vol. 2, pp. 206, 207, 212, 309, 434.

‡ W. W. R. Ball, Mathematical Recreations and Problems, 2d ed., 1892, p. 26. *Proc. Lond. Math. Soc.*, Series 2, vol. 1 (1904), p. 175. BULLETIN, vol. 11, p. 543, and vol. 12, p. 449.

taneously to the London Mathematical Society;* and it was found that the two calculations were in exact agreement. There is therefore no doubt that F_7 is composite, but the actual factors are unknown. The authors have since carried out a similar calculation for F_8, each doing a half of the whole work. As this is probably by far the largest calculation in connection with the theory of numbers which has been yet performed, some particulars of the methods employed may be of interest.

The test employed by the authors in the case both of F_7 and F_8 depends on the following theorem:

If $a^x \equiv 1 \pmod p$ is true when $x = p - 1$, but is not true when x is any factor of $p - 1$, then p is prime.

This theorem appears to have been first given by Lucas in 1876, and is called by him the reciprocal of Fermat's theorem.† The proof is very simple; if p is composite, then $\phi(p) < p - 1$; and if the assumption of the theorem is satisfied, then $a^x \equiv 1 \pmod p$ for both $x = \phi(p)$ and $x = p - 1$, and therefore also for $x = \delta$, the greatest common divisor of $\phi(p)$ and $p - 1$, which contradicts the assumption. Therefore, under the conditions of the theorem, p cannot be composite.

Applying this to the case of Fermat's numbers, and taking $a = 3$, it is clear that if F_n is prime,

$$3^{\frac{1}{2}(F_n - 1)} \equiv -1 \pmod{F_n},$$

for $F_n \equiv -1 \pmod 3$, and so 3 is a quadratic non-residue of F_n. And since $\frac{1}{2}(F_n - 1)$ is a power of 2, no value of x less than $\frac{1}{2}(F_n - 1)$ will make

$$3^x \equiv -1 \pmod{F_n}.$$

Therefore by Lucas's theorem, if

$$3^{\frac{1}{2}(F_n - 1)} \equiv -1 \pmod{F_n},$$

F_n is prime, and if not, F_n is composite.

The index of 3 in this congruence may be reduced; for

$$-1 \equiv 2^{2^n} \pmod{F_n},$$

and we therefore obtain, by successive extractions of the square root,

$$3^{2^{(2^n - n - 1)}} \equiv 2^x \pmod{F_n},$$

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* *Proc. Lond. Math. Soc.*, ser. 2, vol. 3, p. xxi.

† Lucas, Théorie des Nombres; Paris, 1891, p. 441.

where x is odd. Now

$$2 \equiv 2^{2^{n-1}}(2^{2^{n-1}} - 1)^2 \pmod{F_n},$$

and so

$$(A) \qquad 3^{2^{(2^n-n-2)}} \equiv \pm 2^y(2^{2^{n-1}} \pm 1) \pmod{F_n},$$

where y may be taken to be between 0 and 2^{n-1}.

Applying this criterion to F_8, it is necessary to calculate the residue of the 2^{246}th power of 3 (mod F_8).

Mr. Western's method was as follows. Having obtained

$$3^{2^t} \equiv \Sigma a_n x^n \pmod{F_8} \qquad (n = 0, 1, \cdots, 7),$$

where $x = 2^{32}$, and $a_0, a_1, \cdots, a_7$, are positive or negative numbers less than $\frac{1}{2}x$ (*i. e.* containing at most 10 digits), the residue on the right is squared; since $x^8 \equiv -1$, we obtain

$$3^{2^{t+1}} \equiv \Sigma A_n x^n \pmod{F_8},$$

where

$$A_0 = a_0^2 - 2a_1a_7 - 2a_2a_6 - 2a_3a_5 - a_4^2,$$

$$A_1 = 2(a_0a_1 - a_2a_7 - a_3a_6 - a_4a_5), \quad \text{etc.};$$

thus division by F_8 is performed simultaneously with the operation of squaring. The numbers $A_0, A_1, \cdots$ are each less than $2x^2$ (and generally less than x^2); each A is then divided by x, giving

$$A_r = b_r + xc_r,$$

where

$$|b_r| < \tfrac{1}{2}x, \quad |c_r| < 2x,$$

and then

$$3^{2^{t+1}} \equiv (b_0 - c_7) + (b_1 + c_0)x + \cdots + (b_7 + c_6)x^7.$$

The coefficients in this are then adjusted, so as to make each of them numerically less than $\frac{1}{2}x$, and we finally get

$$3^{2^{t+1}} \equiv \Sigma a_n' x^n \pmod{F_8}.$$

The whole process is then repeated. The calculations of A_r, b_r, c_r, were wholly performed on a 10-figure arithmometer, and the b_r and c_r alone were written down.

In such a calculation as this, in which a single error would vitiate all the subsequent work, care must be taken to detect any mistakes in each stage, before proceeding to the next stage. The test employed by Mr. Western was

$$\sum a' \equiv (a_0 + a_1 + a_2 + a_3)^2 - (a_4 + a_5 + a_6 + a_7)^2$$
$$+ 2(a_0 + a_1 + a_2 + a_3)(a_4 + a_5 + a_6 + a_7) - 4a_3(a_5 + a_7)$$
$$- 4a_6(a_2 + a_3) - 4a_7(a_1 + a_2) - 2(c_7 + d) \pmod{k},$$

where d is the multiplier of x^8 carried forward from the last term $(b_7 + c_6)x^7$ to the first term, and k is any factor of $x - 1$. The values of k used were 3, 5, and 17.

Dr. Morehead expressed the residue of $3^{2^{126}}$, calculated by Mr. Western, in the form $ax + b$, where $x = 2^{128}$ and $|a|$, $|b| < \frac{1}{2}x$; then squaring and replacing x^2 by -1, he obtained

$$3^{2^{127}} \equiv 2abx + (b - a)(b + a) \pmod{F_8}.$$

The products $2ab$, $(b - a)(b + a)$ were then calculated and expressed, by division by x, in the forms $\alpha x + \alpha'$, $\beta x + \beta'$ respectively, so that

$$2abx + (b - a)(b + a) = (\alpha x + \alpha')x + \beta x + \beta'$$
$$\equiv (\alpha' + \beta)x + \beta' - \alpha \pmod{F_8},$$

$\alpha, \alpha', \beta, \beta'$ having been so adjusted that $|\alpha' + \beta|$, $|\beta' - \alpha| < \frac{1}{2}x$. Thus was obtained

$$3^{2^{127}} \equiv a'x + b' \pmod{F_8},$$

and, by repetition of this process, the residues of

$$3^{2^{128}}, 3^{2^{129}}, \ldots, 3^{2^{246}}. *$$

The test formula (A) shows that if F_8 were prime we should have, in the residue of $3^{2^{246}}$, $|a| = |b|$. This residue was found to be

$$(107\ 2093\ 3158\ 0550\ 8180\ 4331\ 6350\ 5866\ 1999\ 4098)x$$
$$+ (\ 34\ 1778\ 3881\ 0697\ 6545\ 8021\ 2127\ 5588\ 1034\ 4254),$$

and therefore F_8 is composite.

This result is especially interesting as completing a chain of

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* The part of the calculation carried out by Dr. Morehead was checked at each stage by applying the test

$$-ab + b^2 - a^2 \equiv 4a' + b' - \alpha \pmod{9},$$

and similar tests for the moduli 11, 19, 41, 101. Two 8-figure calculating machines were used together for the greater part of the calculation and checking at each stage.

five composite Fermat numbers, F_5, F_6, F_7, F_8, F_9, following the first five (and only known) Fermat primes, 3, 5, 17, 257, 65537, and as leaving F_{10}(a 309-place number) the smallest Fermat number whose status is unknown. All the Fermat numbers $F_5, \cdots, F_{12}$, except F_{10}, are now known to be composite.

If ζ is a primitive 2^{n+1}th root of 1, so that $\zeta^{2^n} = -1$, F_n is the norm of $2 - \zeta$ in the field of ζ. Accordingly the problem of factoring F_n is the same as that of factoring $2 - \zeta$. It is possible that Fermat may have observed this, and may have assumed that it was improbable that $2 - \zeta$ could have complex factors of the form

$$a_0 + a_1\zeta + \cdots + a_{2^n-1}\zeta^{2^n-1}.$$

In fact, when $n \geqq 2$, the field contains an infinite number of units, and so $2 - \zeta$ can be expressed in an infinite number of ways as the product of two complex numbers of the field, one of these being a unit, and the other having F_n as its norm. But whether, for any value of n, $2 - \zeta$ can be expressed as the product of two actual numbers of the field, neither being a unit, is not known. It seems probable that the prime factors of $2 - \zeta$, if any, are always ideals.

Actual complex numbers containing the complex factors of known factors of Fermat's numbers may be easily found; for instance, ζ being a primitive 64th root of 1, and p being 641, the smaller factor of F_5,

$$p = 1 + 2^7 + 2^9.$$

But $2 \equiv \zeta \pmod{\pi}$, π being one of the prime factors of p in the field of ζ, so

$$1 + \zeta^7 + \zeta^9 \equiv 0 \pmod{\pi}.$$

The norm of $1 + \zeta^7 + \zeta^9$ is 193.641, as shown by Reuschle,* who states that the prime factors of 641 are ideal.

Again the smaller factor of F_6 is

$$p = 274{,}177 = 1 - 2^8 - 2^{12} + 2^{14} + 2^{18},$$

so ζ being as before, and π being a prime factor of p in the field of ζ, we have $4 \equiv \zeta \pmod{\pi}$, and so,

$$1 - \zeta^4 - \zeta^6 + \zeta^7 + \zeta^9 \equiv 0 \pmod{\pi}.$$

* Tafeln complexer Primzahlen, p. 455.

The norm of the number on the left is found to be p. It seems impracticable to determine whether or not p has actual prime factors in the field of 128th roots of 1, but this is very improbable, as the class number in that field is a multiple of 21,121.*

The use of complex numbers appears to be of no assistance in the problem of determining whether F_n is prime or composite.

AN EXTENSION OF CERTAIN INTEGRABILITY CONDITIONS.

BY PROFESSOR J. EDMUND WRIGHT.

SUPPOSE there are n functions $a_1, a_2, \cdots, a_n$ of n independent variables $x_1, x_2, \cdots, x_n$, satisfying the conditions

$$\frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = 0$$

for all values of p and q. It is well known that the functions a must all be first derivatives of a single function V. Similarly, if there are $\frac{1}{2}n(n+1)$ functions a_{pq} such that $a_{pq} = a_{qp}$, satisfying the relations

$$\frac{\partial a_{pq}}{\partial x_r} = \frac{\partial a_{pr}}{\partial x_q}$$

for all values of p, q, r, then the a's must be second derivatives of a single function.

The following question arises in connection with an application of the theory of invariants of quadratic differential forms :

Suppose there are $n(n+1)$ functions H_{pq}, K_{pq} such that $H_{pq} = H_{qp}$, $K_{pq} = K_{qp}$, satisfying the conditions

$$\frac{\partial}{\partial x_r}(H_{pq}) + K_{pq}\frac{\partial Y}{\partial x_r} = \frac{\partial}{\partial x_p}(H_{qr}) + K_{qr}\frac{\partial Y}{\partial x_p},$$

for all values of p, q, r; Y being a given function of the variables; what are the conditions on the functions H, K?

We first consider the case of $2n$ functions $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$, satisfying the conditions

* Reuschle, Tafeln, p. 461.

$$(1) \qquad \frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = b_q \frac{\partial Y}{\partial x_p} - b_p \frac{\partial Y}{\partial x_q}.$$

Take three equations of the type (1), those for (p, q), (q, r), (r, p), differentiate the first with respect to r, the second with respect to p, the third with respect to q, and add. The quantities a are eliminated, and we have the result

$$(2) \quad (b_{qr} - b_{rq}) Y_p + (b_{rp} - b_{pr}) Y_q + (b_{pq} - b_{qp}) Y_r = 0,$$

where additional suffixes denote differentiation.

Now the equations (1) are unaltered if we replace b_p by $b'_p + \lambda Y$, where λ is an arbitrary function of the variables, and functions b and λ can be determined to satisfy the two equations

$$b_1 = \lambda Y_1 + \frac{\partial b}{\partial x_1}, \qquad b_2 = \lambda Y_2 + \frac{\partial b}{\partial x_2},$$

for elimination of λ gives a single equation for b, and any solution of this, combined with one of the above equations serves to determine λ.

We may thus in equations (1), (2), assume b_p replaced by b'_p, where b'_1 and b'_2 are first derivatives of a function b. Also we write

$$b'_p - \frac{\partial b}{\partial x_p} = b''_p.$$

In equation (2) give p, q, r, the values 1, 2, 3. It becomes precisely

$$J\begin{pmatrix} \lambda, & b''_3 \\ x_1, & x_2 \end{pmatrix} = 0,$$

and therefore b''_3 is a function of $Y, x_3, x_4, \cdots, x_n$ only.

We can therefore find a function $F(Y, x_3, x_4, \cdots, x_n)$ such that $b''_3 = (\partial F/\partial x_3)_0$ where the suffix indicates that Y is kept constant. Hence

$$b''_3 = \frac{\partial F}{\partial x_3} - \frac{\partial F}{\partial Y} Y_3;$$

also

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial Y} Y_1, \qquad \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial Y} Y_2,$$

and therefore

$$b_1 = \left(\lambda - \frac{\partial F}{\partial Y}\right) Y_1 + \frac{\partial}{\partial x_1}(b + F),$$

$$b_2 = \left(\lambda - \frac{\partial F}{\partial Y}\right) Y_2 + \frac{\partial}{\partial x_2}(b + F),$$

$$b_3 = \left(\lambda - \frac{\partial F}{\partial Y}\right) Y_3 + \frac{\partial}{\partial x_3}(b + F),$$

or, changing the notation, we have found functions b and λ such that $b_p = \lambda Y_p + b'_p$, and $b'_p = \partial b/\partial x_p$ for $p = 1, 2, 3$.

If we now apply (2) for the three sets of values (1, 2, 4), (2, 3, 4), (3, 1, 4), we get

$$J\begin{pmatrix} Y, b''_4 \\ x_1, x_2 \end{pmatrix} = 0, \quad J\begin{pmatrix} Y, b''_4 \\ x_2, x_3 \end{pmatrix} = 0, \quad J\begin{pmatrix} Y, b''_4 \\ x_3, x_1 \end{pmatrix} = 0,$$

and hence b''_4 is a function of $Y, x_4, x_5, \cdots, x_n$ only. As before we may modify λ and b, so as to make $b'_p = \partial b/\partial x_4$ for $p = 1, 2, 3, 4$, and the process may be continued so that finally we have

$$(3) \qquad b_p = \lambda Y_p + \frac{\partial b}{\partial x_p}$$

for all values of p.

Again, from (1),

$$\frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = \frac{\partial b}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial b}{\partial x_q}\frac{\partial Y}{\partial x_p}$$

$$= \frac{\partial}{\partial x_p}\left(b\frac{\partial Y}{\partial x_q}\right) - \frac{\partial}{\partial x_q}\left(b\frac{\partial Y}{\partial x_p}\right),$$

or

$$\frac{\partial}{\partial x_q}\left(a_p + b\frac{\partial Y}{\partial x_p}\right) = \frac{\partial}{\partial x_p}\left(a_q + \frac{\partial Y}{\partial x_q}\right),$$

and therefore

$$a_p = -\,b\frac{\partial Y}{\partial x_p} + \frac{\partial Z}{\partial x_p},$$

where Z is a new function. The complete solution of (1) is therefore given by

$$(4) \qquad a_p = -\,b\frac{\partial Y}{\partial x_p} + \frac{\partial Z}{\partial x_p}, \quad b_p = \lambda\frac{\partial Y}{\partial x_p} + \frac{\partial b}{\partial x_p},$$

where Z, b, λ, are three arbitrary functions.

Now consider the equation

$$(5)\qquad \frac{\partial}{\partial x_r}(H_{pq}) + K_{pq}\frac{\partial Y}{\partial x_r} = \frac{\partial}{\partial x_q}(H_{pr}) + K_{pr}\frac{\partial Y}{\partial x_q}.$$

Keep p fixed, and let $H_{pq} = a_q$, $K_{pq} = -b_q$. We now have equation (1), and hence

$$(6)\qquad H_{pq} = -B_p\frac{\partial Y}{\partial x_q} + \frac{\partial Z_p}{\partial x_q},$$

$$(7)\qquad K_{pq} = \lambda_p\frac{\partial Y}{\partial x_q} + \frac{\partial B_p}{\partial x_q},$$

where λ_p, B_p, Z_p, denote $3n$ as yet arbitrary functions.

Again, $H_{pq} = H_{qp}$, and therefore from (6)

$$\frac{\partial Z_p}{\partial x_q} - \frac{\partial Z_q}{\partial x_p} = B_p\frac{\partial Y}{\partial x_q} - B_q\frac{\partial Y}{\partial x_p}.$$

This equation is of the same type as (1), and hence

$$(8)\qquad Z_p = -B\frac{\partial Y}{\partial x_p} + \frac{\partial C}{\partial x_p},\quad B_p = \nu\frac{\partial Y}{\partial x_p} + \frac{\partial B}{\partial x_p}.$$

Similarly from the condition $K_{pq} = K_{qp}$ we have the equations

$$(9)\qquad B_p = -\lambda\frac{\partial Y}{\partial x_p} + \frac{\partial \eta}{\partial x_p},\quad -\lambda_p = -\mu\frac{\partial Y}{\partial x_p} + \frac{\partial \lambda}{\partial x_p}.$$

It follows without difficulty that $\nu = -\lambda$, $\eta = B$, and hence, substituting in (6) and (7) we have the final results

$$H_{pq} = \lambda\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial B}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial B}{\partial x_p}\frac{\partial Y}{\partial x_q} - B\frac{\partial^2 Y}{\partial x_p\partial x_q} + \frac{\partial^2 C}{\partial x_p\partial x_q},$$

$$K_{pq} = \mu\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial \lambda}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial \lambda}{\partial x_q}\frac{\partial Y}{\partial x_p} - \lambda\frac{\partial^2 Y}{\partial x_p\partial x_q} + \frac{\partial^2 B}{\partial x_p\partial x_q}.$$

The $n(n+1)$ quantities H, K, thus depend on the four arbitrary functions λ, μ, B, C.

The above relations may also be written

$$H_{pq} = \frac{\partial^2 A}{\partial x_p\partial x_q} + Y\frac{\partial^2 B}{\partial x_p\partial x_q} + \lambda\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q},$$

$$K_{pq} = \frac{\partial^2}{\partial x_p\partial x_q}(B - \lambda Y) + Y\frac{\partial^2 \lambda}{\partial x_p\partial x_q} + \mu\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q}.$$

Bryn Mawr,
May, 1909.

NECESSARY CONDITIONS THAT THREE OR MORE PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER SHALL HAVE COMMON SOLUTIONS.

BY PROFESSOR C. A. NOBLE.

(Read before the San Francisco Section of the American Mathematical Society, September 26, 1908.)

IN a paper inspired by Hilbert's lectures in 1900, Yoshiye (*Mathematische Annalen*, volume 57) considers, among others, the following problem in the calculus of variations: To find the necessary conditions that the integral

$$\int_{u_0}^{u_1} [\lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty')]\,du$$

shall vanish independently of the path of integration, whereby the two equations

$$F(x, y, z, p, q, r, s, t) = 0, \quad G(x, y, z, p, q, r, s, t) = 0$$

shall be satisfied. λ, μ, ν are arbitrary functions of u; p, q, r, s, t have the usual signification, i. e., $p = \partial z/\partial x$, $q = \partial z/\partial y$, etc.; the accents denote differentiation with respect to u.

The conditions which result upon consideration of this problem are that the two equations

$$\nu^2 F_r + \mu\nu F_s + \mu^2 F_t = 0, \quad \nu^2 G_r + \mu\nu G_s + \mu^2 G_t = 0$$

shall have a common solution in $\mu:\nu$, i. e., that the determinant

$$\begin{vmatrix} F_r & F_s & F_t & 0 \\ G_r & G_s & G_t & 0 \\ 0 & F_r & F_s & F_t \\ 0 & G_r & G_s & G_t \end{vmatrix}$$

shall vanish; and furthermore, that for the value of $\mu:\nu$ which satisfies these two equations the relation

$$\mu[F_t(G_y) - G_t(F_y)] + \nu[F_r(G_x) - G_r(F_x)] = 0$$

shall be satisfied. F_x, F_y, etc., in the foregoing, denote par-

tial derivatives with respect to x, y, etc.; the symbols (F_x), (F_y) are abbreviations for

$$F_x + F_z p + F_p r + F_q s \quad \text{and} \quad F_y + F_z q + F_p s + F_q t$$

respectively, and similarly for (G_x) and (G_y).

If one adds to the auxiliary conditions, in Yoshiye's problem, the third partial differential equation

$$H(x, y, z, p, q, r, s, t) = 0,$$

the first variation of the following integral must vanish:

$$\int_{u_0}^{u_1} [\lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty') + \xi F + \eta G + \zeta H]\, du$$

where ξ, η, ζ are Lagrange multipliers. The Lagrange equations for the determination of the functions $x, y, z, p, q, r, s, t, \lambda, \mu, \nu, \xi, \eta, \zeta$ are as follows:

1) $(\lambda p)' + (\mu r)' + (\nu s)' + \xi F_x + \eta G_x + \zeta H_x = 0,$

2) $(\lambda q)' + (\mu s)' + (\nu t)' + \xi F_y + \eta G_y + \zeta H_y = 0,$

3) $\lambda' + \xi F_z + \eta G_z + \zeta H_z = 0,$

4) $\mu' - \xi F_p - \eta G_p - \zeta H_p + \lambda x' = 0,$

5) $\nu' - \xi F_q - \eta G_q - \zeta H_q + \lambda y' = 0,$

6) $\mu x' - \xi F_r - \eta G_r - \zeta H_r = 0,$

7) $\mu y' + \nu x' - \xi F_s - \eta G_s - \zeta H_s = 0,$

8) $\nu y' - \xi F_t - \eta G_t - \zeta H_t = 0,$

9) $z' - px' - qy' = 0,$ 10) $p' - rx' - sy' = 0,$

11) $q' - sx' - ty' = 0,$

12) $F = 0,$ 13) $G = 0,$ 14) $H = 0,$

Eliminating λ', μ', ν' from 1) and 2) by means of 3), 4), and 5), we obtain

15) $$\mu r' + \nu s' + \xi(F_x) + \eta(G_x) + \zeta(H_x) = 0,$$

16) $$\mu s' + \nu t' + \xi(F_y) + \eta(G\ ) + \zeta(H_y) = 0.$$

Substituting in

$$\frac{dF}{du} = F_x x' + F_y y' + F_z z' + F_p p' + F_q q' + F_r r' + F_s s' + F_t t' = 0$$

for x', y', z', p', q', r', t' their values taken from 6), 8), 9), 10), 11), 15), 16), we obtain

$$\frac{\xi F_r + \eta G_r + \zeta H_r}{\mu}(F_x) + \frac{\xi F_t + \eta G_t + \zeta H_t}{\nu}(F_y)$$

$$- F_r \cdot \frac{\nu s' + \xi(F_x) + \eta(G_x) + \zeta(H_x)}{\mu} + F_s s'$$

$$- F_t \cdot \frac{\mu s' + \xi(F_y) + \eta(G) + \zeta(H_y)}{\nu} = 0.$$

Two similar equations arise when we employ $dG/du = 0$ and $dH/du = 0$. Solving in these three equations for s', and using the further abbreviation

$$\mu[F_t(G_y) - G_t(F_y)] + \nu[F_r(G_x) - G_r(F_x)] \equiv (F,\ G),$$

whereby $(F,\ F) \equiv 0$ and $(F,\ G) = -(G,\ F)$, we obtain

$$\alpha) s'\{\nu^2 F_r - \mu\nu F_s + \mu^2 G_t\} = (F,\ F)\xi + (G,\ F)\eta + (H,\ F)\zeta,$$

$$\beta) s'\{\nu^2 G_r - \mu\nu G_s + \mu^2 G_t\} = (F,\ G)\xi + (G,\ G)\eta + (H,\ G)\zeta,$$

$$\gamma) s'\{\nu^2 H_r - \mu\nu H_s + \mu^2 H_t\} = (F,\ H)\xi + (G,\ H)\eta + (H,\ H)\zeta.$$

From the last three equations we can deduce the following: Since $F = 0$ and $G = 0$ must have solutions in common, the left hand members of α), β) must have a common root in $\mu : \nu$; and for this common root $(F,\ G)$ must vanish. It follows that for this same $\mu : \nu$ $(H,\ F)$ and $(H,\ G)$ also vanish, hence also that both members of γ) vanish. In other words, the left hand members of α), β), γ) have a common root in $\mu : \nu$, the condition for which is

$$\left\| \begin{matrix} F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 \\ 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 \\ 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 \\ 0 & G_r & G_s & G_t & 0 & H_r & H_s & H_t \end{matrix} \right\| = 0.$$

For this common root in $\mu : \nu$ we have $(F, G) = 0, (G, H) = 0$ and $(H, F) = 0$, which yields

$$\frac{F_r(G_x) - G_r(F_x)}{F_t(G_y) - G_t(F_y)} = \frac{G_r(H_x) - H_r(G_x)}{G_t(H_y) - H_t(G_y)} = \frac{H_r(F_x) - F_r(H_x)}{H_t(F_y) - F_t(H_y)}.$$

If we inquire as to the possibility of common solutions to the four partial differential equations of second order

$$F = 0, \quad G = 0, \quad H = 0, \quad K = 0,$$

we obtain, by analogous procedure, the following four equations

$$\alpha') \quad \begin{aligned} & s'\{\nu^2 F_r - \mu\nu F_s + \mu^2 F_t\} \\ & \qquad = (F, F)\xi + (G, F)\eta + (H, F)\zeta + (K, F)\theta, \end{aligned}$$

$$\beta') \quad \begin{aligned} & s'\{\nu^2 G_r - \mu\nu G_s + \mu^2 G_t\} \\ & \qquad = (F, G)\xi + (G, G)\eta + (H, G)\zeta + (K, G)\theta, \end{aligned}$$

$$\gamma') \quad \begin{aligned} & s'\{\nu^2 H_r - \mu\nu H_s + \mu^2 H_t\} \\ & \qquad = (F, H)\xi + (G, H)\eta + (H, H)\zeta + (K, H)\theta, \end{aligned}$$

$$\delta') \quad \begin{aligned} & s'\{\nu^2 K_r - \mu\nu K_s + \mu^2 K_t\} \\ & \qquad = (F, K)\xi + (G, K)\eta + (H, K)\zeta + (K, K)\theta. \end{aligned}$$

Since $F = 0$, $G = 0$, and $H = 0$ have common solutions, the left hand members (and hence the right hand members) of α'), β'), γ') vanish for the same $\mu : \nu$; and for this $\mu : \nu$ we have

$$(F, G) = 0, \quad (G, H) = 0, \quad (H, F) = 0,$$

consequently also

$$(K, F) = 0, \quad (K, G) = 0, \quad (K, H) = 0;$$

and hence the left hand member of δ') vanishes for this same $\mu : \nu$. We obtain therefore, as a necessary condition that

$$F = 0, \quad G = 0, \quad H = 0, \quad K = 0$$

may have common solutions

$$\begin{Vmatrix} F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 & K_r & K_s & K_t & 0 \\ 0 & G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 & K_r & K_s & K_t \end{Vmatrix} = 0.$$

And since

$$(F, G) = (F, H) = (F, K) = (G, H) = (G, K) = (HK) = 0$$

for the same value of $\mu : \nu$, we have, in addition to the vanishing of the above matrix,

$$\frac{F_r(G_x) - G_r(F_x)}{F_t(G_y) - G_t(F_y)} = \frac{F_r(H_x) - H_r(F_x)}{F_t(H_y) - H_t(F_y)} = \frac{F_r(K_x) - K_r(F_x)}{F_t(K_y) - K_t(F_y)}$$
$$= \frac{G_r(H_x) - H_r(G_x)}{G_t(H_y) - H_t(G_y)} = \frac{G_r(K_x) - K_r(G_x)}{G_t(K_y) - K_t(G_y)} = \frac{H_r(K_x) - K_r(H_x)}{H_t(K_y) - K_t(H_y)}.$$

Obviously the plan is general, and one could write down the necessary conditions that a system of n partial differential equations of the type above considered should have solutions in common.

NOTE ON DETERMINANTS WHOSE TERMS ARE CERTAIN INTEGRALS.

BY PROFESSOR R. G. D. RICHARDSON AND MR. W. A. HURWITZ.

(Read before the American Mathematical Society, September 14, 1909.)

THE object of the present note is to prove two simple identities involving a determinant whose elements are certain integrals, and to mention some special cases. Determinants of the form considered present themselves in problems connected with linear differential and integral equations and the calculus of

variations.* The theorems proved include in particular the well-known Schwarz's and Bessel's identities and inequalities, which are of value in the theory of linear integral equations.† A determinant involving infinite series in the same way in which the determinants here considered involve integrals is used by Schmidt in his theory of points and vectors of a function space.‡ Expressions of the same character present themselves in the problem of finding, according to the criterion of the method of least squares, the nearest approximation to an arbitrary function in terms of given functions, not necessarily orthogonal.

1. *Let* $\phi_1, \phi_2, \cdots, \phi_n$; $\psi_1, \psi_2, \cdots, \psi_n$ *denote functions which are limited and integrable in a certain region* R *of an* m*-dimensional space, or if unlimited, are such that the product of any function of the first set by any function of the second set is integrable; let* D_i, $\bar{D}_j$ *denote the determinant obtained by replacing respectively the* i*th row of the determinant*

$$D = \begin{vmatrix} \int_R \phi_1\psi_1 dR & \int_R \phi_1\psi_2 dR & \cdots & \int_R \phi_1\psi_n dR \\ \int_R \phi_2\psi_1 dR & \int_R \phi_2\psi_2 dR & \cdots & \int_R \phi_2\psi_n dR \\ \cdot & \cdot & \cdot & \cdot \\ \int_R \phi_n\psi_1 dR & \int_R \phi_n\psi_2 dR & \cdots & \int_R \phi_n\psi_n dR \end{vmatrix}$$

by $\psi_1, \psi_2, \cdots, \psi_n$ *or the* j*th column by* $\phi_1, \phi_2, \cdots, \phi_n$, *and let* Δ_{ij} *be the co-factor of that term of* D *which stands in the* i*th row and the* j*th column. Then*

$$\int_R D_i \bar{D}_j dR = D\Delta_{ij}. \tag{1}$$

* For further identities which arise when the functions involved are known to be solutions of certain differential equations, see an article by Professor Richardson entitled "Das Jacobische Kriterium der Variationsrechnung und die Oscillationseigenschaften linearer Differentialgleichungen 2. Ordnung," shortly to appear in the *Mathematische Annalen*.

† Cf. Erhard Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Mathematische Annalen*, vol. 63 (1907), p. 433.

‡ "Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten," *Rendiconti del Circolo Matematico di Palermo*, vol. 25 (1908), p. 62.

The proof follows directly from the following applications of well-known theorems on the expansion of determinants:

$$\sum_{q=1}^{n}\Delta_{rq}\int_R \phi_p\psi_q dR = \begin{cases} 0, & p \neq r, \\ D, & p = r, \end{cases}$$

$$\sum_{q=1}^{n}\Delta_{iq}\psi_q = D_i, \quad \sum_{p=1}^{n}\Delta_{pj}\phi_p = \bar{D}_j.$$

For we have

$$\int_R D_i\bar{D}_j dR = \sum_{p=1}^{n}\sum_{q=1}^{n}\Delta_{iq}\Delta_{pj}\int_R \phi_p\psi_q dR$$

$$= \sum_{p=1}^{n}\Delta_{pj}\left[\sum \Delta_{iq}\int_R \phi_p\psi_q dR\right] = \Delta_{ij}D,$$

which was to be proved.

2. *Let $R_1, R_2, \cdots, R_n$ denote n equal regions in different m-dimensional spaces and $\phi_i(R_j)$, $\psi_i(R_j)$ the functions ϕ_i, ψ_i defined in the region R_j. Then*

$$(2) \quad \int_{R_1}\cdots\int_{R_n}\begin{vmatrix}\phi_1(R_1) & \cdots & \phi_1(R_n) \\ \cdot & \cdot & \cdot \\ \phi_n(R_1) & \cdots & \phi_n(R_n)\end{vmatrix} \cdot \begin{vmatrix}\psi_1(R_1) & \cdots & \psi_1(R_n) \\ \cdot & \cdot & \cdot \\ \psi_n(R_1) & \cdots & \psi(_nR_n)\end{vmatrix}$$
$$\times\, dR_1 \cdots dR_n = n\,!\,D.$$

In order to prove the theorem we note that the sign of any term $\phi_{i_1}(R_{k_1})\cdot\phi_{i_2}(R_{k_2})\cdots\phi_{i_n}(R_{k_n})$ in the first determinant of the integrand, where each of the sequences $i_1, i_2, \cdots, i_n$; $k_1, k_2, \cdots, k_n$ is a permutation of the sequence of integers $1, 2, \cdots, n$, is the power of -1 whose index is the number of transpositions required to build the substitution $\binom{i_1 i_2 \cdots i_n}{k_1 k_2 \cdots k_n}$ — that is, to bring the term considered into the principal diagonal. Similarly the sign of the term $\psi_{j_1}(R_{k_1})\cdot\psi_{j_2}(R_{k_2})\cdots\psi_{j_n}(R_{k_n})$ in the second determinant is the power of -1 whose index is the number of transpositions required to build the substitution $\binom{j_1 j_2 \cdots j_n}{k_1 k_2 \cdots k_n}$. The result of integrating the product of these two terms is

$$\int_{R_{k_1}}\phi_{i_1}(R_{k_1})\psi_{j_1}(R_{k_1})dR_{k_1}\cdots\int_{R_{k_n}}\phi_{i_n}(R_{k_n})\psi_{j_n}(R_{k_n})dR_{k_n}$$

$$= \int_R \phi_{i_1}\psi_{j_1}dR\cdots\int_R\phi_{i_n}\psi_{j_n}dR,$$

to which is attached a sign equal to the power of -1 whose index is the sum of the numbers of transpositions required to build the substitutions $\binom{i_1 i_2 \cdots i_n}{k_1 k_2 \cdots k_n}$ and $\binom{j_1 j_2 \cdots j_n}{k_1 k_2 \cdots k_n}$; this sum furthermore is equal to the number of transpositions required to build the substitution $\binom{i_1 i_2 \cdots i_n}{j_1 j_2 \cdots j_n}$. The process of integration therefore leads to a term of D with the proper sign attached. By using every permutation of the sequence $(1, 2, \cdots, n)$ for each of $(i_1, i_2, \cdots, i_n)$, $(j_1, j_2, \cdots, j_n)$, $(k_1, k_2, \cdots, k_n)$ we obtain the result of integrating every term in the product of the two determinants on the left of (2). By using every permutation of this sequence for each of $(i_1, i_2, \cdots, i_n)$, $(j_1, j_2, \cdots, j_n)$, we obtain every term of D once and only once for each choice of $(k_1, k_2, \cdots, k_n)$. We have thus the value of D repeated as often as the number of the permutations of the sequence $(1, 2, \cdots, n)$, — that is, $n!$ times — which proves the theorem.

3. Both theorems reduce to interesting identities under various specializations. For example, if we take* $\psi_1 = k\phi_1$, $\psi_2 = k\phi_2, \cdots, \psi_n = k\phi_n$, we have from (1)

$$(3)\quad D\Delta_{11} = \int_R k \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \int k\phi_2\phi_1 dR & \int k\phi_2^2 dR & \cdots & \int k\phi_2\phi_n dR \\ \cdot & \cdot & \cdot & \cdot \\ \int k\phi_n\phi_1 dR & \int k\phi_n\phi_2 dR & \cdots & \int k\phi_n^2 dR \end{vmatrix}^2 dR,$$

where

$$D = \begin{vmatrix} \int k\phi_1^2 dR & \int k\phi_1\phi_2 dR & \cdots & \int k\phi_1\phi_n dR \\ \int k\phi_2\phi_1 dR & \int k\phi_2^2 dR & \cdots & \int k\phi_2\phi_n dR \\ \cdot & \cdot & \cdot & \cdot \\ \int k\phi_n\phi_1 dR & \int k\phi_n\phi_2 dR & \cdots & \int k\phi_n^2 dR \end{vmatrix},$$

$$\Delta_{11} = \begin{vmatrix} \int k\phi_2^2 dR & \cdots & \int k\phi_2\phi_n dR \\ \cdot & \cdot & \cdot \\ \int k\phi_n\phi_2 dR & \cdots & \int k\phi_n^2 dR \end{vmatrix};$$

* In this case the restriction imposed above reduces to the condition that $k\phi_i\phi_j$ is integrable $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n)$.

and from (2)

$$(4) \qquad D=\frac{1}{n!}\int_{R_1}\cdots\int_{R_n} k(R_1)\cdots k(R_n)$$

$$\times\begin{vmatrix}\phi_1(R_1) & \cdots & \phi_1(R_n)\\ \cdot & \cdots & \cdot\\ \phi_n(R_1) & \cdots & \phi_n(R_n)\end{vmatrix}^2 dR_1\cdots dR_n.$$

It follows that, if k is nowhere negative in R, D is not negative, and that if k is everywhere positive in R, D cannot vanish unless the functions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly dependent. This result is obviously a generalization of the theorem known as Schwarz's inequality, in which the case $m = 1$, $n = 2$ is considered.

4. Let us consider further two sets of functions $\phi_1, \phi_2, \cdots, \phi_n$; $\psi_1, \psi_2, \cdots, \psi_n$ satisfying the relations*

$$\int_R \phi_i\psi_j dR = 0 \quad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n;\ i \neq j).$$

If we write (1) for the sets of $n + 1$ functions $\phi_0 = f, \phi_1, \phi_2, \cdots, \phi_n$; $\psi_0 = g, \psi_1, \psi_2, \cdots, \psi_n$, we have for $i = j = 0$ the result

$$(5) \qquad \int_R \left[Af - \sum_{p=1}^{n} A_p\phi_p \int_R f\psi_p dR\right]$$

$$\times\left[Ag - \sum_{p=1}^{n} A_p\psi_p \int_R g\phi_p dR\right] dR$$

$$= A\left[A\int_R fg dR - \sum_{p=1}^{n} A_p \int_R f\psi_p dR \int_R g\phi_p dR\right],$$

where

$$a_p = \int_R \phi_p\psi_p dR, \quad A = a_1\cdot a_2\cdots a_n,$$

$$A_p = a_1\cdot a_2\cdots a_{p-1}\cdot a_{p+1}\cdots a_n.$$

The further assumption that $\phi_i = \psi_i$ and that $a_i = 1$ $(i = 1, 2, \cdots, n)$, i. e., that we are dealing with a normalized set

* E. g., solutions of the differential equations $L(\phi_i) + \lambda_i\phi_i = 0$, $M(\psi_j) + \lambda_j(\psi_j) = 0$ (where L and M are adjoint differential expressions of the second order in any number of independent variables) in a closed region R, which vanish on the boundary of R.

of orthogonal functions, leads, if applied to the case $f=g$ to Bessel's identity

$$(6) \int_R \left[f - \sum_{p=1}^{n} \phi_p \int_R f\phi_p dR \right]^2 dR = \int_R f^2 dR - \sum_{p=1}^{n} \left[\int_R f\phi_p dR \right]^2,$$

from which Bessel's inequality immediately follows.

5. Theorems analogous to those of the present note involving finite sums or infinite series in place of integrals may be proved in a similar manner.

GÖTTINGEN,
June, 1909.

ON THE TACTICAL PROBLEM OF STEINER.

BY PROFESSOR W. H. BUSSEY.

(Read before the American Mathematical Society, February 24, 1906.)

THE study of tactical configurations known as triple systems had its origin in two problems proposed independently by J. Steiner * and T. P. Kirkman. † The Steiner problem, which is the more general and includes the other, is as follows:

For what values of n is it possible to arrange n elements in sets of three, called triads, so that every set of two elements is contained in one and only one triad? If n is a number for which there is such an arrangement in triads, are there other arrangements that cannot be obtained from it by a mere permutation of the elements? When such an arrangement in triads has been made, is it possible to arrange the n elements in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three that is not a triad is contained in one and only one tetrad? When such an arrangement in tetrads has been made, is it possible to arrange the n elements in sets of five, called pentads, so that no triad or tetrad is contained in a pentad, and so that every set of four that is not a tetrad and does not contain a triad is contained in one and only one pentad? In general, when an arrangement in *k-ads* has been made, is it possible to arrange the n elements in sets of $k+1$, called $(k+1)$-*ads* so that no *l-ad* $(l \leqq k)$ is contained in a $(k+1)$-*ad*, and so that every set of k elements that is not a

* *Journal für die reine und angewandte Mathematik*, vol. 45, p. 181.

† *The Lady's and Gentleman's Diary* for 1850. For other references to the literature of Kirkman's fifteen school girls problem see Ball's Mathematical Recreations and Essays, 4th edition, page 121.

k-ad and does not contain an *l-ad* ($l < k$) is contained in one and only one $(k+1)$-*ad*?

The part of the problem that relates to triads has been completely solved.* The other parts have been little studied.

If an arrangement of n elements in triads, tetrads, pentads, etc., is possible, the number of *k-ads* for $k = 3, 4, 5, \cdots$ is given by the formula

$$N_k = \frac{1}{k!} n(n-1)(n-3) \cdots (n - [2^{k-2} - 1]).$$

This formula was suggested by Steiner. It may be proved without much difficulty by complete induction.

This paper has to do with the case in which n is a number of the form $2^j - 1$. Its object is to show that it is possible to arrange such a number of elements in *k-ads* for $k = 3, 4, 5, \cdots, j+1$. The formula gives $N_k = 0$ when $k > j + 1$.

Consider the $2^{k+1} - 1$ elements $(x_1, x_2, x_3, \cdots, x_{k+1})$, each x being 0 or 1 and the element $(0, 0, 0, \cdots, 0)$ being excluded. For convenience the language of geometry is used and each of the elements is called a point. The $2^{k+1} - 1$ points are said to constitute a finite geometry of k dimensions, or, more briefly, a k-space.† Consider also the linear homogeneous congruence, modulo 2,

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{k+1}x_{k+1} \equiv 0, \tag{1}$$

in which each coefficient is 0 or 1 and at least one of them is not zero. The points of the k-space that satisfy such a congruence are said to constitute a $(k-1)$-space; the points that satisfy two linearly independent congruences of the type (1) are said to constitute a $(k-2)$-space; and, in general, the points that satisfy $(k-l)$ linearly independent congruences of type (1) are said to constitute an l-space. The number of solutions of a set of congruences of type (1) may be counted without much difficulty and the number of points in an l-space, $l < k$, found to be $2^{l+1} - 1$. In particular, the number of points in a plane (2-space) is seven, and the number in a line (1-space) is three. A single point constitutes a 0-space. The points common to two l-spaces, if there are any, constitute an r-space, where $0 \leqq r \leqq l - 1$. A set of $l+1$ points which are

* *Encyclopédie des Sciences mathématiques*, vol. 1, p. 80.

† See Veblen and Bussey, "Finite projective geometries," *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 241–259. In particular, see § 2.

not all contained in the same $(l-1)$-space is contained in one and only one l-space. The $l+1$ points of such a set, if taken l at a time, determine a number of $(l-1)$-spaces whose points constitute a set that may conveniently be called a *simplex* * of order l. The $l+1$ points are called vertices. A convenient symbol for a simplex of order l is $S(l+1)$. Any $i+1$ of the vertices of a $S(l+1)$ are the vertices of a simplex $S(i+1)$ whose points are all contained in the $S(l+1)$.

THEOREM. *The number of points in a simplex of order l is one less than the number of points in the l-space determined by its $l+1$ vertices.*

By actual count, the theorem is true for $l \leqq 3$. The rest of the proof consists in showing that it can be proved for a simplex $S(m+1)$ if it be assumed true for every simplex $S(l+1)$ for which $l < m$. This is done by arranging the points of the simplex $S(m+1)$ in the m following sets. The sets are not mutually exclusive.

1. The $m+1$ vertices of the simplex $S(m+1)$.

2. The points of the ${}_{m+1}C_2$ lines determined by the vertices taken two at a time.

3. The points of the ${}_{m+1}C_3$ planes determined by the vertices taken three at a time.

4. The points of the ${}_{m+1}C_4$ 3-spaces determined by the vertices taken four at a time.

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$i+1$. The points of the ${}_{m+1}C_{i+1}$ i-spaces determined by the vertices taken $i+1$ at a time.

. . . .

. . . .

n. The points of the ${}_{m+1}C_m$ $(m-1)$-spaces determined by the vertices taken m at a time.

[Note: The symbol ${}_{m+1}C_j$ means the number of combinations of $m+1$ things taken j at a time.]

The set numbered $i+1$, i being any one of the numbers $1, 2, 3, \cdots, m$, consists of the points contained in ${}_{m+1}C_{i+1}$ i-spaces each of which is determined by $i+1$ of the vertices of the simplex $S(m+1)$ or, in other words, by the $i+1$ vertices of a simplex $S(i+1)$ which is contained in the simplex $S(m+1)$. By hypothesis, each of these i-spaces contains one and only one

* The word is used in geometry of n-dimensions to denote the configuration analogous to the triangle in the plane or the tetrahedron in 3-space.

point not contained in the simplex $S(i+1)$ that determines it. But that one point is a point of the simplex $S(m+1)$ by the very definition of simplex. Therefore, if one begins to count with the first set and counts through the sets in order, the number of points in the set numbered $i+1$ that have not been counted in any previous set is ${}_{m+1}C_{i+1}$. It follows that the number of points in the simplex $S(m+1)$ is

$$\sum_{j=1}^{m} {}_{m+1}C_j = 2^{m+1} - 2,$$

which is one less than the number of points in the m-space determined by the $m+1$ vertices of the simplex.

From this theorem it follows that the $l+1$ vertices of a simplex of order l determine uniquely another point, namely, the one point of the l-space determined by the simplex that is not also a point of the simplex. It is convenient to call this point the point complementary to the simplex. The triads, tetrads, pentads, etc. of the Steiner problem are found as follows: Every simplex $S(2)$ determines a triad consisting of its two vertices and the complementary point; every simplex $S(3)$ determines a tetrad consisting of its three vertices and the complementary point; and, in general, every simplex $S(l-1)$, $l \leqq k+2$, determines an *l-ad* consisting of the $l-1$ vertices and the complementary point. There are no *l-ads* for $l > k+2$.

When $n = 2^6 - 1 = 63$, it is possible to arrange the n elements in triads, tetrads, pentads, hexads, and heptads. There is no arrangement of the 63 elements in l-ads for $l > 7$. This special case was involved in Steiner's investigation of the configuration of the 28 double tangents of a quartic curve * and led him to propose for solution the "Combinatorische Aufgabe" which I have called "The tactical problem of Steiner."

ON THE SO-CALLED GYROSTATIC EFFECT.

BY PROFESSOR ALEXANDER S. CHESSIN.

(Read before the American Mathematical Society, April 24, 1909.)

IN computing the resisting couple of gyrostats or the so-called "gyrostatic effect" it is customary to assume that it is equal to $C\lambda\omega \sin\theta$, where C, λ, ω and θ denote respectively the moment of inertia of the gyrostat about its geometrical axis, the angular

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* *Journal für die reine und angewandte Mathematik*, vol. 49, pp. 265–272.

velocity of spin, the angular velocity of precession and the angle of the geometrical axis with the axis of precession. It is proposed here to give the exact value of this couple.

The gyrostatic effect is due to the action of what I have called the *convective* and the *turning* forces in the motion of a body relatively to a moving system (XYZ).* Let ω be the angular velocity of this system; p, q, r its components along X, Y, Z. To simplify the results we will assume that the center of gravity of the body is at the origin of these axes. Then the principal moments M and M' of the convective and the turning forces are given by their components

$$M_x = \sum_i (y_i F_{iz} - z_i F_{iy}), \quad M'_x = \sum_i (y_i F'_{iz} - z_i F'_{iy}),$$

$$M_y = \sum_i (z_i F_{ix} - x_i F_{iz}), \quad M'_y = \sum_i (z_i F'_{ix} - x_i F'_{iz}),$$

$$M_z = \sum_i (x_i F_{iy} - y_i F_{ix}), \quad M'_z = \sum_i (x_i F'_{iy} - y_i F'_{ix}),$$

where

$$\frac{1}{m_i} F_{ix} = z_i \frac{dq}{dt} - y_i \frac{dr}{dt} + p(px_i + qy_i + rz_i) - \omega^2 x_i,$$

$$\frac{1}{m_i} F_{iy} = x_i \frac{dr}{dt} - z_i \frac{dp}{dt} + q(px_i + qy_i + rz_i) - \omega^2 y_i,$$

$$\frac{1}{m_i} F_{iz} = y_i \frac{dp}{dt} - x_i \frac{dq}{dt} + r(px_i + qy_i + rz_i) - \omega^2 z_i,$$

$$\frac{1}{m_i} F'_{ix} = -2(q\dot{z}_i - r\dot{y}_i), \quad \frac{1}{m_i} F'_{iy} = -2(r\dot{x}_i - p\dot{z}_i),$$

$$\frac{1}{m_i} F'_{iz} = -2(p\dot{y}_i - q\dot{x}_i).$$

Let now $(\Xi H Z)$ be a system of axes coinciding with the principal axes of inertia of the body.

By a series of transformations which it does not seem worth

* "On relative motion," *Transactions American Mathematical Society*, vol. 1.

while to reproduce here we obtain the following expressions for M and M':

$$M_\xi = A\dot{\omega}_\xi - (B - C)\omega_\eta\omega_\zeta, \quad M_\eta = B\dot{\omega}_\eta - (C - A)\omega_\zeta\omega_\xi,$$
$$M_\zeta = C\dot{\omega}_\zeta - (A - B)\omega_\xi\omega_\eta,$$
$$M'_\xi = (A + B - C)Q\omega_\zeta - (A - B + C)R\omega_\eta,$$
$$M'_\eta = (B + C - A)R\omega_\xi - (B - C + A)P\omega_\zeta,$$
$$M'_\zeta = (C + A - B)P\omega_\eta - (C - A + B)Q\omega_\xi,$$

where A, B, C and P, Q, R are the principal moments of inertia of the body and the components of its angular velocity Ω in the relative motion.

When two of the moments of inertia, as in the case of gyrostats, are equal, *i. e.*, $A = B$, the expressions given above will be simplified by the following selection of axes: axis P, trace of plane ΞH on plane XY; axis Q at right angle to P in the plane ΞH (to the right of P relatively to R); and R coincident with Z. Introducing Euler's angles θ, ϕ, ψ we shall now have

$$M_1 = A\dot{\omega}_1 - (A - C)\omega_2\omega_3, \quad M'_1 = (2A - C)\omega_3\phi' \sin\theta - CR\omega_2,$$
$$M_2 = A\dot{\omega}_2 - (C - A)\omega_3\omega_1, \quad M'_2 = -(2A - C)\omega_3\theta' + CR\omega_1,$$
$$M_3 = C\dot{\omega}_3, \quad M'_3 = C(\omega_2\theta' - \omega_1\phi' \sin\theta),$$

using the symbols 1, 2, 3 in lieu of P, Q, R to indicate the corresponding components.

Were we to select the axes (XYZ) so that the Z axis coincide with the axis (ω), we would have $\omega_1 = 0$, $\omega_2 = \omega \sin\theta$, $\omega_3 = \omega \cos\theta$, and therefore

$$M'_1 = (2A - C)\omega\phi' \sin\theta \cos\theta - C\omega R \sin\theta,$$
$$M'_2 = -(2A - C)\omega\theta' \cos\theta, \quad M'_3 = C\omega\theta' \sin\theta.$$

These results show that even assuming that the system (XYZ) revolves with a constant angular velocity and that this velocity (ω) is very small (so that we may neglect terms of the order of ω^2), the commonly accepted value of the gyrostatic couple is incorrect. The last formulas reduce to $|M'| = C\omega\lambda \sin\theta$ if we assume that $\theta' = \phi' = 0$, *i. e.*, that the gyrostat axis is invariably fixed in the system (XYZ).

A CONTINUOUS GROUP RELATED TO VON SEIDEL'S OPTICAL THEORY.

BY DR. ARTHUR C. LUNN.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

THE determination of the various aberrations of an axially symmetric optical instrument according to the method systematized by Petzval and von Seidel rests on the computation of certain power series whose coefficients are functions of what may be called the paraxial magnitudes of the system. These are the quantities which describe what would be the course of the rays of light through the ideal instrument, to which the actual instrument is an approximation, and which was defined by Abbe* as producing an exact collineative transformation of the object into the image. This ideal transformation coincides with the classic first approximation of geometric optics, a general analytic representation of which was given by Gauss,† together with a method of computation using continued fractions. A set of formulas more convenient in practice was introduced by von Seidel,‡ using as defining coordinates of a ray in a meridian plane the optical height, or distance of the ray from any point of the axis, measured along the normal to the latter, and the angle, or in the ideal collineation the tangent of the angle, between the ray and the axis.

If these coordinates be called x, θ respectively, von Seidel's paraxial equations are equivalent to

$$\Delta\mu\theta = xk\Delta\mu, \quad \Delta x = 0, \quad D\theta = 0, \quad Dx = -t\theta,$$

where μ is the refractive index of a medium, k the curvature of a refracting spherical surface, t the distance of transmission through a homogeneous medium; Δ, D denote the changes corresponding respectively to refraction at a surface where μ changes and to propagation through a medium of constant μ separating two consecutive surfaces. It will be shown here that these equations may be viewed as transformations generat-

* Czapski-Eppenstein, Theorie der optischen Instrumente nach Abbe, chap. ii.

† Dioptrische Untersuchungen, Ges. Werke, V, p. 243.

‡ *Astr. Nach.*, Nos. 1027–1029 (April, 1855).

ing a certain three-parameter group, whose properties thus admit of optical interpretation. In particular the determination of the complete system of invariants seems to mark the exact scope of possible application of Abbe's concept of "optical invariant" in the formulation of optical computations.

Let $\mu\theta = u$, then the transformations and their generating differentiators are

$$(1) \qquad u' = u + \alpha x, \quad x' = x, \quad U_1 = x\frac{\partial}{\partial u}, \qquad (\alpha = k\Delta\mu)$$

$$(2) \qquad x' = x + \beta u, \quad u' = u, \quad U_2 = u\frac{\partial}{\partial x}, \quad \left(\beta = -\frac{t}{\mu}\right)$$

each of which defines a one-parameter group in the variables x, u, the respective parameters α, β being arbitrary because the equations apply to systems having arbitrary curvatures and axial spacing of the refracting surfaces.

The first group relates to successive refractions at a set of surfaces in contact, and corresponds to the ordinary theory of thin lenses close together. For example, the additive property of the focal powers of such lenses is the interpretation of the additive combination of the parameters α in the compounding of the transformations, while the invariance of x means simply that the primitive and refracted rays intersect at the refracting surface.

The second group relates to the optical effect of media stratified in parallel planes. The invariance of u means that such a system has no focal power according to the definition of Gauss, and the additive combination of parameters corresponds to the foreshortening effect of each stratum according to its thickness and refractive index.

The dual character of the two transformations under simultaneous interchange of x, u and α, β was recognized by von Seidel in his use of odd and even subscripts respectively for refracting surfaces and intervening media. It suggests its own phrasing for various statements. For instance, a principal focus is a point where $x = 0$ on a ray transformed from a primitive ray having $u = 0$, or parallel to the axis.

The successive commutators formed from U_1 and U_2 are

$$(3) \qquad (U_1 U_2) = U_3 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad (U_1 U_3) = -2U_1, \quad (U_2 U_3) = 2U_2.$$

There is thus generated a three-parameter group whose equations may be obtained* by integration of

$$\frac{dx'}{dt} = \lambda_2 u' + \lambda_3 x', \qquad \frac{du'}{dt} = \lambda_1 x' - \lambda_3 u',$$

giving

$$x' = A_1\lambda_2 e^{\omega t} + A_2\lambda_2 e^{-\omega t},$$
$$u' = A_1(\omega - \lambda_3)e^{\omega t} - A_2(\omega + \lambda_3)e^{-\omega t},$$
$$\omega^2 = \lambda_3^2 + \lambda_1\lambda_2,$$

where the constants A are to be determined in terms of the initial coordinates x, u. A convenient final form, containing the three parameters a, b, c is

$$(4) \qquad x' = xC + (cx + bu)S, \qquad u' = uC + (ax - cu)S,$$

where $C = \cosh\tau$, $S = \sinh\tau/\tau$, $\tau^2 = c^2 + ab$, and $a = \lambda_1 t$, $b = \lambda_2 t$, $c = \lambda_3 t$. The groups of U_1 and U_2 individually are obtained by putting respectively $b = c = \tau = 0$, $a = \alpha$, and $a = c = \tau = 0$, $b = \beta$.

The combination of two transformations such as (4) with parameters (a, b, c) and (a', b', c') is equivalent to a single such transformation with parameters (a'', b'', c'') determined by

$$(5) \qquad \begin{aligned} C'' + c''S'' &= (C' + c'S')(C + cS) + ab'S'S, \\ b''S'' &= bS(C' + c'S') + b'S'(C - cS), \\ C'' - c''S'' &= (C' - c'S')(C - cS) + a'bS'S, \\ a''S'' &= aS(C' - c'S') + a'S'(C + cS), \end{aligned}$$

which may be replaced by

$$(6) \qquad \begin{aligned} C'' &= CC' + SS'\{cc' + \tfrac{1}{2}(ab' + a'b)\}, \\ c''S'' &= cSC' + c'S'C + \tfrac{1}{2}SS'(ab' - a'b), \\ b''S'' &= bSC' + b'S'C + SS'(bc' - b'c), \\ a''S'' &= aSC' + a'S'C + SS'(ca' - c'a). \end{aligned}$$

Here the first equation determines the set of possible values of the auxiliary τ'', then a'', b'', c'' are given by the last three

* Lie-Engel, Theorie der Transformationsgruppen, I, p. 70.

equations; and a direct computation shows that if $\tau^2 = c^2 + ab$ and $\tau'^2 = c'^2 + a'b'$ then also $\tau''^2 = c''^2 + a''b''$, verifying the group property. The solution fails only if $S'' = 0$, which occurs when τ'' has the form $2n\pi i$ where n is an integer not zero. With τ in the neighborhood of such a value it is more convenient to use a/τ, b/τ, c/τ as parameters. It should be noticed that some complex values of the parameters give real transformations, and that if (a, b, c) and (a', b', c') are real numbers, still (a'', b'', c'') are not necessarily such. Moreover, (a, b, c) may be real but τ pure imaginary, if $c^2 + ab$ is negative.

The group of U_3 alone, obtained by putting $a = b = 0$, $c = \tau$, has the form

$$(7) \qquad x' = xe^{\tau}, \quad u = ue^{-\tau},$$

where the parameters combine additively and xu is invariant.

The general transformation (4) can be built up in the form $T_2 T_1 T_2'$, where each T_i is a transformation of the one-parameter group of U_i. For, if the factors have parameters β, α, β' respectively, the identification rests upon the solution of the equations

$$1 + \beta\alpha = C + cS, \quad \alpha = aS, \quad 1 + \beta'\alpha = C - cS,$$
$$\beta + \beta' + \beta\beta'\alpha = bS.$$

The first three give

$$\alpha = aS, \quad \beta = (C - 1 + cS)/aS, \quad \beta' = (C - 1 - cS)/aS,$$

and these values satisfy the fourth equation also, because of the relation $\tau^2 = c^2 + ab$. An alternative factorization is the dual form $T_1 T_2 T_1'$, with parameters α, β, α', obtained by interchange of α and β and also of a and b.

The first form interprets the general transformation in terms of a single refracting surface as giving the relation of the coordinates of a refracted ray at distance t' behind the surface to those of the primitive ray at distance t in front of the surface. A special case of interest is where x' is independent of u, or $\beta + \beta' + \beta\beta'\alpha = 0$. This is equivalent to $\mu/t + \mu'/t' = (\mu' - \mu)/r$, the ordinary equation of conjugate distances with respect to the surface. Such transformations occur for instance when the successive positions of a ray traversing any system are located with respect to the planes of the aperture diaphragm and its successive images.

The second form interprets the general transformation in terms of two separated refracting surfaces, and gives the relation of the coordinates of the primitive and refracted rays at points in the first and third media in contact with the surfaces; giving thus an analogy to the theorem of Gauss that any system is in the first approximation equivalent to a single thick lens. The special case, dual to the one mentioned in the preceding paragraph, is where u' is independent of x, or $\alpha + \alpha' + \alpha\alpha'/\beta = 0$. This is equivalent to the condition that the second principal focus of the first surface coincides with the first of the second surface, so that the lens is "afocal" or telescopic.

The three-parameter group in two variables thus far considered has no invariant other than a constant. But in the estimation of certain optical errors, such as astigmatism and distortion, which depend on the location of the apertures limiting the pencils of rays, it is necessary to consider not only the systems of rays proceeding from the object and transformed into those converging upon the images formed by the successive refractions, but also a fictitious system of rays proceeding from the "entrance pupil" or initial aperture diaphragm, real or virtual, and refracted into those converging upon its successive images in the same optical system. If the rays of this second system be specified by coordinates v, y, then these will be variables transformed cogrediently with u, x, by equations similar to (4) with the same parameters, belonging thus to a three-parameter group in the four variables generated by

$$U_1 = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}, \quad U_2 = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y},$$

$$U_3 = (U_1 U_2) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

so that a fundamental system will consist of a single proper invariant.

To find its form, let $\rho = ux + vy$, $\sigma = uy - vx$. Then the U's expressed in terms of the variables (x, y, ρ, σ) are

$$U_1 = (x^2 + y^2)\frac{\partial}{\partial \rho}, \quad U_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$U_2 = \frac{1}{(x^2 + y^2)}\left\{ (\rho x + \sigma y)\frac{\partial}{\partial x} + (\rho y - \sigma x)\frac{\partial}{\partial y} + (\rho^2 + \sigma^2)\frac{\partial}{\partial \rho} \right\},$$

whose form shows directly that an invariant must be independent of x, y, ρ, and therefore a function of σ only.

If u, x belong to a ray passing through the margin of the image and the axial point of the diaphragm, and v, y similarly to a ray passing through the center of the image and the edge of the diaphragm, a geometric construction shows that the expression $uy - vx$ is equivalent to the product of refractive index, lateral radius of image, and tangent of angular semi-aperture of central pencil. The invariant σ is therefore identical with the expression which occurs in the well-known equation pointed out in a special case by Lagrange, but first found as general by Helmholtz.* The result here shows that this is essentially the only relation expressible in terms of invariants which gives a general property of the paraxial transformation.

CHICAGO, ILL.,
April, 1909.

SHORTER NOTICES.

Analytische Geometrie der Ebene. By C. RUNGE. Leipzig, B. G. Teubner, 1908. 198 pp.

SHALL we regard elementary analytic geometry, the analytic geometry we teach in a first course to freshmen or sophomores, as a body of doctrine with which it is useful for the student to become acquainted, or shall we rather regard it as an instrument with whose use he is to be made so familiar that he shall always be ready to employ it even in a quite new problem? This is a question which every teacher of analytic geometry and every writer of a text-book on the subject is called upon to face. Upon its answer the nature of the text-book written or selected for use will depend. Few persons, it is true, would go to such an extreme as to adopt without qualification either of the views above referred to. Those who regard it as their main object to inculcate a beautiful and important doctrine would deem it essential that the student gain the power of making the application of the general theorems learned to concrete cases, and the teacher who regards elementary analytic geometry primarily as a method whose use is to be taught would not neglect the opportunity of explaining

* Handbuch der physiologischen Optik, 1. Auflage, p. 50.

and illustrating the use of this method, so far as possible, by means of problems which are in themselves of lasting interest and importance. With this reservation, however, the above classification serves to separate teachers and writers of text-books fairly well into two opposing camps.

In the former camp it is customary to take as one of the largest and most essential parts of the body of doctrine to be inculcated the theory of the conic sections; and the subjects of conjugate diameters, poles and polars, the general equation of the second degree, etc., form essential chapters with whose contents the student is supposed to become familiar. On the other hand such a teacher will usually be content, in the main, with numerical problems, which serve to test the pupil's understanding of the formulas and principles developed.

The teacher who adopts the second standpoint will also usually contrive, as has been suggested above, to familiarize his students with many facts concerning conic sections; but he will do this incidentally, his main concern being all the time to train his student to do things for himself. He will, therefore, regard the numerical problem as the lowest and least useful of all problems (except, indeed, for the pupil incapable of rising higher), while the problem which requires the student to handle his instrument in a slightly new way will be regarded as the highest type, to be used sparingly on account of its difficulty, but as invaluable because it gives to the good student the very best training possible in becoming a real master in the use of his method.

There is still a third current which has been running very strongly of late both in this country and in England, which, so far as it is concerned with the subject with which we are here dealing, has as its motto: "Analytic geometry is a necessary evil. Let us have as little of it as possible." The devotees of this cult, who commonly regard all mathematics as merely a tool, naturally take the same point of view with regard to analytic geometry, but, being engineers or in the employ of engineers, they believe that they have no interest in anything beyond the very simplest geometrical relations, and that consequently this particular tool needs only very slight development at their hands. In the rudimentary condition in which they are willing to leave it, numerical problems are about all that one can venture upon, and this is all they ask.

Professor Runge's book has an individuality and interest of

its own and refuses to fit in perfectly to the pigeon-holes of any classification. This is just what one familiar with his other writings would expect; and yet some of his statements have at first a very familiar ring. We quote the opening words of the preface:

"This book has grown out of lectures which for many years I was in the habit of delivering at the technological school at Hannover, and in writing it I have had constantly in mind the needs of the engineer. It seems to me that the analytic geometry taught in technological schools should, first of all, be a tool for following up geometrical relations by means of numerical computations."

The programme here laid out is carried through, so far as the simplest properties of the straight line and circle are concerned, in Part II, which extends from page 20 to page 63. This part is perhaps the most distinctive and interesting of the whole book. The problems taken up are of the very simplest types: A line is determined by two points, to compute the ordinates of those points on this line which have given abscissas. Or again: To compute the coordinates of the point of intersection of two lines each of which is determined by a point through which it passes and the angle it makes with the axis of x. Each of these problems (they are the first two, and are typical) is treated at length by several different methods, four or five pages being devoted to each problem. This seems surprising until on closer examination we notice that questions of analytic geometry proper are here quite pushed into the background by pure questions of convenient forms of numerical computation. What method will be best if logarithms are to be used? If we use the slide-rule? If we use a computing machine? How can the numerical work be best arranged? These are all good questions, and we Americans may well wish that systematic courses on numerical computation were more frequent in this country. It seems, however, to the reviewer that such questions should hardly be considered at the very beginning of a course on analytic geometry, as they are then in danger of overshadowing the fundamental principles of the subject.

Perhaps, however, in spite of its title and general appearance, it is hardly fair to treat this book as one from which the beginner will get his first acquaintance with analytic geometry. It should rather be regarded as supplementing in a most interesting way the more traditional treatises from which students

will continue to derive most of their working knowledge. Moreover the insistence on the numerical side of the subject, to which we have referred, is only one feature of the book. Long sections of a decidedly theoretical character follow in which affine and perspective transformations of the plane play an important part. Indeed the ellipse, hyperbola, and parabola are introduced as the images of the circle under these transformations, and the whole theory of these curves is made to depend on this point of view. Homogeneous coordinates are also treated at length in the last chapter.

On the whole the book departs less essentially than one would at first suppose from the traditional German text-book in which the subject is presented rather from the point of view of *kennen* than *können*. It is still a body of doctrine which is presented, though the choice of subjects is somewhat unusual, including as it does, besides the subjects already mentioned, a section on the computation of stresses in frameworks of light rods, and an introduction to some of the most elementary aspects of the use of vectors. Even the questions of numerical computation may fairly be regarded as part of this scheme rather than as an attempt to put the student on his own mettle. To a teacher collecting material for a course on numerical computation the section to which reference has just been made will be found most useful. MAXIME BÔCHER.

Gruppen- und Substitutionentheorie. By Dr. EUGEN NETTO. Sammlung Schubert LV. G. J. Göschen, Leipzig, 1908. viii + 175 pp.

IN conformity with the general plan of the Sammlung Schubert Professor Netto aims to give in this book an introduction to the theory of groups of finite order. He has succeeded admirably in his purpose. Those readers who are not already familiar with the details of the theory will find Chapter II particularly valuable in fixing for them the fundamental notions of the subject, if they take the pains to work through the details. Indeed we do not know if there is another place where this particular phase of the subject is treated so happily. But this is by no means the only good chapter. They are all excellent and the book as a whole is a fine example of clear and attractive exposition.

The author introduces some new notation which is of value in the interest of brevity. On page 35 the greatest common sub-

group of the groups $G, H, K, \cdots$ is denoted by the symbol $\}G, H, K, \cdots\{$. The largest subgroup of a group G within which a given subgroup J of G is invariant is called the "Zwischengruppe" of J in G (page 54). New notations in the interest of precision of statement are found on page 58.

The book is not free from misprints, some of which we note. In lines 8 and 7 from the bottom of page 45 H and G should be replaced by H and Γ respectively. The inequality sign in line 7 from the bottom of page 88 should be reversed. The c_λ on page 102, line 3, should be replaced by σ_λ. In line 5 from the bottom of page 103 J_1 should be in the place of H_1. The latter part of the first formula on page 169 should be

$$(p^\kappa - 1)\, p^{\kappa-1}\, (p^{\kappa-1} - 1)\, p^{\kappa-2}\, r_{\kappa-2}.$$

Netto's definition of a group (page 1) is exactly Dickson's definition of a semigroup.* If only a finite number of elements are under consideration, as is the case practically throughout the book, the two notions coincide. But some of the illustrations of groups given on page 2 are, according to Dickson, and also according to de Séguier,† not groups at all, but semigroups.

The book contains a few slight inaccuracies that should be noticed. The statement at the beginning of § 116 applies obviously only to groups of composite order. Moreover, the word transitive in this statement is redundant. The summing up of § 54, page 71, should not apply to the factors of composition $r/r_1, r_1/r_2, \cdots, r_{\nu-1}/r_\nu$. The statement near the bottom of page 102 that there is a $(p^\beta, 1)$ isomorphism between G and Γ is not true in general. The $(s, 1)$ isomorphism that is mentioned near the bottom of page 139 in reality exists between G and D, and not between G and S, as the author states it. As is well known, there are five primitive groups of degree 5. In the enumeration of these on page 156, the cyclic group is omitted.

The definition given of the *class* of a substitution group (page 128) implies that this term is applied only to k-fold transitive groups, where $k > 2$. This is not the definition given by Jordan.‡

The statement on page 123, in regard to solvable groups of

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* *Trans. Amer. Math. Soc.*, vol. 6 (1905), p. 205.

† Éléments de la théorie des groupes abstraits, p. 8.

‡ *Liouville's Journal*, ser. 2, vol. 16 (1871), p. 408. A somewhat different definition is given by Jordan in the *Comptes Rendus* of the Paris Academy, vol. 73 (1871), p. 853.

order $p_1^{a_1}p_2^{a_2}$ could, in view of Burnside's results,* have been made much more general.

This enumeration of a few points in which we think the book might be improved should not be understood as detracting from our statement at the beginning of this review commending the book. In publishing so excellent a treatment of the subject Professor Netto has performed a service of value to the mathematical public.

W. B. FITE.

Einführung in die höhere Mathematik. Von EMANUEL CZUBER. Teubner, 1909. 382 pp.

THIS book is an amplification of the lectures on differential and integral calculus given by Professor Czuber at the technical school in Vienna. The treatment has been extended so as to form a good introduction to higher mathematics, adapted for students other than technical students. The subjects are developed with much care and rigor.

There are essentially three divisions, viz., Functions of real variables, Algebra, and Analytical geometry. The opening chapter is on real and imaginary numbers. The number concept is developed for use in the functions of a real variable. The second chapter is a short, concise, and elegant presentation of infinite series and products. Besides simple demonstrations the subject is made easier for the student by the many well-chosen examples to illustrate the points in question.

Chapters III and IV begin the theory of functions of real variables. The general idea of a function, limit of a function, and continuity are the principal topics discussed. Here again we find many well-chosen examples. The function

$$f(x) = \lim_{n=\infty} \frac{nx + 2}{nx^2 + 1}$$

is given to show the difference between the value of a function given by direct substitution and that obtained by the limiting process. The substitution of $x = 0$ gives $f(0) = 2$, while if we first proceed to the limit and then put $x = 0$ we have $f(0) = \pm \infty$.

The example

$$f(x) = \frac{x}{1 + e^{1/x}}$$

is a good illustration of a function which is continuous for $x = 0$, but whose derivative is discontinuous at this point.

* *Proceedings Lond. Math. Society*, series 2, vol. 1 (1904), p. 388.

$f(x) = x[x]$, ($[x]$ indicating the largest integer contained in x), is an example of a function which is discontinuous at points for which x is an integer, but whose derivative is continuous.

$f(x) = x[1/x]$ is an example of a function which is discontinuous at points where x is an integer and whose derivative is also discontinuous at these points.

The chapter on the applications of the differential calculus contains nothing more than the applications to indeterminate forms and maxima and minima. It seems rather unfortunate that Taylor's and Maclaurin's series should be omitted entirely.

The part of the book dealing with algebra contains a chapter of forty pages on determinants, and a chapter on algebraic equations which treats of resultants, discriminants, and the solution and discussion of numerical equations.

The last 140 pages are devoted to analytical geometry. It is rather remarkable to note that geometric loci are discussed on the fourth page, the equations of the conics, strophoid, cissoid, Cassinian ovals, and four cusped hypocycloid being derived as examples in loci. The particular equations of the line and conic are then taken up and discussed in detail. The treatment of analytical geometry is satisfactory indeed.

The book as a whole is well adapted to the purpose for which it was written, but as is usually the case with the European text it does not contain a sufficient number of exercises and problems which are left for the student. Throughout there are many footnotes, mostly of a historical nature, which are sufficient to arouse an interest in the history of the subject.

C. L. E. MOORE.

Récréations Mathématiques et Problèmes des Temps Anciens et Modernes. Par W. W. ROUSE BALL. Deuxième édition française traduite d'après la quatrième édition anglaise et enrichie de nombreuses additions par J. FITZ-PATRICK. Paris, A. Hermann, 1907. 8vo. 3 parts. 5 francs each.

THE subject of mathematical recreations has always occupied a prominent position in the history of science. Zeno, Alcuin, Bachet, Fermat, Lucas, — these are only a few of the hundreds of names that might be mentioned of those who have contributed to this interesting field. Many of these men have been mathematicians of no small repute, for in reality the border line between recreative and serious mathematics is purely imaginary. To the mathematician all mathematics is a recreation ; it is to

him what color is to Sorolla or form to Michelangelo or rhythm to a reader of Poe, and the teacher is a poor one who does not appreciate this fact. It is because mathematics is itself a subject full of interesting situations, of wonder, and of rhythm that more students enjoy it than some of our pedagogical agitators think, and it is because of this that much of the effort to humanize mathematics to-day is really, though well meant, an effort to make it less human.

It is partly for those mistaken teachers who feel that mathematics has not the same interest per se that music or art or literature has, that Mr. Ball prepared this interesting collection upwards of seventeen years ago. But it was also for the mathematician himself, who abuses his nerve system in his love for the more serious side of the science, that the book was written, even as Punch is published not merely for the casual reader but also for the statesman who needs to see his labors in a different light after a night in parliament.

The English work has now passed through four editions and the French translation through two, which testifies anew to the pleasant style and to the wisdom of selection that characterizes Mr. Ball's various publications.

The French edition is considerably more extended than the English original in some respects, filling three volumes. The chief departure from the original is in Chapter I, Some arithmetical questions. This occupied less than forty pages in the English edition, but it makes up the first volume, of over three hundred pages, in the translation. The added material relates largely to the history of numbers and to interesting problems of early and medieval times. The mysticism of numbers, so exhaustively treated from the religious side by Bungus three centuries ago, speculations on the platonic number, curious properties of decimal numbers, the application of algebra to number games, and the elementary theory of numbers in general are some of the features of the French edition that make it well worth placing upon the shelves of any mathematical or general library.

David Eugene Smith.

Lehrbuch der Kristalloptik. Von F. Pockels. Teubners Sammlung XIX. Leipzig, B. G. Teubner, 1906. x + 520 pp.

Although the past five years have seen the publication of a large number of books on optics, the subject is so broad and

may be treated from so many points of view that there has been relatively little duplication in the different works and there is still room for other treatments. Where one book may lay the stress on the analytical theory and be replete with formulas, another may take the physical side and with relatively few formulas establish the chief points of relation between optics and electromagnetism, and yet another may deal with optics from what might be called a dynamical point of view. Pockels's book is none of these, but as its name indicates is concerned with crystalline optics ; in fact it will offer much more of interest to the crystallographer than to the mathematician. Yet so thoroughly has the author covered his field and so encyclopedic is his treatment, that any student of optics, whether primarily interested in crystals or not ,will find frequent occasion to consult the work, if only to ascertain what may be the known experimental facts with regard to the subject.

After a short introduction, the author divides his work into four major parts which treat respectively transparent crystals without rotary properties, crystals with rotary properties, absorbing crystals, and effects of external influences upon optical properties. Of these divisions the first is naturally the longest and indeed contains more than half of all the matter in the book. From the very introduction the presentation is true to the main object of the author, namely, to describe the phenomena of crystalline optics rather than to construct a well-knit theory. The start is made, not with theories, but with a few statements concerning the propagation of light, the wave surface, rays and plane waves, and Huyghens's principle from which many of the essentials of crystalline optics may be obtained without the need of intricate formulas or detailed physical considerations. Thereupon follow definitions of the light vector, of natural and polarized light, and the presentation of trigonometric and exponential representations of the light vector.

The method of the introduction is pursued during the first two chapters of the first main part of the work. The propagation of light in uniaxial crystals is described with the aid of the wave surface and related surfaces derived from Huyghens's principle. Here, as throughout the book, tables of the optical constants of the type of crystals considered are freely introduced. Even the phenomenon of dispersion with an appropriate table of data is mentioned at this early stage. The second chapter is almost an exact parallel of the first, but treats

biaxial crystals. Thus without any complicated analytical or physical investigations and in the short space of about seventy pages the author has succeeded in giving the reader an excellent account of the general question of the propagation of light in crystals and of the geometric method of discussing the propagation.

The third chapter on the theoretical physical foundations for the theory of light is a model of excellent and concise presentation of a difficult and often confused subject. The trouble is that there are so many different theories of light which give results differing very slightly, usually so slightly that there is no crucial experiment sufficiently accurate to distinguish between them. The elastic theory with its various developments is first mentioned. But it is the electromagnetic theory which naturally comes in for the major part of the discussion. In the propagation of electromagnetic waves there are four vectors, the magnetic induction B, the magnetic force H, the electric induction or displacement D, and the electric force E which may possibly be taken as the light vector. Of these the first two B and H are so nearly parallel in bodies which propagate light that it would be hopeless to distinguish between them and useless to consider as different the theories built upon them. Although many theorists use these vectors, Pockels discards them in favor of the electric vectors, as would be expected in view of results obtained from experiments with stationary waves. As between D and E, the author chooses D as his light vector. He then has a few words to say about the electron theories with especial reference to their bearing on dispersion. Although Pockels thus introduces a little physical optics into his work, the mere fact that Zeeman's name does not occur here or elsewhere is sufficient to show how strictly he adheres to his aim of treating the optics of crystals and how carefully he avoids being led off into the general realm of physical optics.

The fourth chapter is on reflection and refraction. To show the detail with which the subject is presented it will be sufficient to mention the titles of the articles on total reflection. They are: general conditions, methods of observation, the limiting cone of rays in the case of reflection from a uniaxial crystal, polarization of the limiting rays, special cases of total reflection from biaxial crystals, singular phenomena connected with conical refraction, determination of optical constants for

biaxial crystals by means of observations on total reflection. It would scarcely be possible to find a more detailed or simpler presentation of all this material. Many works on analytical or physical optics hardly mention these subjects at all except in the most general way. Several chapters further on in the work, the author takes up the physical basis for these results and develops the formulas from the electromagnetic conditions at the interface of two media.

There is little need of prolonging this review with the recitation of the course of the various chapters. Enough has been said to show the method upon which the author has constructed his book and the detail with which he has written. There is one point in which an improvement might be suggested. The plates which exhibit the elaborate and intricate phenomena of interference are all in black and white. This is a great pity; the beautifully modulated color schemes are the chief attraction of the figures and the author's detailed tables of the colors that are found in some special cases by no means take the place of the actual colors on the plates. If the coloring of the plates had to be done by the eye and hand, there might be good excuses for omitting it; but natural color photography is now so well developed that very good photographs of these effects can be obtained and reproduced.

The student of optics, who frequently finds it very hard to lay his hand upon a large and accurate presentation of the phenomena of crystalline optics, will refer constantly to this work and will feel under deep obligations to its author for the pains taken in preparing it.

E. B. WILSON.

NOTES.

THE July number (volume 10, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Projective differential geometry of curved surfaces (fifth memoir)," by E. J. WILCZYNSKI; "On the osculating quartic of a plane curve," by W. W. DENTON; "Note on a system of axioms for geometry," by A. R. SCHWEITZER; "Irreducible homogeneous linear groups in an arbitrary domain," by W. B. FITE; "On the integration of the homogeneous linear difference equation of second order," by W. B. FORD; "On Cantor's theorem concerning the coefficients of a convergent

trigonometric series, with generalizations," by W. F. OSGOOD; "Equivalence of pairs of bilinear or quadratic forms under rational transformation," by L. E. DICKSON; "On a complete system of invariants of two triangles," by D. D. LEIB.

THE July number (volume 31, number 3) of the *American Journal of Mathematics* contains the following papers: "The birational transformations of algebraic curves of genus 4," by A. L. VAN BENSCHOTEN; "On some loci associated with plane curves," by C. H. SISAM; "Plane sections of a Weddle surface," by F. MORLEY and J. R. CONNER; "The differential equation satisfying abelian theta functions of genus 3," by J. E. WRIGHT; "Differential equations admitting a given group," by J. E. WRIGHT; "On the angles of the regular polytopes of four dimensional space," by P. H. SCHOUTE.

THE concluding (July) number of volume 10 of the *Annals of Mathematics* contains: "Thermodynamic analogies for a simple dynamical system," by E. B. WILSON; "Discussion of a method for finding numerical square roots," by C. L. BOUTON; "On the direct product in the theory of finite groups," by J. H. MACLAGAN-WEDDERBURN; "Existence of the generalized Green's function," by W. D. A. WESTFALL; "The gambler's ruin," by J. L. COOLIDGE.

THE series of lectures announced in connection with the celebration, September 6–18, of the twentieth anniversary of the founding of Clark University included the following in mathematics: Professor E. H. MOORE, "The rôle of postulational methods in mathematics;" Professor E. B. VAN VLECK, "The homogeneous linear difference equation of order n with polynomial coefficients, considered from the functional standpoint;" Professor JAMES PIERPONT, "Modern theories of integration" Discussions of the following topics were also announced: "The unification and continuity of mathematics in school and college;" "The effectiveness of mathematical training," opened by Professor J. W. A. YOUNG; "The use and abuse of textbooks in mathematical classes," opened by Dr. J. S. FRENCH.

THE next meeting of the British association for the advancement of science will be held at Sheffield, England, August 31 to September 10, 1910, under the presidency of T. G. BONNEY.

THE section of mathematics and physics of the royal society of Naples announces the following prize problem:

"A systematic exposition of the known concepts regarding configurations of planes and of spaces, putting them into closer relation with the theory of substitutions, and including, if possible, some new contributions."

The value of the prize is 1000 lire. Competing memoirs must be written in Italian, Latin, or French, and be in the hands of the secretary of the academy before June 30, 1910.

THE prize of the Accademia dei Lincei of 10,000 lire, awarded for excellence in mathematical contributions, has been divided between Professor F. ENRIQUES, of the University of Bologna, and Professer T. LEVI CIVITA, of the University of Padua.

AN extensive laboratory for the construction of mathematical models and apparatus is being fitted up for the use of students of the Ecole Normale of the University of Paris, somewhat after the plan of that at Göttingen. It will be in charge of Professor E. BOREL.

ADVANCED courses in mathematics are announced for the winter semester 1909–1910 at the various German universities as follows:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic geometry, four hours; Theory of analytic functions, II, four hours; Geometry of conics, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of curves and surfaces, four hours; Applications of elliptic functions, four hours; Seminar, two hours. — By Professor G. HETTNER: Definite integrals, two hours. — By Professor J. KNOBLAUCH: Differential calculus, with exercises, five hours; Theory of elliptic functions, four hours. — By Professor R. LEHMANN-FILHÈS: Integral calculus, four hours; Determinants, four hours. — By DR. I. SCHUR: Theory of algebraic equations, four hours; Theory of linear differential equations, four hours.

UNIVERSITY OF LEIPZIG — By Professor C. NEUMANN: Theory of potential and spherical harmonics, four hours; Seminar, two hours. — By Professor O. HÖLDER: Elliptic functions, five hours; Theory of finite groups, one hour; Seminar, two hours. — By Professor K. ROHN: Analytic geometry of space, four hours; Descriptive geometry with exercises, four hours;

Seminar, two hours. — By Professor G. HERGLOTZ: Mechanics, four hours; Mechanics of continua, two hours; Seminar, two hours. — By Professor P. v. OETTINGEN: Elements of projective dioptics, one hour. — By Professor F. HAUSDORFF: Differential geometry with exercises, four hours; Algebraic numbers, two hours. — By Professor H. LIEBMANN: Differential and integral calculus with exercises, five hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Differential calculus, five hours; Analytic mechanics, four hours; Seminar on line and spherical geometry, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Theory of algebraic curves, four hours; Seminar on theory of surfaces, two hours. — By Professor A. PRINGSHEIM: Introduction to analytic functions, five hours. — By Professor A. SOMMERFELD: Vector analysis, three hours; Thermo-dynamics, three hours; Seminar, two hours. — By Professor H. BRUNN: Modern development of analysis situs, two hours. — By Professor K. DOEHLEMANN; Descriptive geometry, with exercises, eight hours; Synthetic geometry, with exercises, five hours; Graphical representation, two hours. — By Dr. G. HARTOGS: Theory of abelian functions, four hours. — By Dr. O. PERRON: Advanced calculus, with exercises, five hours; Theory of continued fractions, two hours.

UNIVERSITY OF STRASSBURG. — By Professor H. WEBER: Differential and integral calculus, four hours; Calculus of variations, two hours; Seminar, two hours. — By Professor F. SCHUR: Analytic geometry, four hours; Selected chapters of the theory of surfaces, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Partial differential equations, five hours; Seminar, one hour. — By Professor M. SIMON: Methods in elementary mathematics, four hours. — By Professor S. EPSTEIN: Determinants, four hours.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1909–1910:

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Calculus of variations, integral equations, and Laplace's series, three hours. — By Professor L. DONATI: Thermodynamics, kinetic theory of gases, magneto- and electro-optics, three hours. — By Professor L. PINCHERLE: Elliptic functions, integrals of algebraic differentials, and abelian functions, three hours.

UNIVERSITY OF CATANIA. — By Professor M. DE FRANCHIS : Geometry on algebraic surfaces, three hours. — By Professor G. LAURICELLA : Integral equations, development in series of characteristic functions, with application to vibrating strings and membranes, four and a half hours. — By Professor G. PENNACCHIETTI : Applications of elliptic functions to mechanical problems, four and a half hours.— By Professor C. SEVERINI : Selected topics in differential geometry, three hours.

INSTITUTE OF FLORENCE. — By Professor T. BOGGIO : Applications of integral equations to mathematical physics, three hours.

UNIVERSITY OF GENOA. — By Professor E. E. LEVI : Differential and integral equations, four hours. — By Professor G. LORIA : Theory of geometric transformations, three hours. — By Professor O. TEDONE : Problems of elastic equilibrium, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO : History of mathematics from Newton to Lagrange, three hours. — By Professor A. CAPELLI : Arithmetic theory of algebraic numbers, three hours. — By Professor R. MARCOLONGO : Hydrodynamics, three hours. — By Professor D. MONTESANO : Theory of geometric correspondences, four and a half hours. — By Professor E. PASCAL : Selected chapters in analysis, three hours. — By Professor L. PINTO : Electro-optics and Hertz's waves, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS : Theory of functions and integral equations, four and a half hours. — By Professor U. CISOTTI : Mathematical theory of elasticity with technical application, three hours. — By Professor A. FAVARO : The life and work of Archimedes, three hours. — By Professor P. GAZZANIGA : Theory of numbers, three hours. — By Professor T. LEVI-CIVITA : Equations of dynamics and principles of celestial mechanics, four and a half hours. — By Professor G. RICCI : Absolute differential calculus, equilibrium and motion of solid elastic bodies, four hours. — By Professor F. SEVERI : Theory of continuous groups, three hours. — By Professor G. VERONESE : Synthetic geometry of hyperspace, four hours.

UNIVERSITY OF PALERMO. — By Professor G. BAGNERA : Automorphic functions, three hours. — By Professor M. GEB-

BIA: Propagation of heat, and thermodynamics, four and a half hours. — By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and a half hours. — By Professor A. VENTURI: Figures of planets, particularly of the Earth, with regard to elasticity, three hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI: Theory of potential, electrostatics and magnetism, three hours. — By Professor L. BERZOLARI: Geometry of hyperspace, three hours. — By Professor R. BONOLA: Imaginary in geometry, projective generation of certain curves and surfaces, linear systems of conics and quadrics, three hours. — By Professor F. GERBALDI: Functions of a complex variable and abelian integrals, three hours. — By Professor G. VIVANTI: Theory of algebraic numbers, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Abelian integrals with application to the geometry on an algebraic curve, three hours. — By Professor E. BIANCHI: Calculus of variations and integral equations, four and a half hours. — By Professor U. DINI: Linear differential equations with application to the development of a given function in series, four and a half hours. — By Professor E. A. MAGGI: Advanced theoretic mechanics, Maxwell's theory of electro-magnetic fields, and electrons, four and a half hours. — By Professor P. PISSETTI: Figures and rotations of celestial bodies, and spherical astronomy, three hours.

UNIVERSITY OF ROME. — By Professor G. CASTELNUOVO: Abelian functions and geometric applications, three hours. — By Professor V. CERRUTI: Partial differential equations of the first order, three hours. — By Professor L. ORLANDO: Dynamics of balloons and of aëroplanes, three hours. — By Professor L. SILBERSTEIN: Complements of dynamics, electro-magnetic fields, and optics, three hours. — By Professor V. VOLTERRA: Integral and integro-differential equations with applications, four and a half hours; hydrodynamics and the theory of the tides, three hours.

UNIVERSITY OF TURIN. — By Professor G. PEANO: Mathematical logic, three hours. — By Professor G. SANNIA: Geometric applications of the calculus and intrinsic geometry, three hours. — By Professor C. SEGRE: Cubic surfaces and plane quartics, three hours. — By Professor C. SOMIGLIANA: Optics and electric oscillations, three hours.

THE University of Rochester has received, under the provisions of the will of the late Rear Admiral W. HARKNESS, professor of mathematics, U. S. Navy, almost his entire collection of astronomical instruments and a considerable part of his library.

PROFESSOR M. CANTOR, of the University of Heidelberg, has been elected associate member of the academy of sciences of Heidelberg.

PROFESSOR A. CAPELLI, of the University of Naples, Professor G. DARBOUX, of the University of Paris, and Professor Sir A. G. GREENHILL, formerly of the Ordnance College, Woolwich, have been elected foreign members of the royal institute of Venice.

PROFESSOR E. ALMANSI, of the University of Pavia, and Professor A. GARBASSO, of the University of Genoa, have been elected corresponding members of the Accademia dei Lincei.

PROFESSOR G. DARBOUX has been appointed the official delegate of the French government at the approaching Hudson-Fulton celebration.

THE title of Hofrat has been conferred upon Professor O. HÖLDER, of the University of Leipzig.

PROFESSOR L. MAURER, of the University of Tübingen, has been promoted to a full professorship of mathematics.

DR. R. E. v. MISES, of the technical school at Brünn, has been appointed associate professor of mathematics at the University of Strassburg.

PROFESSOR F. SCHUH, of Delft, has been appointed professor of analysis at the University of Groningen.

DR. C. E. WASTEELS has been appointed associate professor of rational mechanics at the University of Geneva.

PROFESSOR E. v. WEBER, of the University of Würzburg, has been promoted to a full professorship of mathematics.

DR. — BYDZOVSKY has been appointed docent in mathematics at the Bohemian University of Prague.

DR. T. J. I'A BROMWICH, of Cambridge University, has been promoted to an associate professorship of mathematics.

PROFESSOR F. PRYM, of the University of Würzburg, will retire from active service on October 1.

PROFESSOR A. GRÜNWALD, of the German technical school at Prague, has retired from regular teaching.

A MARBLE monument by G. Monteverde of the late Professor LUIGI CREMONA was unveiled with appropriate ceremonies at the engineering school of Rome on June 10.

PROFESSOR C. RUNGE, of the University of Göttingen, has been appointed Kaiser Wilhelm exchange professor of mathematics at Columbia University for the next academic year. The subject of his lectures will be "Graphical methods in physics and technic."

THE honorary degree of doctor of science has been conferred upon Professor E. H. MOORE, of the University of Chicago, by Yale University.

WILLIAMS COLLEGE conferred the degree of doctor of laws upon Professor H. B. FINE, of Princeton University, last June.

PROFESSOR R. MORRIS, of Rutgers College, has been promoted to a full professorship of mathematics and graphics.

PROFESSOR FLOYD FIELD, of the Georgia School of Technology, has been appointed full professor of mathematics and head of the department for 1909–1910.

AT Princeton University DR. E. SWIFT has been promoted to an assistant professorship of mathematics.

PROFESSOR C. N. HASKINS, of the University of Illinois, has accepted an assistant professorship of mathematics at Dartmouth College.

THE following changes are announced at the University of Illinois: Dr. C. H. SISAM, who returns after a year's study at Turin, has been promoted to an assistant professorship of mathematics. Dr. JACOB KUNZ, of the University of Michigan, has been appointed assistant professor of mathematical physics. During the coming year he will give courses in dynamics and in the theory of electrons. Dr. THOMAS BUCK, of the University of Chicago, has been appointed instructor in mathematics.

Professor W. A. Manning, of Stanford University, has exchanged work with Mr. E. W. Ponzer, of the University of Illinois, for the current academic year.

Dr. J. H. Maclagan-Wedderburn has been appointed preceptor in mathematics at Princeton University.

Dr. C. C. Grove has been appointed instructor in mathematics at Columbia University.

Dr. L. Karpinski, of the University of Michigan, will spend next year at study at Columbia University.

Professor Simon Newcomb died at Washington, D. C., July 11, 1909. He was born in Nova Scotia March 12, 1835, and came to the United States when eighteen years old. For thirty-six years he was professor of mathematics at the United States Naval Academy, with especial duties at the Naval observatory at Washington, and for ten years he was professor of mathematics and astromony at Johns Hopkins University, and co-editor of the *American Journal of Mathematics.* For his extensive contributions to astronomy, celestial mechanics and in particular to the theory of the motion of the moon he was the recipient of a large number of honorary degrees, medals, and prizes. He was a member of the National academy of science, foreign member of the Royal society of London, foreign associate of the Institute of France, knight of the order "Pour le mérite" für Kunst und Wissenschaft, and member of many other societies and academies. He was a member of the American Mathematical Society from 1891, and its president during 1896–1898.

At the meeting of the Paris academy of sciences on August 18, a eulogy on Professor Simon Newcomb, late foreign associate of the academy, was pronounced by the permanent secretary, Professor G. Darboux.

Professor V. Cerruti, of the University of Rome, died August 20, 1909, at the age of 59 years. He was director of the School of Engineers, Senator of the Kingdom of Italy, and member of the Accademia dei Lincei and of the Italian society of sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLZA (O.). Vorlesungen über Variationsrechnung. Umgearbeitete und stark vermehrte deutsche Ausgabe der "Lectures on the calculus of variations." 2te Lieferung. Leipzig, Teubner, 1909. 8vo. 6 + 301–541 pp. M. 6.00

BRAND (E.). Ueber Tetraëder deren Kanten eine Fläche zweiter Ordnung berühren. (Diss.) Strassburg, 1908. 8vo. 50 pp.

BÜCHER, neue, über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet. Mitgeteilt Frühjahr, 1909. Leipzig, Hinrich, 1909. 8vo. 23 pp. M. 0.30

CASTELNUOVO (G.). Lezioni di geometria analitica. 2a edizione. Roma-Milano, Segati e C., 1909. 8vo. 8 + 688 pp. L. 15.00

COOLIDGE (J. L.). The elements of non-euclidean geometry. Oxford, Clarendon, 1909. 8vo. 291 pp. Cloth. $5.00

DÖLP (H.). Grundzüge und Aufgaben der Differential- und Integralrechnung nebst den Resultaten. Neu bearbeitet von E. Netto. 12te Auflage. Giessen, Töpelmann, 1909. 8vo. 6 + 216 pp. Cloth. M. 1.80

DROYSEN (P.). Aufgaben aus der analytischen Geometrie. (Progr.) Belgard, 1909. 4to. 24 pp.

DUMAS (S.). Sur le développement des fonctions elliptiques en fractions continues. (Diss.) Zurich, 1908. 8vo. 50 pp.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française. Vol. II: Fonctions de variables réelles. Fascicule 1. Paris et Leipzig, Teubner, 1909. 8vo. 112 pp. M. 3.60

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III: Geometrie. C 5. Spezielle ebene algebraische Kurven, von G. Kohn und G. Loria. Teil 1: Ebene Kurven dritter und vierter Ordnung, von G. Kohn. Leipzig, Teubner, 1909. 8vo. Pp. 457–570.

FINKE (P.). Ueber Scharen von ∞^5 Kurven im gewöhnlichen Raume. (Diss.) Berlin, 1909. 8vo. 36 pp.

FUCHS (L.). Gesammelte mathematische Werke. Herausgegeben von R. Fuchs und L. Schlesinger. Band III. Abhandlungen (1888–1902) und Reden. Berlin, Mayer und Müller, 1909. 4to. 11 + 460 pp. M. 28.00

JAECKEL (W.). Geometrische Untersuchungen über Kurven auf Flächen dritter Ordnung mit vier Knotenpunkten. (Progr.) Ohlau, 1909. 4to. 22 pp.

KEMPF (A.). Tetraëder deren Kanten eine Fläche zweiter Ordnung berühren. (Diss.) Strassburg, 1908. 8vo. 46 pp.

KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. Teil II: Geometrie. Vorlesung, gehalten im Sommersemester 1908. Ausgearbeitet von E. Hellinger. Leipzig, Teubner, 1909. 8vo. 8 + 515 pp. M. 7.50

KOPPE (M.). Die Iteration des Sinus und anderer Funktionen. (Progr.) Berlin, Weidmann, 1909. 8vo. 28 pp. M. 1.00

KOWALEWSKI (G.). Einführung in die Determinantentheorie einschliesslich der unendlichen und der Fredholmschen Determinanten. Leipzig, Veit, 1909. 8vo. 5 + 550 pp. Cloth. M. 16.00

LEUTENEGGER (J.). Lehrbuch der Differential-Rechnung. Zum Gebrauche an höheren Lehranstalten sowie zum Selbststudium bearbeitet. Bern, Francke, 1909. 8vo. 160 pp. Cloth. M. 3.20

LIER (O.). Ueber Flächenscharen die durch Berührungstransformation in Kurvenscharen überführbar sind. (Diss.) Berlin, 1909. 8vo. 69 pp.

MEYER (W. F.). Allgemeine Formen- und Invariantentheorie. I Band, Binäre Formen. (Sammlung Schubert, XXXIII.) Leipzig, Göschen, 1909. 8vo. 8 + 376 pp. Cloth. M. 9.60

NETTO (E.). See DÖLP (H.).

SCHANZ (J.). Der Aufbau des komplexen Zahlengebiets auf der natürlichen Zahlenreihe. (Progr.) Berlin, Weidmann, 1909. 4to. 31 pp. M. 1.00

SCHWERING (K.). Lehrbuch der kleinsten Quadrate. Freiburg, Herder, 1909. 8vo. 8 + 105 pp. Cloth. M. 2.80

TRAMM (W.). Geometrische Diskussion des Hermite'schen Polynoms. (Diss.) Zürich, 1908. 4to. 32 pp.

II. ELEMENTARY MATHEMATICS.

BALTIN (R.) und MAIWALD (W.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie mit zahlreichen Anwendungen aus der Planimetrie und Physik für Seminare und Präparandenanstalten. Nach der Müller-Kutnewskyschen Aufgabensammlung bearbeitet. Teil II. 3te verbesserte Auflage. Leipzig, Teubner, 1909. 8vo. 8 + pp. 111–352. Cloth. M. 2.40

BASSI (A.). Esercizî e problemi di algebra complementare ad uso del secundo biennio degli istituti tecnici. Vol. I, parte 2. Bra, Rocca, 1909. 8vo. Pp. 145–318. L. 1.80

BOOLE (M.E.). Philosophy and fun of algebra. London, Daniel, 1909. 12mo. 90 pp. 2 s.

BRUNO (G. M.). Elementos de geometria, para la enseñanza secundaria y escuelas preparatorias. Paris, Bouret, 1909. 16mo. 400 pp.

COMBEROUSSE (C. de). Cours de mathématiques à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, à l'Ecole centrale des arts et manufactures. 3e édition. Vol. 4: Algèbre supérieure. Paris, Gauthier-Villars, 1909. 8vo. 24 + 832 pp. Fr. 15.00

GRANVILLE (W. A.). Plane and spherical trigonometry and four-place tables of logarithms. New York, Ginn, 1909. 8vo. 11 + 264 + 38 pp. Cloth. $1.25

GUBLER (S. E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. Methodisch bearbeitet. Heft I. 2te Auflage. Zürich, Artistisches Institut Orell Füssli, 1909. 8vo. 52 pp. M. 0.80

HEINRICH (M.). Logarithmen-Tafeln, vierstellig und dreistellig, in neuer Anordnung. Berlin, Grote, 1909. 4to. 10 + 30 pp. Cloth. M. 1.00

KALENS (M.). Die Quadratur des Kreises. Ihre vollständige Auflösung. Neu bearbeitet. Saarbrücken, 1909. 8vo. 12 pp. M. 1.00

KOHL (A.) und ZSCHOCKELT (H.). Aufgaben für Rechnen und Geometrie zum Gebrauche in ländlichen Fortbildungsschulen. Leipzig, Hahn, 1909. 8vo. 3 + 76 pp. Boards. M. 0.50

KREUSCHNER (R.). Das Additionstheorem der Winkelfunktionen und die Aenderungen der Funktionswerte mit dem sich ändernden Argument am neuen Transporteur für Winkelfunktionen. (Progr.) Barmen, 1909. 4to. 11 pp.

LUCKE (F.). Einführung in die Goniometrie. (Progr.) Zerbst, 1909. 4to. 10 pp.

MAIWALD (W.). See BALTIN (R.).

MARTÍ ALPERA (F.). Las primeras lecciones de geometria. Burgos, Rodriguez, 1909. 8vo. 104 pp. P. 1.00

MATHEMATICAL questions and solutions. Edited by C. L. Marks. New Series, Vol. 15. London, Hodgson, 1909. 8vo. 6s. 6d.

PREMIÈRES NOTIONS d'algèbre avec de nombreux exercices à l'usage des écoles primaires; par une réunion de professeurs. Paris, Poussielgue, 1909. 16mo. 88 pp.

ROSSI (A.). Saggio di geometria popolare. 2a edizione. Udine, Patronato, 1909. 8vo. 10 + 67 pp. L. 1.50

WELLS (W.). New plane and solid geometry. London, Heath, 1909. 12mo. 3s. 6d.

WIENECKE (E.). Die Grundlehren der Planimetrie für Lehrerbildungsanstalten in genetischer Darstellung mit reichem Aufgabenmaterial. 2te Auflage. Berlin, Oehmigke, 1909. 8vo. 8 + 174 pp. Cloth. M. 3.50

WROBEL (E.). Uebungsbuch zur Arithmetik und Algebra, enthaltend die Formeln, Lehrsätze und Auflösungsmethoden in systematischer Anordnung und eine grosse Anzahl von Fragen und Aufgaben. Zum Gebrauche an Gymnasien, Realgymnasien und andern höheren Lehranstalten bearbeitet. Teil II. Rostock, Koch, 1909. 8vo. 12 + 484 pp. Cloth. M. 4.90

ZSCHOCKELT (H.). See KOHL (A.).

III. APPLIED MATHEMATICS.

COFFIN (J. G.). Vector analysis: an introduction to vector methods and their various applications to physics and mathematics. New York, Wiley, 1909. 12mo. 19 + 248 pp. $2.50

GRAF (E.). Technische Berechnungen für die Praxis des Maschinen- und Bautechnikers. Ein Handbuch über gelöste Beispiele aus der gesamten Mechanik, der Maschinen-, Holz- und Bautechnik, einschliesslich Eisenbeton- und Brückenbau. Leipzig, Barth, 1909. 8vo. 8 + 374 pp. Cloth. M. 6.80

HAVELOCK (T. H.). See JESSOP (C. M.).

JADANZA (N.). Trattato di geometria pratica. Torino, Bona, 1909. 8vo. 320 pp.

JESSOP (C. M.) and HAVELOCK (T. H.). Elementary mechanics. London, Bell, 1909. 12mo. 286 pp. 4s. 6d.

KÜSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. Im Einverständnis mit der Atomgewichtskommission der deutschen chemischen Gesellschaft für den Gebrauch im Unterrichtslaboratorium und in der Praxis berechnet. 9te vollständig neu berechnete Auflage. Leipzig, Veit, 1909. 8vo. 107 pp. Cloth. M. 2.40

LESTER (O. C.). The integrals of mechanics. New York, Ginn, 1909. 8vo. 6 + 67 pp. Cloth. $0.80

LORENTZ (H. A.). The theory of electrons and its applications to the phenomena of light and radiant heat : a course of lectures delivered at Columbia University, New York, in March and April, 1906. New York, Stechert and Co., 1909. 8vo. 332 pp. $2.50

MAIDA (G.). Sopra un integrale primo delle equazioni del moto di un punto, il quale è cubico rispetto alle componenti della velocità: dissertazione di laurea in matematica. Palermo, Virzì, 1909. 8vo. 28 pp.

MARVA Y MAYOR (J.). Mecánica aplicada en las construcciones. Vol. I. 4a edición. Madrid, Palacios, 1909. 8vo. 968 pp. P. 44.00

THOMPSON (J. J.). Elements of the mathematical theory of electricity and magnetism. 4th edition. Cambridge, University Press, 1909. 8vo. 558 pp. 10 s.

TURNER (G. C.). Graphical methods in applied mathematics. London, Macmillan, 1909. 12mo. 398 pp. 6 s.

WANGERIN (A.). Theorie des Potentials und der Kugelfunctionen. I Band. (Sammlung Schubert, LVIII.) Leipzig, Göschen, 1909. 8vo. 8 + 255 pp. Cloth. M. 6.60

THE SIXTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE sixteenth summer meeting and sixth colloquium of the Society were held at Princeton University during the week September 13–18, 1909. The attendance included the following forty-one members:

Professor G. D. Birkhoff, Professor G. A. Bliss, Mr. R. D. Carmichael, Dr. A. B. Chace, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Dr. L. S. Dederick, Professor L. P. Eisenhart, Professor T. C. Esty, Professor H. B. Fine, Dr. G. F. Gundelfinger, Professor Harris Hancock, Rev. A. S. Hawkesworth, Dr. A. M. Hiltebeitel, Dr. J. G. Hun, Dr. Frank Irwin, Professor Edward Kasner, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Mr. H. F. MacNeish, Professor E. H. Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. W. J. Risley, Professor D. A. Rothrock, Professor I. J. Schwatt, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor Elijah Swift, Professor J. H. Tanner, Mr. C. E. Van Orstrand, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. E. Wright, Professor J. W. A. Young.

An account of the colloquium, which opened on Wednesday morning, will appear later in the BULLETIN. The four sessions of the summer meeting proper occupied the first two days of the week. At the opening session Professor Fine occupied the chair, which was taken at the later sessions by Professor Morley and Vice-Presidents Kasner and Van Vleck. The Council announced the election of the following persons to membership in the Society: Dr. L. S. Dederick, Princeton University; Dr. G. E. Wahlin, University of Illinois; Mr. E. E. Whitford, College of the City of New York. Eleven applications for membership were received.

The sum of five thousand francs was appropriated from the treasury of the Society in support of the publication of the works of Euler and in subscription for a copy of these works for the Society's library. A committee was appointed to prepare and report at the October meeting suitable resolutions on the death of Ex-President Simon Newcomb. The committee

on nominations, consisting of Professors E. B. Van Vleck, Brown, and Kasner, appointed to nominate officers and other members of the Council will also report at the October meeting.

At the closing session resolutions thanking the University for its hospitality and the committee on arrangements for their services were adopted. On Tuesday the members were conducted through the buildings and grounds of the University, and Tuesday evening was marked by a reception at the house of Professor Fine.

The following papers were read at this meeting:

(1) Professor L. P. Eisenhart: "Congruences of the elliptic type."

(2) Mr. Dunham Jackson: "Resolution into involutory substitutions of the transformation of a bilinear form into itself."

(3) Dr. F. W. Reed: "On singular points in the approximate development of the perturbative function."

(4) Professor Virgil Snyder: "Surfaces invariant under infinite discontinuous birational groups defined by line congruences."

(5) Mr. Joseph Lipke: "Natural families of curves in a general curved space," preliminary communication.

(6) Rev. A. S. Hawkesworth: "A new theorem in conics."

(7) Mrs. Anna J. Pell: "Applications of biorthogonal systems to integral equations."

(8) Mr. G. C. Evans: "The integral equation of the second kind, of Volterra, with singular kernel," preliminary communication.

(9) Professor Edward Kasner: "Triply orthogonal systems of surfaces."

(10) Professor Edward Kasner: "Natural families and Thomson's theorem."

(11) Professor G. A. Miller: "The groups which may be generated by two operators s_1, s_2 satisfying the equation $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α and β being relatively prime."

(12) Dr. F. R. Sharpe: "Integral equations with variable limits, with an application to the problem of age distribution."

(13) Mr. R. D. Carmichael: "Note on a new number theory function."

(14) Professor T. E. McKinney: "On a criterion for λ-developments in the theory of equivalence."

(15) Dr. G. G. Chambers: "Groups of isomorphisms of the abstract groups of order p^2q."

(16) Professor W. R. LONGLEY: "Note on some periodic orbits with more than one axis of symmetry."

(17) Professor W. H. BUSSEY: "Tables of Galois fields of order less than 1,000."

(18) Professor W. B. FORD: "On the relation between the sum formulas of Hölder and Cesàro."

(19) Professor OSWALD VEBLEN: "Products of pairs of involutoric projectivities."

(20) Dr. G. F. GUNDELFINGER: "On the geometry of line elements in the plane with reference to osculating vertical parabolas and circles."

(21) Professor P. F. SMITH: "Theorems in the geometry of surface elements in space."

(22) Professor R. G. D. RICHARDSON and Mr. W. A. HURWITZ: "Note on determinants whose terms are certain integrals."

(23) Professor R. G. D. RICHARDSON: "The Jacobi criterion in the calculus of variations and the oscillation of solutions of linear differential equations of the second order."

(24) Professor I. J. SCHWATT: "Methods for the summation of infinite series."

(25) Professor A. B. COBLE: "Cubic space curves that meet the Hessian of a cubic surface in six pairs of corresponding points."

(26) Professor G. D. BIRKHOFF: "On the theory of stability."

(27) Mr. H. W. REDDICK: "Geometric properties of a system of tautochrones."

(28) Dr. W. B. CARVER: "The poles of finite groups of fractional linear substitutions in the complex plane."

(29) Dr. L. S. DEDERICK: "The solution of the equation in two real variables at a point where both the partial derivatives vanish."

(30) Dr. H. T. BURGESS: "On point-circle correlations in the plane."

(31) Professor H. B. NEWSON: "A general theory of linear groups."

(32) Mr. A. R. SCHWEITZER: "A formal extension of Bolzano's series."

Mr. Jackson's paper was communicated to the Society by Professor Bôcher, Dr. Reed's by Professor White, and Mr. Evans's by Professor Osgood. Dr. Burgess was introduced by Professor Osgood. The papers of Dr. Sharpe and Dr. Carver

were presented by Professor Snyder. The papers of Mr. Jackson, Dr. Reed, Mr. Lipke, Mrs. Pell, Mr. Evans, Professor Kasner's second paper, and the papers of Professor Miller, Professor McKinney, Dr. Chambers, Professor Bussey, Professor Ford, Professor Richardson and Mr. Hurwitz, Professor Richardson, Professor Coble, Professor Newson, and Mr. Schweitzer were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Kummer in his study of rectilinear congruences made use of two quadratic differential forms, namely $Edu^2 + 2Fdudv + Gdv^2$, $edu^2 + (f + f')dudv + gdv^2$, of which the first is the square of the linear element of the spherical representation of the congruence. When the Jacobian of these two forms, written $Adu^2 + 2Bdudv + Cdv^2$, is equated to zero, the resulting differential equation defines the developable ruled surfaces of the congruence. This equation admits of real solutions in the case of normal congruences, congruences of Guichard, cyclic congruences, and congruences of tangents to a real surface. But there is a large variety of congruences for which the solutions of this equation are imaginary, and yet little or no study has been made of them. It is with these congruences of the elliptic type that Professor Eisenhart's paper deals. By means of a theorem of Cifarelli it is found that through each line of a congruence there pass two real ruled surfaces such that when they are parametric, the relations $A/E = C/G$, $B = 0$ obtain; they are called the characteristic ruled surfaces of the congruence. These ruled surfaces are found to possess properties analogous to certain well-known properties of the developable surfaces of a congruence of the hyperbolic type.

For a congruence of the elliptic type the abscissas of the focal points, measured from the middle point, are conjugate pure imaginary quantities. If the latter be denoted by ρ and $-\rho$, the locus of the points at the distances $i\rho$ and $-i\rho$ constitute a pair of real surfaces which are called the pseudo-focal surfaces. When these surfaces are associate to one another, the congruence is of the kind discussed by Lilienthal. A further study of these congruences is made in terms of parameters so chosen that $A = C$, $B = 0$, and these results are considered in connection with the congruences formed by the joins of corresponding points on associate surfaces. The paper closes with a discussion

of the congruences whose developables cut the middle surface in the unicursal curves on the latter.

2. In Mr. Jackson's paper it is first pointed out that by combining a theorem recently proved by E. B. Wilson with an old result of Frobenius one can at once infer that a necessary and sufficient condition that a linear transformation be resolvable into the product of two involutory transformations is that there exist a non-singular bilinear form which this transformation carries over into itself. A new proof of the theorem of Wilson just referred to is then given.

3. Poincaré has shown how the higher terms in the development of the perturbative function may be derived approximately without computing the intervening terms. Applications of the theory to particular cases, principally in coplanar orbits, have been made by Coculesco, Féraud, and Hamy. Dr. Reed's paper considers the general case of elliptic orbits with any mutual inclination. The method is in part dependent upon numerical assumptions, but by use of a criterion for the admissibility of those singular points upon which the approximations depend the application can be made to orbits of any elements whatever. The solution of an ideal Jupiter-asteroid problem of general type is given in detail.

4. If the equation of a quadric contains a rational parameter to some degree higher than the first, and the two systems of generators are rationally separable, each system Σ, T will define a line congruence having the envelope of the family for focal surface ϕ. Each quadric will touch the focal surface in the points of an elliptic space quartic C_4. If a generator of Σ touches ϕ in P_1, P_2, the operation of interchanging P_1, P_2 is involutional. Through P_2 passes a tangent to ϕ belonging to T, touching ϕ again in Q_2. The operation ΣT is now of infinite order.

For special relations among the coefficients, one, two, or three linear factors can be removed from ϕ. Professor Snyder established the following theorem:

Surfaces of order four or any number greater than five can be constructed which are invariant under an infinite discontinuous group of birational transformations.

All the congruences of order 2 except those of class 7 are shown to belong to this category. The method is applied to

two other congruences and the fundamental points of the transformation are determined.

5. In a recent article in the *Transactions*,* a complete geometric characterization of the families of curves defined as the extremals connected with variation problems of the form $\int F ds = minimum$ (where F is any point function and ds is the element of arc in the space considered) is given for euclidean spaces of two and three dimensions; this may be easily extended to all spaces of constant curvature. In the present preliminary paper, Mr. Lipke gives a complete geometric characterization of natural families of curves in a curved space of two dimensions (general surface). The ∞^2 curves of any natural family in such a space have the following properties: (A) The locus of the centers of geodesic curvature of the ∞^1 curves through a point is a straight line; (B) among the ∞^1 geodesic circles (circles of constant geodesic curvature) which osculate the ∞^1 curves through a point, there exist two hyperosculating circles which are orthogonal. The natural families are merely a special case of a much broader class of curves characterized by property (A). The latter class also includes families of isogonal trajectories, which may be associated with and are gotten from the natural families by a curvature transformation.

6. Mr. Hawkesworth's paper is in abstract as follows: If two or more triangles to which any conic is in common escribed have each a vertex upon the same axis, and if in each case a circle be described passing through the triangle's other two vertices and with its center upon the axis, then the resultant system of circles will have the conic's other axis for their common radical axis passing in common through the two foci, or focoids, as the case may be, lying upon it, while their "limiting points" upon the first and given axis and line of their centers will be, in turn, the conic's focoids or foci. With other corollaries, the very important one follows that if any triangle cut by any transversal be given, then there is an easily found unique conic that has the transversal for an axis and is circumscribed by the triangle.

7. By considering a given complete biorthogonal system in

* "Natural families of trajectories: conservative fields of force." By Edward Kasner, *Trans. Amer. Math. Society*, April, 1909.

connection with some of Hilbert's methods Mrs. Pell obtains the following results: Either the no -homogeneous integral equation

$$\phi(s) = f(s) + \int_a^b K(s, t)\psi(t)dt,$$

where $\phi(s)$ and $\psi(s)$ are to be determined as associated functions with respect to the given biorthogonal system, has a solution $[\phi(s), \psi(s)]$; or the homogeneous integral equation

$$\phi(s) = \int_a^b K(s, t)\psi(t)dt$$

has. If the unsymmetrical kernel $L(s, t)$ is the associated function of a symmetrical kernel there always exist characteristic values λ_i and corresponding characteristic solutions

$$\phi_i(s) = \lambda_i \int_a^b L(s, t)\phi_i(t)dt, \qquad \psi_i(s) = \lambda_i \int_a^b L(t, s)\psi_i(t)dt.$$

The solutions $[\phi_i(s), \psi_i(s)]$ form a biorthogonal system equivalent to the given one. Any function $f(s)$ expressible in the form

$$f(s) = \int_a^b L(s, t) f_1(t)dt$$

can be developed in a uniformly convergent series of the form

$$f(s) = \sum_{i=1}^{\infty} \phi_i(s) \int_a^b \psi_i(s) f(s)ds.$$

9. Professor Kasner obtains a set of general relations involving the curvatures of the three surfaces and the curvatures of of their curves of intersection. The more general theory of triply orthogonal congruences of curves is also studied. Corollaries are given for plane curves.

10. In Professor Kasner's second paper it is shown that the orthogonality properties involved in Thomson's theorem are sufficient to characterize natural families of trajectories. Stronger converse results are also obtained.

11. Professor Miller's paper appears in full in the present number of the BULLETIN.

12. The integral equation of the second kind with variable limits which have a constant difference has been solved by P. Hertz and C. Herglotz (*Mathematische Annalen*, volume 65). In Dr. Sharpe's paper a more direct and shorter solution is obtained by considering the limit of a finite difference equation with constant coefficients. The complete solution is found for the case in which a certain transcendental equation has equal roots. An application is made to the problem of the age distribution of population. It is shown that the final distribution is ultimately independent of the initial distribution.

13. By means of Euler's ϕ-function Mr. Carmichael defines $\lambda(n)$ as follows: $\lambda(p^a) = \phi(p^a)$ when p is an odd prime; $\lambda(2^a) = \phi(2^a)$ when $a = 1$ or 2; $\lambda(2^a) = \frac{1}{2}\phi(2^a)$ when $a > 2$; $\lambda(2^a p_1^{a_1} \cdots p_i^{a_i})$ = the lowest common multiple of $\lambda(2^a)$, $\lambda(p_1^{a_1}), \cdots, \lambda(p_i^{a_i})$, $p_1, \cdots, p_i$ being different odd primes. If $\lambda(n)$ is the exponent to which a belongs modulo n, then a is called a primitive λ-root of $x^{\lambda(n)} = 1 \pmod{n}$. The following are some of the theorems obtained:

For any given n the congruence $x^{\lambda(n)} \equiv 1 \pmod{n}$ is satisfied by every x prime to n. In every congruence $x^{\lambda(n)} \equiv 1 \pmod{n}$ a solution g exists which is a primitive λ-root, and for any such g there are $\phi\{\lambda(n)\}$ primitive λ-roots congruent to powers of g. If $\lambda(n) > 2$ the product of primitive λ-roots congruent to powers of g is congruent to $1 \pmod{n}$. If x_1 is the largest value of x satisfying the equation $\lambda(x) = a$, any other solution x_2 is a factor of x_1. Let a be that divisor of α for which $\lambda(x) = a$ has a greatest solution x_1 greater than such a solution when for a any other divisor of α is taken. Then x_1 is the largest divisor of $z^\alpha - 1$ for every z prime to the divisor. Finally it is shown that there are values of composite n for which the relation $e^{n-1} \equiv 1 \pmod{n}$ is true when e is any number prime to n, and a method is given for finding such values of n.

14. In a paper "Concerning a certain type of continued fractions depending on a variable parameter" in the *American Journal of Mathematics*, volume 29, number 3, Professor McKinney showed that when X_0 is properly equivalent to a "critical value" x_0, then $\lambda(X_0)_{\lambda'}$ may be ultimately like $\lambda(x_0)_{\lambda'}$ or ultimately like the "associate" of $\lambda(x_0)_{\lambda'}$. In that paper no criterion was established for deciding between these two cases. It is the object of the present paper to set up a criterion for an important class of critical values.

15. The paper by Dr. Chambers takes up the non-abelian abstract groups of order p^2q as determined by Hölder, and finds for each its corresponding group of isomorphisms and the group of cogredient isomorphisms. Independent generators for these groups of isomorphisms are obtained in terms of the generators of the original groups, and sufficient relations among these generators are determined to define the groups of isomorphisms.

16. The discussion of the periodic orbits of a particle which is subject to the newtonian attraction of finite bodies whose motion is supposed to be known, begins naturally with Jacobi's restricted problem of three bodies. In this case all the periodic orbits of the particle in the plane of motion of the finite bodies have one axis of symmetry, namely, the line joining the finite bodies. The symmetrical properties are important, both in the numerical processes of Darwin * and the analysis employed by Moulton.† In generalizing the problem by introducing a greater number of finite bodies, the next step is to consider three spheres moving in circles according to the equilateral triangle solution of Lagrange. This problem has been treated in detail.‡ The periodic orbits of the particle have no axis of symmetry, or they have one such axis, depending upon the distribution of mass. Professor Longley's note considers certain cases in which the particle is subject to the attraction of more than three finite bodies revolving in circles about their center of mass. Some of the periodic orbits of the particle about one of the finite bodies have more than one axis of symmetry, and, as a consequence, it is possible to make some simplifications in the analysis used for the preceding cases.

17. In the BULLETIN, volume 12 (1905), pages 21–38, Professor Bussey published a paper entitled "Galois field tables for $p^n \leqq 169$." The present paper is an extension of that work and contains tables of all Galois fields of order p^n such that $n > 1$ and $169 < p^n < 1,000$. The limit 1,000 is that set by Jacobi in his Canon Arithmeticus, which contains tables of indices for all primes less than 1,000. These tables of indices are tables of all Galois fields of prime order. When this paper is published, there will be in print tables of all Galois fields of order less than 1,000.

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* *Acta Mathematica*, vol. 21 (1897), p. 99.
† *Transactions Amer. Math. Society*, vol. 7 (1906), p. 537.
‡ *Ibid.*, vol. 8 (1907), p. 159.

18. Professor Ford's paper is devoted to a proof of the following theorem: "If a given series (convergent or divergent) is summable and of indeterminacy r in Hölder's sense, it is summable and of indeterminacy r in Cesàro's sense, and conversely." The direct part of this theorem has already been established by Knopp in his dissertation "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze" (Berlin, 1907), while the converse has been proved for the special cases in which $r = 1, 2$ by Bromwich (*Mathematische Annalen*, volume 65 (1908), pages 363–365). The above theorem supplements these investigations by establishing both the direct and converse theorems for all values of r. It follows that the sum formulas of Hölder and Cesàro are coextensive in their applicability to given series and determine one and the same sum. The paper has been offered to the *American Journal of Mathematics*.

19. Professor Veblen's paper has to do with a class of theorems of which Pascal's theorem on conic sections is one and of which another is the following: If a simple hexagon is inscribed in a surface of the second degree, each pair of opposite sides is met by a pair of lines conjugate to both sides, and the latter three pairs of lines are met by one pair of lines.

20. The investigations made by Dr. Gundelfinger in this paper arise from an interpretation in the plane of the geometry within a linear line complex by means of a transformation due to Sophus Lie.* This transformation takes a point of space into a line element of the plane; curves of the complex into unions; complex lines, in particular, into vertical parabolas; — hence the tangents to a complex curve transform into the vertical parabolas which osculate the corresponding union.

From a classification of curves and of surfaces with respect to the group G_{10} of projective transformations in space which leave the linear line complex invariant we obtain a classification of "line element loci" of ∞^1 and of ∞^2 elements with respect to the Γ_{10} of contact transformations in the plane — that is we classify ordinary differential equations of the first order with respect to the contact relations of the ∞^2 vertical parabolas which osculate their integral curves.

The fact that the linear line complex is invariant also under the correlation which it defines when regarded as a null system

gives rise to a theory of "reciprocal line element loci" of much interest. Furthermore, the various types of congruences of complex lines, characterized by the nature of their focal surface the two sheets of which are reciprocal configurations, give rise to all possible arrangements of ∞^2 vertical parabolas in the plane which are distinct in regard to the singular solution of the corresponding differential equation of the second order.

By using a second transformation of Lie's † which differs from the above in that complex lines go over into circles instead of vertical parabolas, a second interpretation is effected giving rise to many interesting theorems on the osculating circle.

21. In recent investigations of Scheffers and Lilienthal some interesting theorems are demonstrated on the osculating circles of the integral curves of an ordinary differential equation of the first order. Professor Smith's paper recasts these results in such a way that generalization to partial differential equations is made possible. It is found that the rôle played by the osculating circles is assumed in space by osculating vertical parabolic bands, that is, bands on the surface of a parabolic cylinder between any two parallel vertical planes indefinitely near. It is shown that the osculating vertical parabolic bands of the characteristic bands of an arbitrary partial differential equation of the first order belong to a second partial differential equation — the reciprocal of the former. Exceptional cases are easily characterized. The results follow readily from the projective geometry within a null-system in five-dimensional space, a subject which has recently received the attention of Professor Eiesland.

22. The paper by Professor Richardson and Mr. Hurwitz appeared in full in the October number of the BULLETIN.

23. Hilbert has recently discovered * the connection between the calculus of variations and the theories of differential and integral equations and has emphasized the fact that the former theory is the wider in its scope. The self-adjoint differential equation of the second order $(pu')' + qu + \lambda ku = 0$ containing the parameter λ may be regarded as the Lagrange equation of a calculus of variation problem with a certain quadratic sub-condition or with this quadratic and certain linear sub-conditions. The questions concerning the necessary and sufficient

* Geometrie der Berührungstransformationen, p. 238.
† Ibid., p. 245.

conditions for the existence of a minimum, in particular the significance of the Jacobi criterion are discussed in this paper by Professor Richardson. The chief theorem may be stated as follows: The Jacobi criterion demands that the solution $u(x) = U_1(x)$ of the problem

$$\int_0^1 (pu'^2 - qu^2)dx = min, [p(x) > 0,\ q(x) \leqq 0,\ u(0) = u(1) = 0],$$

with the quadratic sub-condition

$$\int_0^1 k(x)u^2dx = 1,$$

shall not vanish in the interval (0, 1), that the solution $u(x) = U_2(x)$ of the same problem with the quadratic and one linear sub-condition

$$\int_0^1 kU_1(x)u(x)dx = 0$$

shall vanish once in the interval, and in general that the solution $u(x) = U_{n+1}(x)$ of the problem with the quadratic and n linear sub-conditions

$$\int_0^1 k(x)U_1(x)u(x)dx = 0,\ \cdots,\ \int_0^1 k(x)U_n(x)u(x)dx = 0$$

shall vanish exactly n times in the interval.

This paper is the first of a series to appear in the *Mathematische Annalen.*

24. Professor Schwatt developed certain methods for the summation of special form of infinite series.

25. Professor Coble's paper deals with two kinds of cubic space curves associated with a general cubic surface.

The curve of the one kind has six skew lines of a double six of the surface as axes. It contains ∞^2 sets of five osculating planes whose ten meets are on the surface. There are 72 systems of these curves, each system containing ∞^2 curves.

The curve of the other kind meets the Hessian of the surface in six pairs of corresponding points. It contains ∞^2 sets of five

* "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen"; *Göttinger Nachrichten*, 1 und 2 Mittheilungen 1904, 4 und 5 Mittheilungen 1906.

points any three of which are an apolar triad of the surface. There are 16 systems of these curves, each system containing the ∞^2 curves on a set of five points found among the poles of the planes of the Sylvester pentahedron.

26. A solution $(x_1, x_2, \cdots, x_n)$ of the system of ordinary differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_n) \quad (i = 1, 2, \cdots, n)$$

is defined to be stable if $x_1, x_2, \cdots, x_n$ remain finite for $t > t_0$. This paper by Professor Birkhoff is a study of such solutions, especially with reference to their periodic or quasi-periodic character.

27. The totality of tautochrones in a plane, due to a force whose rectangular components are functions only of the coordinates of the point at which the force acts, forms a system of ∞^3 curves. In the July number of the BULLETIN Professor Kasner gave the intrinsic equation of such a system. In the present paper Mr. Reddick deduces the cartesian equation of the system, a differential equation of the third order, and investigates its geometric properties. The necessary and sufficient conditions that any plane system of ∞^3 curves shall be a family of tautochrones are obtained.

28. In his Lectures on the icosahedron, Klein shows that any finite group of fractional linear substitutions on a complex variable can be reduced to a canonical form such that each substitution will correspond (by the ordinary stereographic projection) to a rotation of the sphere which has the unit circle of the complex plane as a great circle. The poles of each substitution are what we may call "skew-inverse" points with respect to the unit circle.

The following questions, which are in fact equivalent, naturally arise:

(1) If a finite group of fractional linear substitutions is *not* in this canonical form, do its operations correspond in the same way to rotations of a sphere having some other circle of the complex plane as a great circle?

(2) If upon all pairs of skew-inverse points with respect to a circle in the complex plane, one operates with an arbitrary fractional linear substitution, will they go into skew-inverse points with respect to some circle; and if so, what circle?

In the present paper Dr. Carver answers the second question; and hence also the first.

29. In an earlier paper,* Dr. Dederick gave sufficient conditions for the existence of a solution of the equation $F(x, y) = 0$, in the form $y = f(x)$ in the neighborhood of a point where $\partial F/\partial x = \partial F/\partial y = 0$, and for the existence and continuity of dy/dx in the case where the equation for formally determining dy/dx has a simple root. The present paper extends these results to the higher derivatives of y with regard to x, and to the case of a multiple root of the equation for dy/dx. The results may be summed up in the statement: The ordinary process for determining the character of the branches of the plane curve $F(x, y) = 0$ at a singularity, and the method of differentiating $F(x, y)$ totally with regard to x or with regard to a parameter t, in order to obtain the successive derivatives of y with regard to x or of x and y with regard to t, are justified if the partial derivatives of $F(x, y)$ which are used are continuous at and near the point in question.

30. The problem of the determination and discussion of point-circle correlations in the plane from the standpoint of the geometry of contact transformations is treated in Dr. Burgess's paper. The method used involves those questions in the geometry on a quadric which arise in point-plane correlations in space. Algebraic difficulties make it necessary to limit the scope of the paper to a complete solution of the involutory cases — a solution effected by elementary divisors. The Cayley numbers for a curve on a quadric are interpreted for its stereographic projection, and formulas given by which the Cayley numbers for the transformed curve may be found. The æquatio-directrix of the transformation suggests two problems: (1) the study of the properties of a single transformation, and (2) the study of a system of transformations in one parameter. In certain cases, this system turns out to be a group.

31. A general theory of linear groups which applies equally to continuous and discontinuous groups in any number of variables was announced in Professor Newson's paper. If no restrictions are laid upon the variation of the elements of the matrix M of a linear transformation T in n variables, the continuous group G_{n^2-1} of ∞^{n^2-1} collineations in space of $n-1$

* Presented at the April meeting of the Society.

dimensions is obtained. A subgroup of G_{n^2-1} is obtained when the elements of M satisfy certain conditions, as *e. g.*, the well-known conditions defining the orthogonal group. Professor Newson's fundamental theorem lays down the necessary and sufficient conditions which its elements of M must satisfy in order to have a subgroup of G_{n^2-1}. He defines a complete family of automorphic forms ϕ_i which are homogeneous and symmetric functions in from 1 to n sets of n variables each. His theorem is: A necessary and sufficient condition for the existence of a subgroup of G_{n^2-1} is that the elements of M satisfy a set of equations $\phi_i = l_i$ consisting of a complete family of automorphic forms in the elements of the rows or columns of M, each equated to the corresponding coefficient of the family.

Families of lower degrees define continuous subgroups of G_{n^2-1}; after a certain degree is reached the subgroups become discontinuous; above a certain other degree the conditions are satisfied only by the identical transformation. The paper will be published in the *Kansas University Science Bulletin*.

32. Mr. Schweitzer contrasted the formal properties of Bolzano's linear series with his exposition of the series of Vailati (the system ${}^1R_1$) and showed how to extend Bolzano's series to n dimensions ($n = 1, 2, 3, \cdots$) by considering simple modifications of the axioms of dimensionality and extension in his system ${}^nR_n$. Application of the author's n-dimensional open and closed chains is made.

F. N. COLE,
Secretary.

THE GROUPS WHICH MAY BE GENERATED BY TWO OPERATORS s_1, s_2 SATISFYING THE EQUATION $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α AND β BEING RELATIVELY PRIME.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 13, 1909.)

SINCE s_1s_2 and s_2s_1 are of the same order and α, β are relatively prime, it results that this common order is prime to both α and β. Hence s_1s_2 and s_2s_1 are generated by either $(s_1s_2)^\alpha$ or $(s_2s_1)^\beta$, and the cyclic group generated by s_1s_2 coincides with the one generated by s_2s_1. A direct consequence of this is that the group generated by s_1s_2 is invariant under the entire group G

generated by s_1, s_2. It is also evident that G is generated by s_1s_2, s_1, and that every group which involves an invariant cyclic subgroup and is generated by this subgroup and an additional operator may also be generated by two operators satisfying the conditions imposed upon s_1, s_2 in the heading of this paper. These facts establish the theorem :

The totality of the groups which may be generated by two operators satisfying the condition $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α and β being relatively prime, coincides with the totality of those generated by two operators which are such that one of them is transformed into a power of itself by the other.

The groups which may be generated by two operators t_1, t_2 having a common square have been called the generalized dihedral rotation groups.* That these groups are included in the category satisfying the conditions imposed in the heading of the present paper results directly from the facts that

$$t_1t_2^{-1} = (t_2^{-1}t_1)^{-1}, \quad \{t_1, t_2\} \equiv \{t_1, t_2^{-1}\}.$$

Hence this category of groups may be regarded as composed of those groups which result from a second generalization of the dihedral rotation groups; the first generalization corresponds to the special values $\alpha = 1$, $\beta = -1$. As $(s_1s_2)^\alpha = (s_2s_1)^\beta$ expresses only one condition between s_1, s_2 this equation must be satisfied by the generators of an infinite number of distinct groups of finite order for every pair of values of α, β.†

One of the most interesting features of the relation $(s_1s_2)^\alpha = (s_2s_1)^\beta$ is that it may be used as one of the two conditions which are satisfied by the two generators of only a finite number of groups. This fact may be established as follows : If $s_1^a = s_2^b$, it is evident that s_1^a is invariant under G and hence $\{s_1^a, s_1s_2\}$ is an abelian group. When a and b are relatively prime this abelian group coincides with G ; for otherwise the quotient group of G with respect to this abelian group would have an order prime to b, and s_2 would correspond to an operator differing from the identity in this quotient group. This is impossible since s_2^b would have to correspond to identity. Having proved that G is abelian whenever α, β and a, b are two pairs of relatively prime numbers, it is not difficult to prove that there is only a finite number of distinct operators which can satisfy both of these conditions for given values of

* *Archiv der Mathematik und Physik*, vol. 9 (1905), p. 6.

† *Amer. Jour. of Mathematics*, vol. 31 (1909), p. 167.

the relatively prime pairs α, β and a, b. We shall prove this fact in the following paragraph.

Since G is abelian $(s_1s_2)^\alpha = (s_2s_1)^\beta = (s_1s_2)^\beta$, and hence $(s_1s_2)^{\alpha-\beta} = s_1^{\alpha-\beta}\, s_2^{\alpha-\beta} = 1$, or $s_1^{\alpha-\beta} = s_2^{\beta-\alpha}$. Combining this equation with $s_1^a = s_2^b$, there results $s_1^{a(\alpha-\beta)} = s_2^{a(\beta-\alpha)} = s_2^{b(\alpha-\beta)}$, and hence

$$s_2^{(a+b)(\beta-\alpha)} = 1 = s_1^{(a+b)(\alpha-\beta)}.$$

As the orders of s_1, s_2 are limited and these operators must be commutative, this proves that *only a finite number of groups can be generated by two operators which satisfy both of the equations*

$$(s_1s_2)^\alpha = (s_2s_1)^\beta \quad \text{and} \quad s_1^a = s_2^b,$$

where α, β and a, b represent two pairs of relatively prime numbers. For instance, when these numbers are 4, 5 and 2, 3 G is the group of order 5. That is, if s_1, s_2 satisfy both of the equations

$$(s_1s_2)^4 = (s_2s_1)^5, \; s_1^2 = s_2^3,$$

they must generate the group of order 5. This result establishes close contact between the present note and the paper "On groups which may be defined by two operators satisfying two conditions," *American Journal of Mathematics*, volume 31 (1909), page 167.

A NOTE ON IMAGINARY INTERSECTIONS.

BY PROFESSOR ELLERY W. DAVIS.

In the plane let there be a conic C and a line L. Set up a system of coordinates such that L is the line infinity, its pole O with regard to C is the origin, the axes OX and OY are conjugate with regard to C, while X and Y are their intersections with L. Furthermore let $x = \pm 1$, $y = \pm 1$ be tangents to C through Y and X respectively. Then $x =$ a constant passes through Y, while $y =$ a constant passes through x. All these lines are to be determined by the fact that any four convergents form a harmonic set when the constants in the right member are a harmonic set of numbers. In brief, C, L, and the coordinates are projectively transformed from a circle $x^2 + y^2 = 1$, the line infinity, and a rectangular system whose origin is the center of the circle. The equation of any line in the transformed coordinates is precisely the same as that of which it is the projection in the rectangular coordinates.

In the transformed coordinates, let us represent the point with complex coordinates $(x' + ix'', y' + iy'')$ by a vector from (x', y') to $(x' + x'', y' + y'')$. If we change to another pair of conjugate axes OX_1, OY_1 the coordinates are changed to $x_1' + ix_1''$, $y_1' + iy_1''$, but the vector is left unchanged since its end points are.

Consider, however, the intersections of C with L. These are given by $x' = \infty$, $x'' = 0$, $y' = 0$, $y'' = \pm ix'$. That is, the points x', $\pm y''$ are the intersections of $y = \pm x$ with L.

But the lines $y = \pm x$ change when we change the axes as above. Thus the vectors taken to represent the intersections of C with L also change.

If with Cauchy we represent $(x' + ix'', y' + iy'')$ by a vector joining the real points on the circular rays through it, we meet the same difficulty. For these real points are $(x' + y'', y' - x'')$ and $(x' - y'', y' + x'')$; that is, in the case under consideration, they are $(\infty, 0)$ and $(0, 0)$ so that the direction changes as OX does.

Von Staudt's representation is, however, unimpaired; since his imaginary intersection is the involution of all the point pairs cut out on L by the involution of all the line pairs $y = \pm x$. The vector representation should then be regarded as a symbol, adapted to the particular coordinates used, for the more complete Von Staudt representation.

UNIVERSITY OF NEBRASKA,
July 16, 1909.

MAUROLYCUS, THE FIRST DISCOVERER OF THE PRINCIPLE OF MATHEMATICAL INDUCTION.

BY DR. G. VACCA.

Introductory Note. — Soon after the publication of my review of Voss's address (BULLETIN, volume 15, page 405), wherein I considered at some length the history of mathematical induction, I received a note from Professor Moritz Cantor of Heidelberg, in which he called my attention to Dr. Vacca's research on this same historical topic. As Dr. Vacca's research was not accessible to me, I wrote to him for information and received, in reply, the following notes, which will doubtless be of general interest to American readers. — FLORIAN CAJORI.

Many years ago I published in the *Formulaire de Mathé-*

matique of Professor Peano an account of the first discovery of mathematical induction, as due to the Italian Maurolycus. But this paper seems to have had only a small diffusion. I think it useful, therefore, to give, with some more details, a short account of this important discovery.

I.

Franciscus Maurolycus was born in Messina in the year 1494, and died in the same city in the year 1575. His works on the Greek mathematical authors, Euclid, Archimedes, Apollonius, Theodosius, and many others, have been of the greatest importance for the transmission of Greek science to Europe. But the most original of his works is the treatise on arithmetic "Arithmeticorum libri duo" written in the year 1557 and printed in Venice in the year 1575 in the collection "D. Francisci Maurolyci Opuscula mathematica."

In the Prolegomena to this work he points out that neither in Euclid nor in any other Greek or Latin writer (among them he enumerates Iamblichus, Nicomachus, Boetius) is there, to his knowledge, a treatment of the polygonal and polyhedral numbers, and he reproaches Jordanus for having been content with a useless repetition of what was written by Euclid.

"Nos igitur [he says] conabimur ea, quae super hisce numerariis formis nobis occurrunt, exponere: multa interim faciliori via demonstrantes, et ab aliis authoribus aut neglecta, aut non animadversa supplentes."

This new and easy way is nothing else than the principle of mathematical induction. This principle is used at the beginning of the work only in the demonstration of very simple propositions, but in the course of the treatise is applied to the more complicated theorems in a systematic way.

For instance, he demonstrates at first that: "omnis quadratus cum impari sequente coniunctus, constituit quadratum sequentem (Prop. 13)." [In modern symbols: if a is a number, then $a^2 + (2a + 1) = (a + 1)^2$].

Using this result, he now demonstrates that:
"Ex aggregatione imparium numerorum ab unitate per ordinem successive sumptorum construuntur quadrati numeri continuati ab unitate, ipsisque imparibus collaterales (Prop. 15)" [in modern symbols:

$$1 + 3 + 5 + 7 + \cdots + (2a + 1) = (a + 1)^2].$$

"Nam per P 13, unitas imprimis cum impari sequente, facit quadratum sequentem, scilicet 4. Et ipse 4 quadratus secundus, cum impari tertio, scilicet 5, facit quadratum tertium, scilicet 9. Itemque 9 quadratus tertius, cum impari quarto, scilicet 7, facit quadratum quartum, scilicet 16. Et sic deinceps in infinitum, semper P 13 repetitam propositum demonstratur."

The form adopted by Maurolycus with the object of persuading the reader of the truth of his demonstrations is generally the following. He applies the reasoning to the first numbers, very often to the first five, and then he concludes with some one of these phrases:

"et eodem syllogismo pro quovis alio assignato loco, utemur ad roborationem propositi (pag. 30, Prop. 65)."

"et argumentatio a quinto loco ad alia quaevis loca transferetur ad conclusionem propositi (pag. 31, Prop. 66).

"et a quinto loco transfertur syllogismus ad quemvis alium, ut propositio conclusit (pag. 33, Prop. 67)."

There are in this work many other points of interest for the history of mathematical knowledge, but I hope to be able to write something on it at another occasion.

II.

But now we have a question before us. Was Pascal unaware of the book of Maurolycus?

In his Traité du triangle arithmétique, printed perhaps in the year 1657, he never mentions Maurolycus, notwithstanding that, in my opinion, this treatise is only an application of the method discovered by Maurolycus. But Pascal, shortly after, being engaged in the polemic concerning the cycloid, in the well-known letter "Lettre de Dettonville à Carcavi" had to demonstrate a proposition concerning the triangular and pyramidal numbers. He says then: *

"CELA EST AISÉ PAR MAUROLIC."

It is strange to point out that not even the name of Maurolycus has been included in the Table analytique of the old edition of the works of Pascal, and more strange that the editors of the new edition of the "Oeuvres" of Pascal † in a very incomplete

* B. Pascal, Oeuvres complètes, tome troisième, Paris, Hachette, 1889, page 377.

† Blaise Pascal, Oeuvres par Léon Brunschwigg et Pierre Boutroux. Paris, Hachette, 1908, in 3 vols.

historical note before the reimpression of the Traité du triangle arithmétique (volume 3, pages 435–444) never mention the name of one of the greatest European mathematicians of the sixteenth century.

GENOA, ITALY,
June, 1909.

DARWIN'S SCIENTIFIC PAPERS.

Scientific Papers. By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. I, *Oceanic Tides and Lunar Disturbance of Gravity*, xiv + 463 pp.; Vol. II, *Tidal Friction and Cosmogony*, xvi + 516 pp. Cambridge University Press, 1907, 8. Royal 8vo.

IF one were in need of an example to illustrate the English use of the term "applied mathematics," it would hardly be possible to find a better one than that furnished by the scientific papers of Sir George Darwin. Many investigations ranging from the purest of pure mathematics to the observational portions of the phenomena of nature have at times been placed under this title in our journals and treatises, and after all our language is governed by general usage and not by arbitrary rules. But in the use of an imported term it would seem better, in spite of the customs of late years, to take the foreign value rather than the domestic as the basis of definition. This definition implies an actual or suggested relation between a problem set forth on arbitrary hypotheses and the observed phenomena of matter in space. The course which the argument follows — the laying down of hypothetical laws approximating as nearly as may be to those of nature, the translation into and development of those laws by means of symbols, and the final transference from the symbols back to the phenomena, is well known. The skill of the applied mathematician is chiefly shown in his management of the second of these stages so as to produce as much as possible in the third.

One cannot read any of Professor Darwin's papers without noting this attitude towards his work, and it is markedly shown in the excellent prefaces with which he introduces the papers contained in the two volumes under consideration. In them he gives a clear synopsis of each memoir, with observations on the results obtained. These observations are almost always on

the physical aspects of the problems he investigates and but little is said of the mathematical methods and the long and laborious developments which were sometimes necessary to arrive at the ends he had in view. Incidentally, it may be mentioned that it is quite possible to obtain a general idea of the contents of the memoirs from the prefaces alone, a more complete summary being given at the end of each of the longer memoirs. An interesting and important part of the preface to volume II is the author's summary of the present attitude of those best capable of judging towards the more speculative portions of his work.

The papers published by Sir George Darwin up to the present time lie chiefly in the domain of hydrodynamics and elasticity. Part I of the first volume consists mainly of memoirs dealing with the practical problem which has as its final object the prediction of the times and heights of the tides at any port. There are two general methods of attacking this question. The first, known as the equilibrium theory, is based on the supposition that the position which the water on the earth has at any instant is the same as that which it would have if the centers of the earth and moon were for that instant in their actual positions but relatively at rest. In other words, in the equations of motion we neglect all the effective forces (mass accelerations), except that due to the rotation of the earth round its axis. Even with this simplification the application of the theory to actual tidal effects is still remote, on account of the considerable effect which the distribution of land and water has on the height and time of the tide at any place. The "corrected equilibrium theory" attempts to take some account of this distribution by the evaluation of certain integrals over the water boundaries. This matter is dealt with numerically by Darwin in conjunction with H. H. Turner in paper 8. The boundaries in general are supposed to follow lines of latitude and longitude, $l = \text{const.}$, $\lambda = \text{const.}$; Professor Turner has obtained a closer approximation by making some of them follow the lines $\pm al \pm b\lambda = \text{const.}$, where a, b are small integers. It would be interesting to compare these results with integrals computed by following Love's harmonic division of the earth's surface.*

The only practical use which is made of the dynamical theory is the discovery of the periods of the principal tides on the sup-

* Presidential address, Section A, B. A. A. S. 1907.

position that the times and heights of the water at any place may be expressed as a sum of harmonic terms, each of which constitutes a "tide." Theoretically, the amplitude and phase of each tide are also determinate but owing to mathematical difficulties, they are always obtained from the observations. The number of terms to be taken into account is of course infinite, but for most places only ten or twelve need be considered, the observations themselves showing that the sum of the remainder may be neglected. Sir George Darwin has succeeded in putting this problem into an economical form. First, the determination of the constants, the same for all time for a particular port, from the observations of the tides for a year or more; and second, the approximate determination of the tide, when these constants and the position of the moon are given, by methods which are within the grasp of the ordinary navigator. More than half the first volume is devoted to these questions.

An interesting point in connection with the terms or tides of long period (a fortnight or a month) is the possibility that they would be considerably affected by an elastic yielding of the solid earth to the tide-generating forces. This was subjected to computation by Sir George Darwin and thence he deduced evidence that the earth's effective rigidity is at least as great as that of steel. The volume closes with the papers which contain an account of the experiments performed by himself and Horace Darwin to measure the lunar disturbance of gravity. This attempt did not meet with success at the time, but quite lately it has been found possible by means of improved apparatus to observe this disturbance.

It is in the second volume that we find the memoirs in which Sir George Darwin has developed the evolution theory of tidal friction which has its chief application in the past history of the earth and moon. In tracing this history back, several stages are to be noted. The first or present stage is that in which we consider an earth practically rigid and more or less covered by water. The attractions of the moon and sun cause tides which give rise to secular effects on account of friction in the motion of the water relative to the earth. The effect is two-fold: it increases the period of rotation of the earth on its axis and also the period of rotation of the moon round the earth. The changes in these two periods are not the same, and the combined effect to an observer situated on the earth is an apparent shortening of the period of rotation of the moon

round the earth. Some other body must be observed in order to separate the two effects: Professor Newcomb had hoped to examine the observations of Jupiter's satellites for this purpose. Since it directly concerns our measure of time, the question is of great importance and it may be advisable to collect observations of every kind which can be affected and examine them with this end in view.

A certain disagreement between observation and gravitational theory in the mean period of the moon's motion has long been known under the name of the "unexplained secular acceleration of the moon's mean motion" and it has most generally been ascribed to this differential effect of tidal friction. For many years its amount was considered to be such as to change the mean longitude by six seconds of arc in one century from the epoch of reckoning. Later, Newcomb's investigations changed it to four and later still to about two seconds. More careful research into the records of ancient eclipses, which form the chief means for obtaining the number, by Newcomb, Cowell, and others have shown how little dependence can be placed on the majority of them, and the number of eclipses to be rejected for this reason has continually increased. It is now doubtful whether we can state that there is any residual effect greater than one second to be ascribed to tidal friction or other unknown cause. This, of course, does not invalidate Darwin's work, since the cause and therefore the effect, though small, is real; it simply alters the time estimates for the total changes. Other effects are changes in the inclination and eccentricity of the lunar orbit.

The second stage is that of a viscous or imperfectly elastic earth in which the main tides are bodily, and it is in this stage that the past history of the earth and moon has been most fully developed by the author. The results are too well known to need restatement here. Volume II contains the memoirs in which they were developed, the paper embodying his remarkable discussion of the series of changes in the orbit of the moon due to the tidal distortion of the earth being that numbered 6. Looking backward in time, we observe the moon brought comparatively near to the earth.

In the third stage the earth is treated as a liquid body in which the effects of viscosity are probably small compared with those of inertia, in contradistinction to the second stage for which the reverse is the case. The problem is now that of a rotating

liquid mass disturbed by a body moving round it and at a distance which may be nearly of the same order of magnitude as the tidal distortion. In the fourth, and in this development the final, stage the two bodies form one liquid rotating mass ; the separation is supposed to have been effected by the period of rotation being too short for secular stability. The problem is therefore a consideration of the forms and stabilities of the equilibrum of rotating liquid masses under gravitation. The memoirs dealing with these stages will be given in volume III.

We have then a series of problems the connection between which is continuous from the physical point of view, but for which the mathematical representation is different. These differences of representation may be due to the mathematical difficulties which are presented in an attempt to treat them all as successive stages arising from giving to the time successive values in the same set of formulas, but more probably they are due to the real impossibility of obtaining a single formula adapted to numerical computation for the whole series of changes. We have an analogy in the representation of an analytic function containing singular points in the finite part of the plane; the same power series will not serve for every region of the plane. To continue the analogy, we may say that the different regions for which the formulas are available have been to a certain extent mapped out and the formulas more or less accurately obtained, but the connections between these regions are narrow and the labor of finding values for the function in these connecting portions, so as to make the numerical results continuous, becomes very great. Thus the gaps in Darwin's work which are most serious from the point of view of evolution are those which occur between the successive stages. But though mathematical analysis has not succeeded in bridging them, the crossing is less difficult, if less accurate, by argument from general physical principles.

The method for the solution of these problems adopted by Sir George Darwin is nearly always based on that used to solve the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

When the body approximates to a sphere the potentials are expanded in a series of spherical harmonics and these harmonics, in order to satisfy the hydrodynamical equations, have coefficients

which depend on the time. Much careful consideration is necessary in making the choice of those harmonics which will produce sensible effects. And the time element is an important factor. If the harmonics are periodic with respect to the time, we obtain oscillatory changes the magnitudes of which are directly seen, and they produce no secular changes in the system. But if they are not periodic, the changes produced will vary very considerably with the length of the interval over which they are discussed. We cannot then neglect non-periodic terms even when the coefficients may be small over intervals of time comparable with those of the periods of the larger terms whenever it is desired to discuss secular changes. This is one of the chief pitfalls in many of the problems of celestial mechanics. The necessity for using infinite series of functions the coefficients of which do not follow any simple law carries with it the necessity for stopping the series at some point, and in the great majority of cases we are not able to discover the error committed by so doing. Thus a doubt will continually arise as to whether some neglected term or terms may not considerably modify the results over the chosen interval of time. The doubts may become greater when the constants of the solution have to be determined from observations which should extend far back into the past to obtain the accuracy demanded.

As will be gathered from the above remarks, the arrangement of the papers is mainly by subjects, so that each volume is to a certain extent complete in itself. A chronological list of the papers published to date is given. The third volume containing papers on figures of equilibrium of rotating liquids will embody the investigations which lead to the possibility that the moon separated off from the earth by fission ; while the fourth volume will consist of the memoir on periodic orbits, addresses, and miscellaneous papers not included under the preceding titles.

ERNEST W. BROWN.

YALE UNIVERSITY,
July 28, 1909.

SHORTER NOTICES.

Vorlesungen über die Elemente der Differential- und Integralrechnungen und ihre Anwendungen zur Beschreibung von Naturerscheinungen. Von HEINRICH BURKHARDT. Leipzig, Teubner, 1907. 8vo. xi + 252 pp.

THIS volume has been prepared to meet the needs of a growing body of students who are finding the calculus useful in certain sciences which do not call for its fuller development — notably chemistry, mineralogy, and statistics. The author has aimed so to choose the material for exposition that the volume may serve as a first course in the calculus for others who, from choice or necessity, require an advanced course. The problem is complicated by seeking to avoid all "arithmetization" of material such as would be necessary for a complete and rigorous treatment of the fundamental limit

$$\lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^n .$$

This limit is, in a sense, the pons asinorum of the calculus and Professor Burkhardt feels that the chemist can not be gotten over it easily, and that even the mathematician finds difficulty with it when he is introduced to it too abruptly and not in connection with a complete discussion of the concept of limit. (Preface, page v.) The avoidance of arithmetic necessitates, from the author's point of view, leaving out all reference to convergence and to infinite series. Thus the usual series are spoken of as formulas and treated as approximations to the value of a function for certain of its arguments.

It is certainly interesting and instructive to any teacher of mathematics to know how a mathematician of Professor Burkhardt's attainments seeks to solve the problem he has set for himself.

The book is divided into ten chapters and a Nachtrag. It has a good table of contents and a complete register and is put together in the well known style of the publishers. It begins by a long, but in no wise uninteresting, introduction leading up, by easy stages through the laws of motion, to the notion of differential quotient. In this first chapter one finds the

usual concepts and symbols together with an exposé of the fundamental principles of cartesian geometry as applied to the straight line and to tangents in general.

The second chapter is devoted to the differentiation of rational functions, one section being set aside for the special consideration of the linear fractional function. This enables the author to illustrate the notion of infinitely great and to introduce the hyperbola and its asymptotes.

The differentiation of irrational functions is considered in the third chapter. The rule for differentiating inverse functions is first taken up and, by its aid, the rule for differentiating x^m is shown to hold for any rational m. The chapter closes with the rule for differentiating a function of a function.

The student is now able to differentiate any algebraic function, and in the next chapter he is introduced to the inverse operation. The chapter opens by finding a function whose differential quotient is gt, without previous knowledge of the differential calculus. This is the nearest approach to the treatment of a definite integral as the limit of a sum. Such a treatment would, of course, be impossible from the author's point of view. Integration is defined as the inverse of differentiation, and the integral of x^m is found for all values of m except $m = -1$. We find here the geometrical interpretation of the definite integral as the area beneath the curve whose ordinates, between the limits, are given by the integrand. The trapezoidal formula for calculating approximately the value of a definite integral is followed by its application to the integral

$$\int_1^2 \frac{dx}{x}.$$

Chapter five contains the crux of the whole volume. Professor Burkhardt takes his dilemma by both horns, so to speak, and *defines*

$$\int_1^x \frac{d\xi}{\xi}$$

to be the natural logarithm of x. By means of the geometrical interpretation just given, the student is shown that this new function is defined for all positive values of x and that it is not defined for $x = 0$. The question of its definition for negative values of x is expressly excluded.

The addition theorem for logarithms follows easily from the definition, and a comparison with common or Brigg's logarithms leads to the formula

$$\log x = \frac{\log z}{\log \text{Brigg}\, z} \cdot \log \text{Brigg}\, x.$$

This formula enables one to calculate the natural logarithms of all positive numbers, with the help of a table of common logarithms, as soon as the natural logarithm of *any* number $z \neq 1$ is known. The natural logarithm of 2 was found approximately in the preceding chapter. Thus we are across the bridge and the student has not realized its existence, the approaches have been so cunningly concealed !

The reviewer is reminded of the proof for differentiating the logarithm given in Olney's Calculus (1871) and there credited to the astronomer, J. C. Watson. Olney says of this proof that it "banishes from the calculus the last necessity for resort to series to establish any of its fundamental operations." But the modulus of the common system — there's the rub ! Olney resorts to infinite series to find it and Burkhardt relies upon the trapezoidal formula, which is dangerously near to the same thing. The fact is, it seems a doubtful policy to avoid all reference to arithmetic and substitute therefor the notion of an integral as a function of its upper limit. The student must feel that the method is, to say the least, round about. The question is pedagogically difficult, but teachers will find greater satisfaction in treating it as is done, for example, in Osgood's Calculus even for the student of chemistry or statistics.

The remainder of the chapter is devoted to the integration of rational fractional functions but only for the cases where the denominator has distinct or multiple real roots. Application is made to the integration of simple ordinary differential equations expressing chemical reaction between two or more substances.

The sixth chapter is the longest in the book and in many respects the most interesting. It contains sections upon higher derivatives, the mean value theorem, Maclaurin's and Taylor's formulas. The three important special cases, the binomial formula, the exponential formula, and the logarithmic or Mercator formula, are emphasized and illustrated. Numerical calculation is given a prominent place, and sections on reckoning with small quantities, on approximate solution of nu-

merical equations, and reversion of approximate formulas are given. Section 50, on the generalized mean value theorem and the remainder in the Maclaurin formula, is the nearest approach to a complete treatment of the development of a function in series which Professor Burkhardt allows himself. Maxima and minima values of a function receive only the barest treatment and in only the simplest cases. The same thing is true of the evaluation of indeterminate forms. One may, of course, disagree with Professor Burkhardt's placing of emphasis upon the materials chosen for this chapter. It would seem that maxima and minima values of a function were of sufficient importance, even to the chemist, to deserve fuller notice. But one must agree that the chapter is charmingly written and will commend itself to teachers for its simple straightforward presentation of subjects which many students find tedious and difficult in their formal setting.

Interpolation forms the subject of the seventh chapter. It is first performed by setting up a rational integral function to fit the observations, and a problem is fully worked out with all the necessary numerical calculation of coefficients. In the next place it is performed by finite differences, and lastly where the number of observations is greater than the number of constants to be determined.

Some of the numerical calculation might well have been left to the student and a paragraph or two devoted to logarithmic plotting, since it is just as easy to fit the curve $y = bx^a$ to the observations by plotting the logarithms of x and y and then using the straight edge as it is to fit $y = ax + b$ to the observations in the first place (page 184), and it may turn out to fit more accurately.

Chapters eight and nine are each but twelve pages in length. The first treats of the infinitesimal or differential, and the second of functions of two variables and partial derivatives.

In the tenth chapter, the student is led to see the use of trigonometric functions in describing periodic phenomena. Rules for differentiating and integrating trigonometric functions follow, and by inversion for the circular or *cyclometric* functions. Trigonometric interpolation is considered, as well as three simple cases of vibration, viz., free vibration, damped vibration, and forced vibration, in case the external force is itself periodic.

The Nachtrag is devoted to interpolation by means of exponential functions.

The book, as a whole, commends itself for its simplicity of presentation. The treatment of the logarithm is a doubtful pedagogical expedient but there is no lax rigor about it. The responsibility is shifted to the numerical calculation of logarithms, just as was done in Olney's Calculus over thirty years ago.* A student who has had elementary training in algebra and trigonometry can read the book without difficulty and, in the main, it presents enough of the calculus and its applications to serve that body of students of which we have spoken at the beginning. But it does not represent what we, in America, have come to consider as a first course in the calculus.

L. WAYLAND DOWLING.

Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen. Von VICTOR VON DANTSCHER. Leipzig und Berlin, Teubner, 1908. vi + 79 pp.

IN the preface we are told that these lectures are based upon a course given by Weierstrass in the summer semester of 1872, which was followed by the author of the present volume, and upon an elaboration of a later course given in 1884. The work under review is, however, not a mere reproduction of things given by Weierstrass, but it is the direct outcome of a course given repeatedly at the University of Gratz by Professor von Dantscher. It furnishes an easy and clear introduction to that theory of irrational numbers which was first developed by Weierstrass in his lectures at the University of Berlin, and it has decided pedagogic as well as scientific value.

C. Méray was the first to give a purely arithmetic meaning to the term irrational number,† and the theories developed by him, G. Cantor, Heine, and Dedekind have perhaps become better known than the theory of Weierstrass. This may be partly due to the fact that no expository publication relating to this theory was ever prepared under the direction of Weierstrass, and only the fundamental elements of this theory have been accessible in the works of Kossak, Pincherle, Biermann, and others. The first of these was based upon a course of lectures given by Weierstrass during the winter semester of 1865–6, and it was published in 1872 under the title "Die Elemente

* It should be stated that Olney sought only to avoid infinite series. The Watson proof of the rule for differentiating the logarithm tacitly assumes that $dx^n/dx = nx^{n-1}$ holds for all real values of n.

† Encyclopédie des Sciences mathématiques, tome I, vol. 1, p. 149.

der Arithmetik, Programm-Abhandlung des Werder'schen Gymnasiums."

The subject matter of the present volume may be sketched as follows: After a brief historical introduction and a proof of the fact that it is impossible, in the domain of rational numbers, to extract the nth root of a natural number which is not an nth power of a natural number, the fundamental concept "additive aggregate of an infinite number of positive rational numbers" is introduced, and we are reminded that the terms "additive" and "multiplicative," as applied to aggregates, are not due to Weierstrass. It seems desirable to use the term aggregate instead of the term set (Menge, ensemble) since a set of numbers generally implies that the numbers considered are discrete, while this is not commonly the case as regards the aggregates of numbers used by Weierstrass.* The term "additive" as applied to an aggregate of an infinite number of positive rational numbers implies that every possible sum of a finite number of the "members" of the aggregate is to be formed.

To make it possible to operate with these aggregates it is necessary to give a definition for the equality of two aggregates. In framing such a definition the term part (Bestandteil, partie) of a positive rational number a is used for every positive rational number smaller than a. Hence two rational positive numbers are equal if every part of each is a part of the other, and, similarly, two additive aggregates are said to be equal whenever every part of each is a part of the other. As there are additive aggregates whose equality may be established according to this definition irrespective of whether one of these aggregates is increased or diminished by any finite number, the author restricts himself to the consideration of additive aggregates which are such that every part is less than a given finite number. These aggregates are said to be convergent while all other aggregates are said to be divergent.

The explanation of these fundamental notions is followed by the representation of a convergent additive aggregate as a systematic fraction, and the proof that these aggregates may be combined by addition, subtraction, multiplication, and division after the definitions of these operations have been properly formulated. In view of the fact that the ordinary rules of arithmetic apply to these aggregates, they are called numbers, and the domain of rational numbers is extended by adding to

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* J. Tannery, *Bulletin des Sciences math.*, vol. 32 (1908), p. 103.

it these additive convergent aggregates, which may also be called irrational numbers. It is then proved that, in this enlarged domain, it is possible to extract the nth root of any natural number, and to represent the ratio of two lines, whether they are commensurable or incommensurable.

The subject matter of the rest of the volume may perhaps be sufficiently evident from the general headings of the last three lectures. They read as follows : Additive aggregates of an infinity of positive and negative rational numbers; additive aggregates of an infinity of complex numbers of the form $a + bi$; multiplicative aggregates of an infinity of numbers. The value of the volume is greatly enhanced by illustrative examples, and it may be heartily recommended even to those who are just beginning graduate work in our universities. It need scarcely be added that a clear comprehension of this theory of irrational numbers will clear up many difficulties as regards the theory of absolutely convergent series with numerical terms.

G. A. Miller.

Magic Squares and Cubes. By W. S. Andrews. With Chapters by Paul Carus, L. S. Frierson, C. A. Browne, Jr., and an Introduction by Paul Carus : Chicago, The Open Court Publishing Company, 1908. vi + 199 pp.

Among the Arabians magic squares were known in the ninth century of our era and about this time they played an important rôle in Arabian astrology. A special work on the subject is attributed to an Arabian mathematician named Tâbit ben Korrah who died in 901,* and H. Suter mentions several other early Arabian writers on this subject in his work entitled Die Mathematiker und Astronomen der Araber und ihre Werke. These facts are not in accord with the statement on page 1 of the book under review, which reads as follows : "The earliest record of a magic square is found in Chinese literature dated about 1125 A. D."

The present work is, in the main, a direct reprint of articles which appeared in the *Monist* during recent years. Its author is an electrical engineer who, during his leisure hours, "has given considerable thought to the working out in his own original way the construction of magic squares and cubes of various styles and sizes." As may be inferred from this excerpt

* Encyclopédie des Sciences mathématiques, t. 1, vol. 3 (1906), p. 63.

from the announcement and also from the preceding paragraph, the author made little use of the extensive literature on the subject, but has aimed to give a clear and interesting account of the results to which his own labors and those of his correspondents have led. An important exception is furnished by the chapter on the "Franklin squares," which gives a very interesting account of magic squares constructed by Benjamin Franklin, including a letter in which Franklin says "I make no question, but you will readily allow the square of 16 to be the most magically magical of any magic square ever made by any magician."

The general headings of the various parts of the book are as follows: Introduction by Paul Carus, magic squares, magic cubes, the Franklin squares, reflections on magic squares by Paul Carus, a mathematical study of magic squares by L. S. Frierson, magic squares and Pythagorean numbers by C. A. Browne, some curious magic squares and combinations, notes on the various constructive plans by which magic squares may be classified, and the mathematical value of magic squares. Under the sub-heading "Mr. Browne's square and lusus numerorum" Paul Carus gives instances of numbers which exhibit surprising qualities without being in the form of a magic square.

In 1896 Emory McClintock read a paper before the American Mathematical Society entitled "On the most perfect forms of magic squares with methods of their construction" in which he introduced the term pandiagonal magic squares for a type of squares which were called diabolic by Lucas and are generally known in Europe by the latter term. The paper by McClintock was published in the *American Journal of Mathematics*, volume 19, and constitutes one of the most important American contributions to the subject. Extensive bibliographical data on this subject may be found in volume 1 of the Subject Index of the Royal Society of London Catalogue of Scientific Papers, pages 84 and 85; in Ahrens's "Mathematische Unterhaltung und Spiele," 1901; and in the "Encyclopédie des Sciences mathématiques," tome 1, volume 3, pages 62 to 75. It is of interest to observe that the French edition of the great mathematical encyclopedia devotes thirteen pages to this subject while less than a page is devoted to it in the German edition.

As has been observed above, the few historical references in the present work should not be taken seriously, but in other

respects it can be commended highly to those who are attracted by marvellous relations among natural numbers. The author is looking forward to a second edition in which a number of slight errors will be corrected, and he has had the courtesy to send the reviewer a marked copy, in which the following changes are suggested: The term "perfect square" as used on page 2 is replaced by "regular square." In the second and third lines from the bottom of page 5 "twenty-eight" and "sixteen" are replaced by twenty and eight respectively, and in the first line of page 6 "twelve" is replaced by sixteen. The term "prime number" as used on page 14 and in many other places in the book is replaced by primary number. In the last line on page 65 the expression "first and last" should read last and first. Near the middle of page 179 the statement marked I. should be followed by "with four exceptions." These errors are, however, not sufficiently serious to detract much from the value of the volume.

G. A. MILLER.

Exercices et Leçons d'Analyse. By R. D'ADHÉMAR. Paris, Gauthier Villars, 1908. 208 pp.

THE subtitle of this volume is "Quadratures, équations différentielles, équations intégrales de M. Fredholm et de M. Volterra, équations aux dérivées partielles du second ordre." It will be seen that the topics treated are thoroughly up to date. The book is meant, as the author says in his preface, to supplement the larger Traités and Cours. An introduction of 22 pages presents a brief statement of some of the principal theorems on differential geometry and analysis, together with references for their proofs and for further developments. Then follow chapters on quadratures; the functions of Legendre, Bessel, Euler, etc.; partial differential equations of the elliptic type, including a brief treatment of Fredholm's integral equation; equations of the hyperbolic and parabolic types; and two chapters on miscellaneous problems. The book, in spite of its decidedly fragmentary character, will prove useful both by furnishing a source of interesting problems and by giving the reader at least a superficial idea of many recent developments in analysis. The indications given as to the scope and purport of theorems mentioned (to say nothing of their proofs) are, however, frequently so meagre that the reader seeking to gain information will often be in doubt as to how they should be

interpreted. Occasionally an actually misleading statement occurs, as when on page 116 in discussing the essential difference between the Cauchy-Kowalewski existence theorem for Laplace's equation and the "problem of Dirichlet" it is at least strongly implied that the former theorem does not apply to closed curves, whereas it applies exactly as well to closed curves as to open ones, but in both cases, and this is the essential point, to only a small neighborhood of the curve.

Judiciously used by a person who is able to perceive that he does not understand a thing when that is the actual case, the book will prove a source of inspiration.

MAXIME BÔCHER.

Analytische Geometrie auf der Kugel. Von Dr. RICHARD HEGER. Sammlung Schubert, LIV. Leipzig, G. T. Göschen, 1908. 12 mo. vii + 152 pp.

THERE is a spherical trigonometry ; why not also a spherical analytical geometry? This question interested mathematicians towards the end of the eighteenth century and the beginning of the nineteenth, and there resulted numerous papers published in the periodicals of the time. Certain problems had been solved at an earlier date and others have appeared up to within a decade ago. It is the object of the author of the little volume before us to bring this material together in a convenient form and to arrange it along the lines of the usual text upon plane analytic geometry. The book begins by explaining several coordinate systems upon the sphere. Dr. Heger adopts that one in which the homogeneous coordinates of a point are the sines of the angles whose arcs are drawn perpendicularly from the point to the sides of the spherical triangle of reference. This triangle of reference is assumed to be trirectangular. The homogeneous coordinates of a great circle are taken to be the point coordinates of one of its poles. Many of the formulas are exactly the same as the analogous formulas in plane geometry. For instance, the necessary and sufficient condition that a point (x, y, z) lie upon a great circle (u, v, w) is

$$ux + vy + wz = 0.$$

Small circles and conics in general are represented by quadratic equations. There is a theory of poles and polars and of tangents, all of which is analogous to plane analytic geometry.

The book closes with a section devoted to spherical cubics. There are 152 pages, a bibliography, and a short index.

The book will find its greatest use in technical schools. Spherical trigonometry has come to be counted as one of the technical studies and taught in connection with geodetic surveying or with astronomy. It is the same with spherical analytic geometry from Dr. Heger's point of view, and the field of application is narrower.

Geometry upon the sphere is most interesting when studied as a correspondence between the sphere and some other surface — in particular, the plane. This point of view is hinted at in section 8 in explaining Gudermann's axial coordinates, but no general theory of correspondence between sphere and plane is worked out.

Dr. Heger's analytical geometry amounts to a correspondence between the sphere and the projective plane. There is another geometry upon the sphere arising from a one to one correspondence with the plane whose results are quite as useful, in their way, but which does not come within Dr. Heger's field of view.

L. WAYLAND DOWLING.

Die Elemente der Mathematik. Von ÉMILE BOREL, Professor an der Sorbonne zu Paris. Vom Verfasser genehmigte deutsche Ausgabe, besorgt von PAUL STÄCKEL, Professor zu Karlsruhe i. B. Erster Band: *Arithmetik und Algebra.* Mit 57 Textfiguren und 3 Tafeln. Leipzig und Berlin, B. G. Teubner, 1908. xvi + 431 pp.

This work is a German translation, or rather a "Bearbeitung," in one volume, of the three French booklets published by Borel in 1903. Borel traverses in his texts the ground to be covered in arithmetic and algebra by pupils between the ages of 14 and 17, in accordance with the courses of study laid out officially in 1902. The distinctive feature of this movement lies in the emphasis laid on graphic work, on the concept of a variable and of a function. Stäckel says in his preface to the German edition that, in view of the wide divergence of opinion as to what can be accomplished in this line with elementary pupils, the only way of arriving at an understanding and thereby at an actual realization of the contemplated reform, appears to be in showing by an example just what that reform really aims to achieve and how the subject can be developed

with pupils of the ages named. Such an example is furnished in Borel's texts. As Stäckel remarks, this publication is intended only for teachers. Since the reform movement in France and Germany is essentially the same as in the United States, the book under review, coming from authors of distinction, cannot fail to be of interest to American readers.

FLORIAN CAJORI.

A Treatise on the Mathematical Theory of Elasticity. By A. E. H. LOVE, M.A., D.Sc., F.R.S. Second edition. Cambridge University Press, 1906. xvii + 551 pp.

THE first edition of this important work was published in two volumes in 1892 and 1893. The present edition is a new treatise which contains some extracts from the old one. The object of the book is threefold, namely, to be useful to engineers, to set forth the physical notions and analytical processes which are also used in other branches of physics, and to afford a complete picture of the present state of the science of elasticity. The book commences with an excellent historical introduction which explains the parts taken by various eminent mathematicians in establishing the theory. The first four chapters are concerned with the analysis of strain and stress, the equations of equilibrium and small motion, the expression of the stresses as functions of the strains, and the connection between the mathematical theory and technical mechanics. Chapter V opens with a useful recapitulation of the essential parts of the preceding chapters and the author proceeds to illustrate them by a number of simple examples which are needed for the subsequent development of the subject. In Chapter VI there is a discussion of the elastic constants. The 6 components of stress are linear functions of the 6 components of strain and hence depend on 36 constants. The law of conservation of energy reduces the number of constants to 21. The hypotheses of Navier and Cauchy concerning the constitution of matter (according to which bodies are regarded as made up of material points which are supposed to act on each other so that the mutual action between each pair of points is along the line joining them and is a function of the length of the line) leads to 6 relations which are called Cauchy's relations, so that the number of constants is reduced to 15. As the author remarks in the historical introduction, our views concerning the constitution of matter have changed so that the

argument as to whether the number of constants is 21 or 15 holds a subordinate position. The development of the atomic theory in chemistry, of statistical molecular theory in physics, and the discovery of electrical radiation, have shaken our confidence in the hypothesis of central forces between material points. In the case of crystals possessing certain kinds of symmetry the number of constants may have various values from 21 down to 2 for an isotropic body. Chapter VII discusses the uniqueness of a solution and also Betti's reciprocal theorem, which subsequently proves to be of great importance in developing a general method of solution. The various singular solutions analogous to $1/r$ in ordinary potential theory are next obtained, and an interesting application follows, namely, Hertz's theory of the distribution of pressure between two bodies in contact. Chapter X opens with an excellent resumé of the theory of potential which explains the two lines of attack, namely, the method of a series of harmonic functions and the method of singularities or the use of Green's functions. The author here states that little progress has been made with the existence theorem of elasticity. Shortly after the present work appeared however, Tedone published in the Encyklopädie der mathematischen Wissenschaften a valuable account of the memoirs dealing with the existence theorem and remarked on the probability of using Fredholm's integral equations. Lauricella and others have since successfully applied integral equations to the problem and the new book on the Equations of Fredholm by d'Adhémard contains an account of these researches. Professor Love proceeds to give a full account of Betti's method of integration and of Cerruti's application to the problem in which the boundary is a plane. In the next chapter the problem of the sphere is discussed by means of harmonic functions. Tedone has solved the same problem by the use of Green's functions. After discussing the vibrations of solids, the author takes up the important practical problems of the torsion and flexure of beams and regrets the slowness of engineers in adopting the exact methods of solution. The sections of a beam do not usually remain plane, with the curious result that the greatest stress is not at the point furthest from the center. The remainder of the book is devoted to the theory of deformable bodies and is substantially identical with the second volume of the first edition. In the case of thin rods the theory is fairly simple, although a

long analysis is required to justify the approximation that the bending couple is proportional to the curvature. In the cases of plates and shells the necessary analysis is still more lengthy. The problem is complicated by the fact that the bending is usually accompanied by stretching particularly near the edges. The book concludes with an account of the important practical problem of the stability of cylindrical shells. It is perhaps needless to say that the treatise can be heartily commended both as a text-book and a book of reference. A German edition was published by Teubner in 1907.

F. R. SHARPE.

Les Découvertes modernes en Physique. Par O. MANVILLE. 2ème édition. Paris, A. Heimann et Fils, 1909. 463 pp.

ONLY a year after the first edition of Manville's short book of 182 pages on Les découvertes modernes en physique a second edition was needed. The author evidently did not have to contend with costly electrotype plates in which the publisher would allow few changes, for he has practically written a new book about three times the size of the first — the term second edition is really a misnomer. The new work is divided into two parts, entitled Electricité et matière and Les ions et les électrons dans la théorie des phénomènes physiques — La matière et l'éther. This entire rewriting and expansion of the original is very fortunate. The state of fundamental electrical theory is to a considerable extent still speculative, and experiments which reveal new and sometimes nearly crucial results are still of frequent occurrence. To write at all on this subject brings with it the liability and desirability of rewriting after the lapse of a very short period.

From the title of the work we might be inclined to fear that the author had written a popular and unreliable essay on the wonders of recent discoveries. Fortunately this is by no means the case; many chapters contain considerable hard physics and more or less hard mathematics, which require and repay close application on the part of the reader. The presentation, however, let it be stated, deals with a vast variety of interrelated physical data after the manner of the experimental physicists rather than with the broad mathematical groundwork of electrical theories as treated by such theorists as Larmor, Lorentz, or Minkowski. Sooner or later, theory and experiment in regard to atomic electricity will probably be well knit together;

at present the data are too diverse and too numerous to be worked into any mathematical theory, and the different theories are in need of crucial experiments. In addition to these general observations only a few words as to the contents of the work under review need be added.

The author begins with the discussion of the conduction of electricity by liquids and of the ionic theory of electrolysis. He then takes up the corresponding questions for gases; this, of course, requires a much longer treatment, as it involves cathode rays, X-rays, Lénard rays, and electrons. Then follows the electronic theory of matter with a detailed discussion of radioactivity. This ends the first part of the work and is to all intents and purposes merely a new edition of the original volume. In the second part the author goes over the data above presented and applies the facts and points of view acquired to the connected study of the phenomena involved — first, as regards liquids, second, as regards gases, whether ionized or not, and third, as regards solids. The volume finally closes with a discussion of matter and ether.

To write the second part must have been a much harder task than to write the first; for whereas the fundamental facts as to atomic electricity are tolerably firmly established and generally agreed to, the manner in which those facts shall be interpreted and combined into physical theories is by no means so well settled. For instance in the treatment of electric and thermic conductibility in metals there is the theory of J. J. Thomson and the English school of physicists, that of Drude and the German school, and that of Lorentz. None of these theories is as yet entirely satisfactory; the author carefully presents them all and makes a few comments as to their several defects or excellencies. It will thus be seen that we are in possession of a work which will serve probably better than any other one book to orient the student in regard to the modern physics of electricity and to place him in a position to read critically the original memoirs that have appeared or may appear — in short, to put him quickly and easily where he can and must do some real thinking for himself.

E. B. WILSON.

NOTES.

THE sixteenth annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Boston in convocation week, in affiliation with the sixty-second meeting of the American Association for the advancement of science. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary by December 15. Abstracts intended to be printed in the announcement of the meeting must be submitted by December 8.

THE closing (October) number of volume 10 of the *Transactions of the American Mathematical Society* contains the following papers: "The summability of the developments in Bessel functions, with applications," by C. N. MOORE; "Singular points of ordinary linear differential equations," by G. D. BIRKHOFF; "Automorphisms of order two," by G. A. MILLER; "Resolution into involutory substitutions of the transformations of a non-singular bilinear form into itself," by DUNHAM JACKSON; "On singular points in the approximate development of the perturbative function," by F. W. REED. Also "Notes and errata," volumes 8 and 10; Table of contents, volume 10; and Indices by authors and subject matter of volumes 1–10.

THE opening (October) number of volume 11 of the *Annals of Mathematics* contains: "Theory of floating tubes," by FRANK GILMAN; "Generalized geometric means and algebraic equations," by OTTO DUNKEL; "The geometry of chains on a complex line," by J. W. YOUNG.

THE publishing house of Ginn and Company, New York, announces that the following books are in preparation: "Theoretical mechanics," by P. F. SMITH and W. R. LONGLEY; "Projective geometry," by O. VEBLEN and J. W. YOUNG. The treatise on differential geometry by L. P. EISENHART has recently been published.

IN the press of B. G. Teubner are the following books on mathematics: Encyklopädie, parts of II 2, III 1, IV 1, IV 2, V 2, V 3, VI 1, and VI 2; Encyclopédie, parts of I 1–4, II 2–5, III 1, IV 2, IV 4; "Verzeichnis der Schriften Leonhard Eulers," by G. ENESTRÖM (*Jahresbericht*, Ergänzungsband III); "Didaktik des mathematischen Unterrichts," by A. HÖFLER; "Niedere Zahlentheorie, Teil II," by P. BACH-

MANN; "Elemente der Geometrie," by H. THIEME; "Lehre von den geometrischen Verwandtschaften, Band IV," by R. STURM; "Synthetische Theorie der Cliffordschen Parallelen und der linearen Linienörter des elliptischen Raumes," by W. VOGT; "Analytische Geometrie der Punktpaare, Kegelschnitte und Flächen zweiter Ordnung," by O. STAUDE; "Funktionentafeln mit Formeln und Kurven," by E. JAHNKE; "Vektoranalysis und ihre Anwendung in der theoretischen Physik," by W. v. IGNATOWSKY.

THE annual list of American doctorates published in SCIENCE for the academic year 1908–1909 contains 378 names, of which 189 are credited to the sciences. The following 15 successful candidates offered mathematics as major subject (the titles of the theses are appended): H. E. BUCHANAN, Chicago, "Periodic oscillations of three finite masses about the Lagrangian circular solutions"; T. BUCK, Chicago, "Oscillating satellites near the Lagrangian equatorial triangle points"; H. T. BURGESS, Yale, "Point-circle correlations"; J. R. CONNER, Johns Hopkins, "Basic systems of rational norm-curves"; L. S. DEDERICK, Harvard, "Certain singularities of transformations of two real variables"; A. DRESDEN, Chicago, "The second derivatives of the extremal integral"; G. F. GUNDELFINGER, Yale, "On the geometry of line elements in the plane with reference to osculating vertical parabolas and circles"; G. W. HARTWELL, Columbia, "Plane fields of force invariant under projective transformations"; D. D. LEIB, Johns Hopkins, "On a complete system of invariants of two triangles"; J. V. MCKELVEY, Cornell, "The groups of birational transformations of algebraic curves of genus 5"; W. D. MACMILLAN, Chicago, "Periodic orbits about an oblate spheroid"; E. H. TAYLOR, Harvard, "On some problems in conformal mapping"; H. I. THOMSEN, Johns Hopkins, "Some facts in regard to plane rational curves"; M. O. TRIPP, Columbia, "Groups of order p^3q^2"; Miss M. S. WALKER, Yale, "A generalized definition of an improper multiple integral."

The number of American doctorates in mathematics for each of the last eleven years is 13, 11, 18, 8, 7, 14, 21, 11, 13, 22, 15.

ANOTHER professorship of mathematics has just been created at the University of Paris, with the designation of chair of the theory of functions. Professor E. BOREL has accepted the

new position. There are now twelve full professorships of mathematics at the Sorbonne, the present occupants being Professors Andoyer, Appell, Borel, Boussinesq, Darboux, Goursat, Koenigs, Painlevé, Picard, Poincaré, Raffy, Tannery. There are also two associate professors, Hadamard and Puiseux, and one chargé de conférences, Dr. Blutel.

The following advanced courses in mathematics are announced for the academic year 1909–1910:

Cambridge University. (Michaelmas term begins October 14, Lent term January 17, and Easter term April 25.) — By Professor A. R. Forsyth: Differential equations, three hours (M); Differential geometry, three hours (M); Functions of two or more complex variables, three hours (L). — By Professor Sir G. H. Darwin: Lunar theory, three hours (M); Figure of the earth, three hours (L). — By Professor Sir R. S. Ball: Mathematical astronomy, three hours (M); Applications of geometry to dynamics, three hours (L). — By Dr. E. W. Hobson: Harmonic analyses, three hours (M); Higher dynamics, three hours (L); Lebesgue integrals, three hours (E). — By Dr. H. F. Baker: Theory of functions, three hours (M, L); Curves, three hours (M, E). — By Mr. B. A. Herman: Hydromechanics, three hours (M, L). — By Mr. H. W. Richmond: Algebraic and synthetic geometry, three hours (M, L, E). — By Dr. A. N. Whitehead: Synthetic geometry, three hours (M); Philosophy of mathematics, three hours (L, E). — By Dr. T. J. I'A Bromwich: Theory of potential and applications, three hours (M, L). — By Mr. A. Berry: Elliptic functions three hours (L, E). — By Dr. E. W. Barnes: Linear differential equations, three hours (L). — By Mr. C. T. Bennett: Line geometry, three hours (L). — By Mr. A. Munro: Hydrodynamics and sound, three hours (M). — By Mr. G. J. Leathem: Elementary electron-theory, three hours (M). — By Mr. J. H. Grace: Theory of numbers and invariants, three hours (M, L). — By Mr. G. H. Hardy: Integral equations, three hours (E).

At the session of the Paris academy of sciences held on July 19, 1909, the following prizes were awarded: Binoux prize (fr. 2000) to Professor P. Duhem for his contributions to the history of science; Pierron-Perrin prize (fr. 5000) to Professor E. Mercadier, of the Ecole polytechnique, for his work on

acoustics, elasticity and telegraphy; Montyon prize (fr. 700) to M. DE SPARRE, for his work on rational mechanics; Boileau prize (fr. 1300) to Professor BOULANGER, of the University of Lille, for his treatise on general hydraulics; Lalande prize (fr. 540) to M. BORELLY, of the observatory of Marseilles, for his discoveries of small planets and comets; Vals prize (fr. 460) to M. de la BAUME-PLUVINEL for his contributions to astronomy; Pontécoulant prize (fr. 700) to Professor E. W. BROWN, of Yale University, for his researches relative to the theory of the moon.

AMONG the recent contributions to the Euler fund, we mention the subscription for forty sets of the complete works by the academies of science of Berlin and of St. Petersburg, and a cash subsidy of five thousand francs by the latter; a gift of eight thousand francs by the city of Zürich, and the appropriation of five thousand francs by the American Mathematical Society. At the beginning of September the number of sets subscribed for was 275. In view of these facts, the Swiss scientific society, at its ninety-second annual meeting at Lausanne, September 5–8, 1909, unanimously voted to proceed with the publication of the complete works of Leonhard Euler. The preparation of the different volumes will be assigned to one or more specialists in the respective fields; about twenty have already signified their willingness to serve. A general committee of three will direct the whole undertaking.

THE French association for the advancement of science held its thirty-eighth annual meeting at Lille, August 2–7, under the presidency of Professor Landouzy, dean of the faculty of medicine of the University of Paris, with Professor E. LEBON, chairman of the section of mathematics and astronomy. The following papers were read: "Biography of Poincaré," by E. LEBON; "Modular geometry," by G. TARRY; "Fredholm's equations," by H. POINCARÉ; "Instruction in mathematics," by C. A. LAISANT; "Geometric representation," by N. CHAPELON; "Fermat's theorem," by A. GÉRARDIN; "Plane representation of paraboloids," by F. MICHEL; "Hyperelliptic surfaces," by E. TRAYNART; "Theory of integrals of algebraic functions of several variables," by L. AMOROSO; "The method of indeterminants and the solution of functional equations," by M. CARLIER; "On the theory of curves," by A. PELLET; and twenty papers on astronomy. The next meet-

ing of the association will be held at Toulouse during the summer of 1910, under the presidency of Professor C. M. Gariel, of the University of Paris, the chairman of the section of mathematics being M. Belot.

The third annual meeting of the Italian association for the advancement of science was held at Padua, September 20 to 25, under the presidency of Professor V. Volterra. A notable feature of the meeting was the prominence given to general papers read before joint sections, among which was the address given by Professor F. Severi, on "Hypotheses and reality in geometric science." The following papers were read before the section of pure and applied mathematics. By A. Alessio, "On the reduction of observations for the determination of relative gravity to a rigid bearing of the pendulum"; by U. Cisotti, "Maxwell's stresses and elastic media"; by A. Crocco, "Aërial navigation"; by G. Gallucci, "On irregular configurations N_3"; by T. Levi-Civita, M. Abraham and M. O. Corbino, "On the constitution of electric radiation"; by E. Pascal, "Mechanical integration of differential equations"; by G. Ricci, "On the determination of three-dimensional varieties having prescribed intrinsic properties."

The Italian mathematical society Mathesis, recently reorganized to consider both scientific and pedagogic questions, held its second annual meeting at Padua, September 20–23, under the presidency of Professor F. Severi. The opening address was given by Professor G. Loria on the existing crisis in the secondary schools. After a symposium on proposed reforms and on more systematic preparation of teachers in which a large number participated, Professor Berzolari and Bonola proposed the organization and publication of an encyclopedia of elementary mathematics in Italian. Professor Castelnuovo reported on the work of the international commission on mathematical instruction. The society's prize of 900 francs for the best essay on some elementary subject was awarded to M. Padoa for his essay on the theory of fractions.

Professor J. Petersen, of the University of Copenhagen, has retired from active teaching. Professor N. Nielsen has been promoted to a full professorship and to be head of the department of mathematics.

Dr. H. Hahn, of the University of Vienna, has been appointed associate professor of mathematics at the University of Czernowitz.

PROFESSOR E. NAETSCH, of the technical school at Dresden, has been promoted to a full professorship of mathematics.

THE gold medal of the French association for the advancement of science, for the present year, has been awarded to Professor H. POINCARÉ, of the University of Paris.

DR. J. O. MÜLLER has been appointed docent in mathematics at the University of Bonn.

AT the University of Cambridge Dr. T. J. I'A. BROMWICH has been appointed university lecturer in mathemathics.

PROFESSOR G. A. GIBSON, of the Glasgow and West of Scotland Technical College, has accepted the professorship of mathematics at the University of Glasgow.

PROFESSOR G. A. MAGGI, of the University of Pisa, has been elected corresponding member of the mathematical society of Charkow.

AT the University of Chicago, Professor O. BOLZA has returned from a year's leave of absence and will resume his duties commencing with the autumn quarter; Professor L. E. DICKSON has been granted leave of absence, and will spend the present year in Europe.

IN connection with the celebration of the twentieth anniversary of the founding of Clark University the honorary degree of doctor of laws was conferred upon Professor W. F. OSGOOD, of Harvard University, Professor J. PIERPONT, of Yale University, and Professor E. B. VAN VLECK, of the University of Wisconsin; the degree of doctor of mathematics was conferred upon Professor E. H. MOORE, of the University of Chicago.

AT Northwestern University Professor D. R. CURTISS has been promoted to a full professorship of mathematics. Mr. R. E. WILSON and Dr. J. C. MOREHEAD have been promoted to assistant professorships of mathematics.

PROFESSOR G. W. HARTWELL, of the University of Kansas, has accepted the professorship of mathematics at Hamline University, St. Paul, Minnesota.

PROFESSOR E. L. HANCOCK, of Purdue University, has been appointed professor of applied mechanics at the Worcester Polytechnic Institute.

Professor Charles Haseman, of the University of Indiana, has accepted an associate professorship of mathematics at the University of Nevada, Reno, Nevada.

Professor C. Gundersen, of the Michigan Agricultural College, has accepted the professorship of mathematics at the Agricultural College of Oklahoma, Stillwater Oklahoma.

Professor O. T. Geckler, of the Georgia School of Technology, has been appointed professor of mathematics at Whitman College, Walla Walla, Washington.

Professor A. B. Frizell, of Midland College, and Miss W. Bauer have been appointed instructors in mathematics at the University of Kansas.

Dr. H. F. Burgess has been appointed instructor in mathematics at the University of Wisconsin.

Mr. E. F. A. Carey has been appointed instructor in mathematics at the University of Montana.

Mr. J. C. Rayworth has been appointed instructor in mathematics at Washington University, St. Louis.

Dr. G. F. Gundelfinger has been appointed instructor in mathematics at the Sheffield scientific school of Yale University.

Dr. L. S. Dederick has been appointed instructor in mathematics at Princeton University.

At the University of Illinois, Dr. R. L. Börger has been promoted to the position of associate in mathematics. Mr. G. H. Scott has been appointed assistant in mathematics.

Mr. J. H. Kindle and Mr. J. R. Trimble have been appointed instructors in mathematics at the University of Cincinnati.

Mr. G. H. Palmer has been appointed instructor in mathematics and graphics at Rutgers College.

Mr. F. C. Eaton has been appointed instructor in mathematics at the University of Iowa.

Mr. R. K. Williams has been appointed instructor in mathematics at Indiana University.

AT the University of Maine Mr. S. D. CHAMBERS and Mr. T. L. HAMLIN have been appointed instructors in mathematics.

AT the Western Reserve University, Cleveland, Ohio, Professor C. J. SMITH has been granted leave of absence for the current year. Mr. A. H. FORD has been appointed instructor in mathematics.

PROFESSOR C. REUSCHLE, of the technical school at Stuttgart, died August 17, at the age of 62 years.

PROFESSOR IRVING STRINGHAM, dean of the University of California since 1886, professor of mathematics since 1882, and acting president during the current year, died on October 5, 1909, at the age of 52 years. He was graduated at Harvard in 1877, and took the degree of doctor of philosophy at Johns Hopkins in 1880. Professor Stringham became a member of the American Mathematical Society in 1891, and was a member of the Council in 1902–1905 and Vice-President of the Society in 1906.

CATALOGUES of second-hand books: Bowes and Bowes, 1 Trinity St., Cambridge, England, catalogue No. 326, containing 2576 titles in pure and applied mathematics. — W. Heffer and Sons, Cambridge, catalogue No. 53, containing titles 287–1111 on mathematics and physics. — Meier and Ehrat, Untere Bahnhofstrasse 94, Zurich, Switzerland, catalogue No. 300, about 650 titles in mathematics. — Martin Breslauer, Unter den Linden 16, Berlin W. 64, Germany, catalogue No. 1, alte Rechenbücher, etc., 29 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

DAIBER (E.). Die Formänderung rechter Winkel. (Diss.) Stuttgart, 1909. 8vo. 78 pp.

DIENES (P.). Essai sur les singularités des fonctions analytiques. (Thèse.) Paris, Gauthier-Villars, 1909. 4to. 91 pp.

FINE (H. B.). and THOMPSON (H. D.). Coördinate geometry. New York, Macmillan, 1909. 8vo. 8 + 300 pp. Cloth. $1.60

FRANKENBACH (F. W.). Lineare Erzeugung der Kegelschnitte und auf ihr beruhende Ableitung der Kegelschnittsgleichungen. Ein Beitrag zur Lehre von den Kurven 2ter Ordnung zum Gebrauch an Realanstalten. Liegnitz, Krumbhaar, 1909. 8vo. 49 pp. M. 1.00

GRASSMANN (H.). Projektive Geometrie der Ebene, unter Benützung der Punktrechnung dargestellt. Erster Band: Binäres. Leipzig, Teubner, 1909. 8vo. 12 + 360 pp. Cloth. M. 13.00

HÖEGH (E.). Elementarer Beweis des Fermat'schen Satzes. Rostock, Boldt, 1909. 8vo. 8pp. M. 1.50

MURER (V.). Introduzione alla teoria dei numeri, con numerosi esercizi e con notizie storiche. Livorno, Giusti, 1909. 16mo. 6 + 123 pp. L. 1.00

PERRON (O.). Ueber eine Verallgemeinerung des Stolzschen Irrationalitätssatzes. München, 1909. 8vo. 18 pp.

PRINZHORN (H.). Der Fermatsche Satz und sein Beweis. Magdeburg, Peters, 1909. 8vo. 11 pp. M. 1.00

SCHANZ (J.). Der Aufbau des komplexen Zahlengebiets auf der natürlichen Zahlenreihe. (Progr.) Berlin, 1909. 8vo. 31 pp.

SPEISER (A.). Die Theorie der binären quadratischen Formen mit Koefficienten und Unbestimmten in einem beliebigen Zahlenkörper. (Diss.) Göttingen, 1909.

THOMPSON (H. D.). See FINE (H. B.).

WETTERNIK (J.). Divergente Reihen und deren Anwendung auf lineare Differenzialgleichungen. (Progr.) Wien, 1908. 8vo. 16 pp.

ZIEMKE (E.) Ueber partielle Differentialgleichungen erster Ordnung mit Integralvereinen, die als Punktmannigfaltigkeiten zweifach ausgedehnt sind. (Diss., Greifswald.) Leipzig, 1909.

II. ELEMENTARY MATHEMATICS.

BALSER (L.). See NELL (A. M.).

BAUDOIN (P.). Cahiers d'exécution de dessins géométriques d'après le cours abrégé de géométrie de M. Carlo Bourlet. 1er cahier. Classes de 6e, 5e et 4e B. Paris, Hachette, 1909. 8vo. 48 pp. Fr. 1.50

BAUR (L.). Lehr- und Uebungsbuch der allgemeinen Arithmetik und Algebra. Resultate. 2te Auflage. Stuttgart, Bonz, 1909. 8vo. 60pp. Boards. M. 1.60

BOYMAN (J. R.). Lehrbuch der Mathematik für Gymnasien, Realschulen und andere höhere Lehranstalten. Teil II: Geometrie der Ebene, ebene Trigonometrie und Geometrie des Raumes. Düsseldorf, Schwann, 1909. 8vo. 12 + 520 pp. Cloth. M. 4.80

BÜTZBERGER (F.). Lehrbuch der ebenen Trigonometrie mit vielen Aufgaben und Anwendungen für Gymnasien, Seminarien und technische Mittelschulen, sowie zum Selbstunterricht. 4te verbesserte und vermehrte Auflage. Zürich, Füssli, 1909. 8vo. 12 + 84 pp. Cloth. M. 2.00

CAPPILLERI (K.). Die Einführung der Infinitesimalrechnung in der Realschule. (Progr.) Wien, 1908. 8vo. 33 pp.

CRATHORNE (A. R.). See RIETZ (H. L.).

EDERT (R.) und KRÖGER (M.). Geometrie für Mittelschulen und verwandte Anstalten. Heft 1. Vorkursus und Planimetrie. 2te unveränderte Ausgabe. Hannover, Meyer, 1909. 8vo. 7 + 91 pp. Boards. M. 1.20

FERVAL (H.). Cours de géométrie cotée à l'usage des candidats à l'Ecole spéciale militaire de Saint-Cyr. 3e édition. Paris, Belin frères, 1909. 8vo. 248 pp. Fr. 4.00

GODFREY (C.) and SIDDONS (A. W.). Solutions of the exercises in modern geometry. Cambridge, University Press, 1909. 8vo. 124 pp. 4 s.

HARTL (H.). Lehrbuch der Planimetrie. Für den Unterrichtsgebrauch und für das Selbststudium verfasst. 2te verbesserte Auflage. Wien, Deuticke, 1909. 8vo. 5 + 141 pp. Cloth. M. 2.40

HEILERMANN und DIEKERMANN's Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Neu bearbeitet von K. Knops. Essen, Baedeker, 1909. 8vo. 8 + 265 pp. Cloth. M. 2.80

KARST (L.). Lineare Funktionen und Gleichungen. (Progr.) Lichtenberg bei Berlin, 1909. 4to. 44 pp.

KOPPE und DIEKMANN's Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Teil III. Die Stereometrie, der Koordinatenbegriff, die Kegelschnitte. 3te Auflage, bearbeitet von K. Knops. Essen, Baedeker, 1909. 8vo. 4 + 168 pp. Cloth. M. 2.80

KRÖGER (M.). See EDERT (R.).

MAHLERT (A.). See MÜLLER (H.).

MILNE (W. J.). Key to standard algebra. New York, American Book Co., 1909. 12mo. 432 pp. Cloth. $1.00

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch für das Lyzeum. Fortsetzung des mathematischen Lehr- und Uebungsbuches für höhere Mädchenschulen. 2ter Teil. Leipzig, Teubner, 1909. 8vo. 8 + 339 pp. Cloth. M. 4.40

NELL (A. M.). Fünfstellige Logarithmen der Zahlen und der trigonometrischen Functionen. 13te Auflage von L. Balser. Giessen, Roth, 1909. 8vo. 6 + 84 pp. Cloth. M. 2.00

OTTO (F.) und SIEMON (P.). Lehr- und Uebungsbuch der Geometrie für zehnklassige höhere Mädchenschulen. Leipzig, Hirt, 1909. 8vo. 144 pp. Cloth. M. 1.50

PIERPONT (A. E.). The elements of geometry in theory and practice. Based on the report of the committee appointed by the mathematical association. London, Longmans, 1909. 8vo. 3 s.

RIETZ (H. L.) and CRATHORNE (A. R.). College Algebra. New York, Holt, 1909. 12mo. $1.40

SIDDONS (A. W.). See GODFREY (C.).

SIEMON (P.). See OTTO (F.).

VALLÍN Y BUSTILLO (A. F.). Geometría para los niños. Madrid, Hernando, 1909. 8vo. 122 pp. P. 1.25

III. APPLIED MATHEMATICS.

BLOMFIELD (C. H.). See JONES (A. C.).

GASCUE (F.). Curso elemental de mecánica y construcción. 3a edición. 2 vols. Sama de Langreo, 1909. 8vo. 15 + 333 pp. P. 11.00

GRIMSEHL (E.). Lehrbuch der Physik. Zum Gebrauch beim Unterricht, bei akademischen Vorlesungen und zum Selbststudium. Mit 1091 Figuren im Text, 2 farbigen Tafeln und einem Anhange, enthaltend Tabellen physikalischer Konstanten und Zahlentabellen. Leipzig, Teubner, 1909. M. 16.00

JONES (A. C.) and BLOMFIELD (C. H.). Elementary mechanics of solids and fluids. London, Arnold, 1909. 8vo. 388 pp. 4s. 6d.

LEBLANC (H.). Les mécanismes. Traité élémentaire de cinématique appliquée. 2e édition. Paris, Garnier frères, 1909. 18mo. 672 pp.

MANVILLE (O.). Les découvertes modernes en physique. Première partie. Electricité et matière. Deuxième partie, les ions et les électrons dans la théorie des phénomènes physiques ; la matière et l'éther. Deuxième édition revue et augmentée. Avec 65 figures dans le texte. Paris, Hermann, 1909. 8vo. 463 pp. Fr. 8.00

MARTINI (Z. A.). Teoria matematica del conto corrente e sue applicazioni. Livorno, Giusti, 1909. 16mo. 59 pp. L. 0.50

MARVÁ Y MAYER (J.). Mecánica aplicada á las construcciones. 2 vols. 4a edición. Madrid, Palacio, 1909. 8vo. 2042 pp. P. 40.00

MÖLLER (J.). Nautik. Mit 58 Figuren im Text und auf einer Tafel. Leipzig, Teubner, 1909. M. 1 25

PILKINGTON (W.). Coördinate geometry applied to land-surveying. London, Spon, 1909. 12mo. 44 pp. 1s. 6d.

TESAR (L.). Die Mechanik. Eine Einführung mit einem metaphysischen Nachwort. Mit 111 Figuren. Leipzig, Teubner, 1909.

VERGNE (H.). Contribution à la théorie des ondes liquides. (Thèse.) Paris, Gauthier-Villars, 1909. 4to. 85 pp.

WILSON (V. T.). Descriptive geometry : a treatise from a mathematical standpoint, together with a collection of exercises and practical applications. New York, Wiley, 1909. 8vo. Cloth. $1.50

THE PRINCETON COLLOQUIUM.

THE Sixth Colloquium of the American Mathematical Society was held, at the close of the sixteenth summer meeting, at Princeton University, Princeton, N. J., opening on Wednesday morning, September 15, 1909, and extending until the following Friday.* At the April meeting, 1908, the Council appointed a committee consisting of Professors H. B. Fine, W. F. Osgood, T. F. Holgate and F. N. Cole to arrange for the colloquium. A preliminary circular announcing the general features was issued in May of 1909. The colloquium opened on Wednesday morning, September 15, 1909, in the lecture room of McCosh Hall, Princeton University, the following 28 persons being in attendance:

Professor G. D. Birkhoff, Professor G. A. Bliss, Mr. R. D. Carmichael, Dr. A. B. Chace, Dr. A. Cohen, Dr. G. M. Conwell, Dr. L. S. Dederick, Professor L. P. Eisenhart, Professor T. C. Esty, Professor H. B. Fine, Dr. G. F. Gundelfinger, Dr. Frank Irwin, Professor Edward Kasner, Mr. A. K. Krause, Professor W. R. Longley, Mr. H. H. Mitchell, Professor J. H. Maclagan-Wedderburn, Mr. H. F. MacNeish, Professor E. H. Moore, Professor Frank Morley, Professor G. D. Olds, Mr. W. J. Risley, Professor Virgil Snyder, Mr. C. E. Van Orstrand, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. E. Wright.

Two courses of lectures were given, as follows:

I. Professor G. A. BLISS: "Fundamental existence theorems." Four lectures.

II. Professor EDWARD KASNER: "Geometric aspects of dynamics." Four lectures.

Two lectures were given each morning, one on Wednesday afternoon, and one at noon on Friday, the lecturers alternating. Printed syllabi of both courses had been issued in advance, as usual. Thursday afternoon was devoted to an excursion to Washington's Headquarters, carriages and automobiles being provided by Dean Fine and Mr. Pyne, of the University. The evenings were spent in social conference at the Princeton

* For the history of the preceding colloquia, see the Report of the New Haven Colloquium, BULLETIN, vol. 13 (1906-07), page 71, where complete references are given.

Inn, the general headquarters of the meeting. The hospitality of Princeton University, and particularly of the mathematical department, was gratefully acknowledged by a vote of thanks at the closing meeting.

The following abstracts of the lectures convey a general idea of their content. More detailed reports by the lecturers will appear in later numbers of the BULLETIN.

I. The earlier part of Professor Bliss's course was devoted to a review of the theory of implicit functions, including a detailed account of some of the recent developments in the subject, with their applications in the calculus of variations. The existence theorems for ordinary differential equations were taken up with special reference to the definition of solutions over an extended region and their behavior as functions of the initial constants. A short account of the geometrical theory of partial differential equations of the first order was given, and with this as a guide the results already obtained were applied to show the existence of solutions of such equations, even when the function defining the equation is not analytic. The theory of implicit functions for real variables and some knowledge of the methods of approximation of Cauchy and Picard for ordinary differential equations were presupposed.

II. All physical phenomena take place in space and may therefore suggest geometric investigation. In this connection the attention given to kinetics has been slight in comparison with that devoted to statics and kinematics. Professor Kasner's lectures dealt with geometric aspects of kinetics. Such topics as the following were treated:

Conservative systems, the principle of least action, reduction to geodesics, Thomson's theorem, natural families, conformal transformations (Larmor, Goursat, Darboux). General fields of force, trajectories, geometric explorations. Projective properties and Appell's transformation. Interrelations of catenaries, brachistochrones, and tautochrones. Representation of time (Minkowski); representation of phase in statistical mechanics. Transformations of time, contact transformations. Problem of several bodies. Non-holonome systems and Hertz's geometry of material systems. Optics and elasticity.

VIRGIL SNYDER.

THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

The sixteenth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California, Saturday, September 25, 1909. The following members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor A. O. Leuschner, Professor H. C. Moreno, Professor C. A. Noble, Mr. E. W. Ponzer, Mr. H. W. Stager, Professor A. L. Whitney.

The following officers were elected for the ensuing year: Professor Blichfeldt, Chairman; Professor Noble, Secretary; Professors Hoskins, Lehmer, Noble, Program Committee. The next two meetings are to be held respectively at Stanford University February 26, 1910, and at the University of California September 24, 1910.

The following papers were read at this meeting:

(1) Professor D. N. Lehmer: "On the arithmetical theory of pencils of binary quadratic forms" (preliminary communication).

(2) Professor H. F. Blichfeldt: "On the infinitude of primes in certain arithmetical progressions" (preliminary communication).

(3) Mr. G. F. McEwen: "The motion of a viscous fluid between two parallel planes" (preliminary communication).

(4) Professor L. M. Hoskins: "The strain of a gravitating compressible elastic sphere."

(5) Professor A. O. Leuschner: "An equation giving the geocentric distance in the problem of determining parabolic orbits from geocentric observations."

Mr. McEwen was introduced by Professor Blichfeldt.

Abstracts of the papers are given below in order as numbered in the foregoing list:

1. Professor Lehmer gives a study of the arithmetical properties of the forms included in the pencil $\alpha A + \beta B$, where α and β are integers and A and B binary quadratic forms. The

forms of the pencil must have determinants representable by a certain binary quadratic form $H = Dx^2 + \Theta xy + D'y^2$, where D and D' are the determinants of A and B and Θ is the joint invariant. If $A = (abc)$, $B = (a'b'c')$, the form $J = (ab' - a'b)x^2 + (ac' - a'c)xy + (bc' - b'c)y^2$ is also of fundamental importance in the theory. A pencil may be found having a given form J. A pencil may or may not be found having a given form H, according as H is or is not of the principal genus. The form H is the duplicate of J if J is a primitive form.

2. By elementary algebraic processes, involving the approximate evaluation of certain factorials, Professor Blichfeldt proved that the arithmetical progressions k, $11 + k$, $22 + k$, $33 + k$, $\cdots$ (k prime to 11) contain an infinite number of primes each.

3. Mr. McEwen's paper is in abstract as follows: A viscous fluid is confined between two parallel planes, one being fixed, the other having a displacement in its own plane,

$$x_1 = ae^{-at} \sin \sigma t.$$

Assuming the distance between the planes to be great, the displacement of the fluid at the distance y from the moving plane is

$$x_2 = ae^{-at-\beta y} \sin (\sigma t - \beta y),$$

where $\beta = \sqrt{\sigma\rho/2\mu}$, ρ = density of the fluid, μ = the coefficient of viscosity of the fluid.

A gravity pendulum is hung so that a small plane attached to its lower end is parallel to the plane of vibration of the pendulum and the fluid in which it is immersed. x_3 is the displacement of this plane. x_1', x_2', and x_3' are the maximum values of x_1, x_2, and x_3.

$$\frac{x_2'}{x_1'} = e^{-\beta\left(1+\frac{\alpha}{\sigma}\right)y} \quad \text{and} \quad \frac{x_3'}{x_1'} = \left(1 - \frac{\alpha^2}{2\sigma^2}\right) e^{-\beta\left(1+\frac{\alpha}{\sigma}\right)y},$$

if $x_3 = ae^{-\alpha t} \sin \sigma t$ when the plane is not in the fluid and if α/σ is small.

4. In Kelvin's well-known solution of the problem of the strain of an elastic sphere, the bodily forces are assumed to be known functions of the coordinates of position. When self-

gravitation is considered this solution is inapplicable, except in the case of incompressibility, because the force of attraction acting upon any volume element depends in part upon the change of density produced in that element by the strain and upon the change of density distribution of the attracting mass. A solution of the problem taking account of the actual gravitational forces in the strained configuration is given in Professor Hoskins's paper. The problem is worked out completely for the case in which the strain is due to disturbing forces of the type of tidal or centrifugal forces, and numerical results have been obtained corresponding to several different values of the ratio of the elastic constants. The strain at any distance from the center being specified by two quantities — the ellipticity of the originally spherical surface and the angular displacement of a radius vector inclined 45° to the axis of symmetry — it is found that for a given value of the rigidity modulus, the former of these quantities is decreased and the latter increased by compressibility. The solution has also been generalized so as to apply to the case in which the potential of the disturbing forces is any spherical harmonic of degree not less than 2.

5. In his adaptation of the "short method of determining orbits" to the direct computation of a parabola for comets, Professor Leuschner derives the geocentric distance ρ at a fixed date from the equation

$$(z-p')^2 - \frac{n}{[(z-c)^2+s^2]^{\frac{1}{2}}} - q'^2 = 0,$$

where $z=\rho/R$, and R is the distance of the sun. p', n, q'^2, c, and s are auxiliary quantities depending on observed coordinates and other data. c and s are the sine and cosine of the angle ψ subtended at the observer by the arc between the comet and the sun.

The solutions are given by the intersections of the parabola $y=z'^2$ and the curve $y=n/\sqrt{[z'-c']^2+s^2}-q'^2$ for which z' is real and positive, where $z'=z-p'$ and $c'=c-p'$.

There will be either one or three solutions. Three solutions exist, if

either

$$p'>0, \quad c>0;$$

or

$$p'>0, \quad c<0, \quad 90° < \psi < 125° \ 16';$$

or

$$p' < 0, \quad c > 0, \qquad 0^\circ < \psi < 54^\circ\ 44',$$

and if

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0,$$

where

$$\frac{p}{3} = \frac{1}{81}\left[9(2s^2 + q'^2) - 7c'^2\right]; \quad \frac{q}{2} = \frac{5c'}{9}\left[\frac{p}{3} + \frac{1}{9}\left(m^2 - \frac{11}{10}q'^2\right)\right]$$

and

$$m^2 = c'^2 + s^2.$$

For solution the equation in z is written in the form

$$y' = (\zeta + c')^2 - (\eta - q'^2) = f(\vartheta) = 0,$$

where

$$\zeta = s \tan \vartheta; \quad \eta = \frac{n}{s}\cos \vartheta.$$

A convenient graphical solution is proposed for the solution of $f(\vartheta) = 0$. Then

$$\rho / R = z = s \tan \vartheta + c.$$

Geocentric distances correct to four or five decimals result from the graphical solution. Further decimals may be obtained by a simple differential correction.

In practice no case with three solutions has been encountered.

C. A. Noble,
Secretary of the Section.

THE WINNIPEG MEETING OF THE BRITISH ASSOCIATION.

The seventy-ninth annual meeting of the British Association for the advancement of science was held in Winnipeg August 25 to September 1. Fourteen hundred members and associates were in attendance. The opening event was the address of the President of the Association, Sir J. J. Thomson, on Wednesday evening, August 25, in which he gave an account of some of the more recent developments in physics and in his opening remarks took occasion to urge a closer union between mathematics and physics and to emphasize the advantages of

research in developing the qualities of a student. The following morning the sectional meetings opened with the addresses of the presidents of the various sections. In Section A President Rutherford took as his subject the present position of the atomic theory in physical science. The sectional addresses, as also a more general account of the meeting of the Association, will be found in *Nature* and in *Science;* for abstracts of the various papers presented the reader is referred to the annual Report of the proceedings of the Association. Section A covers the mathematical and physical sciences and the Winnipeg meeting of the section was a particularly successful one, in which however the subject of physics was dominant. There were six sessions in physics with crowded programmes and a large number in attendance. Cosmical physics, included in the Section for convenience, occupied several separate sessions. One session sufficed for the presentation of the papers in pure mathematics, which were the following:

(1) E. H. Moore: "Theorems in general analysis."

(2) E. W. Hobson: "On the present position of the theory of aggregates."

(3) G. A. Miller: "Generalizations of the icosahedral group."

(4) G. A. Bliss: "A new proof of Weierstrass's factor theorem."

(5) J. H. Grace: "On ideal numbers."

(6) P. A. MacMahon: "On a correspondence in the theory of the partition of numbers."

(7) W. H. Metzler: "On a continuant expressed as the product of linear factors."

(8) E. W. Davis: "Imaginary geometry of the conic."

(9) F. Cajori: "On the invention of the slide rule."

(10) J. W. Nicholson: "The asymptotic expansion of Legendre functions."

(11) Report on the calculation of Bessel functions.

1. Professor Moore started out from the statement of the fundamental principle: "The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features." The speaker emphasized the importance of such a general theory for the subject of integral equations. He also considered a number of impor-

tant analytic systems included under a certain general type, with reference to the properties which they had in common.

2. Professor Hobson recalled various points which have been raised in recent controversies relative to the theory of aggregates, and advocated the adoption of a definition of an aggregate of more restricted character than that of G. Cantor.

3. The general groups treated in Professor Miller's paper are those in which the two generating operators t_1, t_2 satisfy one of the three sets of conditions

$$t_1^{\,2} = t_2^{\,5}, \quad (t_1t_2)^3 = (t_2t_1)^3; \quad t_1^{\,2} = t_2^{\,3}, \quad (t_1t_2)^5 = (t_2t_1)^5;$$
$$t_1^{\,3} = t_2^{\,5}, \quad (t_1t_2)^2 = (t_2t_1)^2.$$

Among others Professor Miller established the following important theorems: There are an infinite number of groups each of which may be generated by two operators satisfying one of these pairs of conditions. Each of the possible groups generated by t_1, t_2 contains either the icosahedral group or the group of order 120 which is unsolvable and does not contain a subgroup of order 60, and it must have one of these groups for its commutator subgroup.

4. The theorem of Weierstrass for which Professor Bliss has found a simplified proof is the following: Any convergent series in $p+1$ variables $F(x_1, x_2, \cdots, x_p, y)$ in which the lowest term of y alone is of degree n, can be expressed as a product

$$(y^n + a_1y^{n-1} + \cdots + a_{n-1}y + a_n)\phi(x_1, \cdots, x_p, y),$$

where $a_1, \cdots, a_n$ are convergent series in $x_1, \cdots, x_p$ which vanish with these arguments, while ϕ is a convergent series with a constant term different from zero, in all $p+1$ variables.

5. Dr. Grace's paper treated of the ideals in a field defined by the root of a quadratic equation.

7. Professor Metzler's theorem includes as special cases theorems due to Painvin, Sylvester, and Cayley, and may be stated thus:

$$|x_{i,k}| = \{r + na(\alpha - \beta)\}\{r + na(\alpha - \beta) - 2a\alpha + a\beta\}$$
$$\{r + na(\alpha - \beta) - 4a\alpha + 2a\beta\} \cdots \{r - na\alpha\},$$

where the determinant is of order $n + 1$ and a continuant in which the laws of the elements in the three diagonals are the following :

$$x_{i,i-1} = (n - i + 2)a(\alpha - \beta), \quad x_{i,i} = r - (i - 1)a\beta, \quad x_{i,i+1} = ia\alpha.$$

8. Professor Davis gave a complete representation of the elements of the central conic whose axes are non-similar complex quantities.

9. The object of Professor Cajori's paper was to collect all available data bearing on the invention of the slide rule and to decide between the rival claims of Gunter, Wingate, and Oughtred. He concluded from the evidence that the slide rule was invented by William Oughtred, agreeing in this with Augustus DeMorgan although most writers of the present time attribute the invention to Edmund Wingate. Much of the literature bearing on the subject and consulted by Professor Cajori was not accessible to DeMorgan. The rest of the papers were read by title.

A very pleasant feature in connection with Section A was a smoker held on Tuesday evening, which brought the members of the section into closer personal contact with one another. In a general account of a meeting of the British Association the social and semi-popular scientific functions must not be forgotten, for these have their distinct values in the purposes of the organization. There were several well attended semi-popular scientific lectures in the evenings and quite a number of garden parties and receptions, both private and official, were given in honor of the members of the Association.

At the conclusion of the meeting the officials of the Association together with a number of invited guests, in all a party of nearly two hundred, made an excursion to the Pacific coast over the main line of the Canadian Pacific Railway. Short stops were made at the principal towns en route and at a number of the mountain resorts, and Victoria was reached by boat from Vancouver. The return trip was made by way of Edmonton, from which city the special train was run over the Canadian Northern Railway back to Winnipeg.

The meeting of the British Association next year will be held in Sheffield, England.

J. C. FIELDS.

THE SALZBURG MEETING OF THE DEUTSCHE MATHEMATIKER VEREINIGUNG.*

THE twentieth annual meeting of the Deutsche Mathematiker-Vereinigung was held at Salzburg, Austria, September 19–25, 1909, in affiliation with the eighty-first convention of the Society of German naturalists and physicians.

An informal reception was held in the Kurhaus on the evening of September 19, and the first general session of all the sections and affiliated societies took place in the Aula the following morning.

The five sessions of the Vereinigung were held at the Realschule. The following papers were read:

(1) F. ENGEL, Greifswald, "Hermann Grassmann."

(2) E. WAELSCH, Brünn, "Applications of the theory of invariants of binary forms."

(3) G. PICK, Prague, "Differential equations of the periods of hyperelliptic functions."

(4) Miss E. NOETHER, Erlangen, "On the theory of invariants of forms of n variables."

(5) G. KOHN, Vienna, "On a group of theorems of projective geometry."

(6) R. ROTHE, Clausthal, "On the theory of isothermal surfaces."

(7) R. MÜLLER, Darmstadt, "On the instantaneous motion of similarly varying plane systems."

(8) R. MEHMKE, Stuttgart, "Contributions to the kinematics of rigid spatial systems, and those varying in affinity."

(9) M. GRÜBLER, Dresden, "The criterion for constrained motion of a system of screws."

(10) E. SALKOWSKI, Charlottenburg, "On a remarkable class of deformation surfaces of hyperboloids of revolution."

(11) H. WIENER, Darmstadt, "Application of kinematic considerations to the construction of curves."

(12) E. STÜBLER, Stuttgart, "The systems of accelerations in the motion of a rigid body."

(13) H. WIENER, Darmstadt, "On some new kinematic and geometric models."

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* Translated from Dr. Dintzl's manuscript by Professor VIRGIL SNYDER.

(14) R. Skutsch, Dortmund, "Explanation of some demonstration apparatus."

(15) H. Jung, Hamburg, "Report on the theory of algebraic functions of two variables."

(16) S. Günther, Munich, "Mathematical and physical geography of Leonhard Euler."

(17) E. Hoppe, Hamburg, "The sexagesimal system and the division of the circle of the Babylonians."

(18) L. G. du Pasquier, Zürich, "Concerning the question of the most suitable base for a system of numbers."

(19) W. Velton, Kreuznach, "The development of elliptic functions."

(20) H. Hahn, Vienna, "Report on the theory of linear integral equations."

(21) A. Korn, Munich, "Solution of the linear integral equation by the method of successive approximations."

(22) W. Wirtinger, Vienna, "Concerning conformal representation by means of abelian integrals."

(23) O. Perron, Munich, "On the behavior of the integrals of linear difference equations at infinity."

(24) L. G. du Pasquier, Zürich, "On integral tettarions."

(25) E. Timerding, Strassburg, "Kinematic models."

(26) R. Mehmke, Stuttgart, "The calculating machine 'Euclid.'"

(27) E. Papperitz, Freiberg, "The kinodiaphragmatic projection, a new method of geometric representation."

(28) F. Engel, Greifswald, "On a family of curves invariant to a given differential expression."

(29) E. Müller, Vienna, "Suggestions for shaping a course in descriptive geometry in technical schools and universities."

(30) E. Czuber, Vienna, "The measure of mortality."

With the exception of numbers 3 and 6, abstracts of the papers are given below, the numbers corresponding to those in the list above.

1. Professor Engel gave a brief sketch of the mathematical development of Grassmann, which is all the more remarkable since he studied under no one. It is probably for this reason that his writings have not found more recognition, and have not exerted the influence on mathematical development that the importance and novelty of the ideas contained in them deserve.

2. Professor Waelsch generalized his definition of a spherical

harmonic in three-dimensional space, as the complete transvection of a binary form a_ξ^{2n} on the nth power of the quadratic form $(x+iy)\xi_1^2 - 2iz\xi_1\xi_2 + (x-iy)\xi_2^2$, to apply to four-dimensional space. The harmonic of R_4 appears as the bilateral complete transvection of the bilinear form $a_\xi^n b_\eta^n$ on the nth power of the form

$$(x+iy)\xi_1\eta_1 + (u-iz)\xi_1\eta_2 - (u+iz)\xi_2\eta_1 + (x-iy)\xi_2\eta_2.$$

The methods of binary invariants are not applicable to the euclidean geometry of higher spaces. The theory can therefore be regarded as a specific auxiliary for euclidean geometry of three and of four dimensions.

4. The theorems relating to the composition of forms in a ternary field are known. Their generalization to forms of n variables was impossible because no comprehensive symbolism for such forms existed. Miss Noether has developed such a symbolism by means of products of matrices. The two fundamental theorems, that concerning invariant processes and that regarding the finiteness of the system of forms, readily appear by this method.

5. The fundamental concept of Professor Kohn's paper is the "Wurf" on n elements in R_d $(n > d+2)$. These Würfe are arranged in pairs; for every one of n elements in R_d an associated Wurf of n elements in R_{n-d-2} is determined. By this means a large number of isolated theorems in projective geometry are brought into systematic relation, including Clifford's anharmonics, Rosanes's linearly independent systems of pairs of points, and Sturm's problem of projectivity.

7. Professor Müller considered first the motion through three consecutive phases, determined the center of curvature of the path curve of an arbitrary point of the system, the (1, 2) correspondence between the systems of moved points and the associated centers of curvature, and the curvature of the envelope; second, the corresponding motion through four or more such positions, introducing stationary curvature and Burmester points, as well as the locus of the pole. Finally, the determination of the phases by means of the instantaneous positions of three arbitrary points, and the introduction of branch phases.

8. Professor Mehmke discussed the torsion of an element x

of an orbit, under the most general conditions. A relation between the bisecant of the inflexional curve through x, the torsion of x and its position on the bisecant on the one hand, and the angle of contingence and the binormal component of the hyper-acceleration on the other was found, which are analogous to the formula for the normal acceleration of a plane system. The formula was generalized for certain hyperspaces, and interpretations given for the behavior at singular points.

9. The condition under which a closed system of screws consisting of n members and s screws may still have constrained motion was given by Professor Grübler as $5s - 6n + 7 = 0$, when the axes of the screws are independent. For systems of rotation the criterion is the same. If the axes of revolution intersect, giving rise to spherical motion, the criterion reduces to that of plane systems, $2s - 3n + 4 = 0$. This last relation is also true for translation.

10. The surfaces here considered appear in connection with the Bertrand curves in the following way: The rectifying cuspidal edges of two associated Bertrand curves have the same rectifying surface. After proving this theorem Professor Salkowski considered various special cases. The curves which give rise to cylinders are obtained by regarding a plane as the developable of an asteroid. The latter should now be twisted, keeping the curvature invariant, into a helix. The axes of the curve will become Bertrand curves and the altitudes of the triangles formed by the axes and a tangent will define two deformation surfaces of a hyperboloid of revolution.

11. When a point describes a plane curve the curvature is determined by the velocity and acceleration. Professor Wiener showed by means of numerous examples how to construct the tangent and radius of curvature, when the law of motion is properly chosen. If the circle of osculation be replaced by the parabola of osculation, these constructions can be simplified, and generalized to properties of contact of any order.

12. Professor Stübler investigated the instantaneous motion of a rigid system defined by the vector of the velocity of translation v, the rotation vector n, and the position of the axis of the screw. He showed how to construct the vector defining the acceleration, or conversely, when this is given, to find the

preceding elements. Then followed a construction for the axis of curvature, and the equation for the curve of inflexions, obtained as the locus of the centers of all possible accelerations compatible with the system. The case in which the velocity of rotation is perpendicular to the vector was studied in detail.

13. Professor Wiener exhibited two groups of models, invented by the author, eight to illustrate linkage quadrilaterals, and thirteen to illustrate linkage surfaces. The necessary and sufficient condition that a ruled surface be such a surface with regard to its generators and a system of directrix curves was given. A planigraph constructed on these principles was capable of drawing a complete circle of 64 cm. with an apparatus 40 cm. high.

14. Dr. Skutsch exhibited an apparatus for demonstrating the theorem of Möbius regarding the equilibrium of varying systems, keeping similarity as an invariant property.

15. In the report of Professor Jung, the investigation was restricted to linear systems of curves and the various invariant numbers that appear in functions of two independent variables.

16. It was shown by Professor Günther that the productions of Euler in mathematical geography were extensive and important, including the determination of latitude and longitude, map drawing, and the shifting of the earth's axis.

17. Professor Hoppe showed that the sexagesimal system arose from the efforts of the Babylonians to determine and define directions. The fundamental angle was not a right angle, but that of the equilateral triangle. This was divided into ten equal parts, and later into 60 equal parts. The theory is confirmed by numerous writings in the cuneiform characters.

18. Dr. du Pasquier arrived at the conclusion that the most suitable basis number is 4.

19. Starting from the differential equation defining the logarithm of an elliptic function, Dr. Velten derived the expansion in series of the Jacobi function.

20. On account of the limited time available for preparation Dr. Hahn confined his report to the Fredholm equation. He considered the solution by means of the Neumann series, the

derivation of the equation from an algebraic one by a limiting process, the memoirs of Plemelj and Goursat on the unsymmetric kernel, those of Hilbert and Schmidt on the symmetric kernel, and finally the method of Hilbert of an infinite number of variables.

21. The paper of Dr. Korn is an extension of that in the *Comptes Rendus* of June 24, 1907, by means of the methods of Poincaré, published in the *Palermo Rendiconti* in 1904, analogous to that employed in the solution of the differential equation $\Delta\phi + \kappa^2\phi = f$. Results already found by Hilbert, Fredholm, and others, were rediscovered, and an explanation of an infinite kernel of degree $\lambda < 1$ was given.

22. Professor Wirtinger first showed that every Riemann surface of an abelian integral can be uniquely obtained by the repetition of a finite number of convex rectilinear polygons. The process was carried through for normal elliptic integrals of the second kind, and for hyperelliptic integrals of the first kind. In the second case the dissection of the surface may not be unique, so that each sheet may be mapped upon a plane hexagon having given properties. From this standpoint it follows immediately that the transformation group of the periods can for $p = 2$ be generated by two operators.

23. When in the difference equation of order r

$$D_{\nu+r} + a_1^{(\nu)} D_{\nu+r-1} + a_2^{(\nu)} D_{\nu+r-2} + \cdots + a_r^{(\nu)} D_\nu = 0$$

$$(a_r^{(\nu)} \neq 0) \quad (\nu = 0, 1, 2, \cdots)$$

the coefficients $a_i^{(\nu)}$ approach a power of ν, say $a_i^{(\nu)} \doteq \nu^{\kappa_i}$, then the integrals approach $\nu\,!$. The exponents of $\nu\,!$ are functions of κ only, and can be immediately found by a simple geometric construction. Dr. Perron discussed equations which are related to the above form, or can be generalized from it, and applied his results to linear differential equations.

24. The tettarions discussed by Dr. du Pasquier are defined as a special class of complex numbers that are related to linear substitutions. The idea of the field was developed and some fundamental theorems established regarding tettarion integers.

25. Professor Timerding exhibited his model, recently manufactured by Schilling, to illustrate systems varying in affinity.

In them the Nurnberger rails are applied to construct triangles and tetrahedra.

26. Professor Mehmke mentioned, as two principal advantages of the calculating machine in question, the construction of the gear and the automatic division.

27. The procedure explained by Professor Papperitz and illustrated by the lantern consists essentially in the varying role of rapidly moving models which serve to generate a picture, to translate it, and to develop it. For example, consider two diaphragms with different width of slit, in rapid rotation. The intersections of the strips of light will generate a variety of curves. On the other hand, by means of a fixed slit, a model can be projected which will give rise to as great a variety of space curves on the changing surface of the model. Curves can also be generated by appropriate shadows.

28. The calculus of variations associates with every differential expression $w(x, y, y')dx$ a covariant system of curves, the extremals, for which the first variation of the integral $\int wdx$ vanishes. Professor Engel points out that the extremals are only the first links of an endless chain of systems of curves of this kind. If ∞^2 curves c be defined by $y'' = \phi(x, y, y')$ and $\int wdx$ defines the length of arc along a curve c, and if two particular curves of the system are fixed by having given tangents at a given point, and finally if ds_0 is the difference of their length of arc at the point of intersection, then a relation $\psi(x_0, y_0, y_0', dx_0, dy_0, dx_0', dy_0', ds_0) = 0$ exists. If this relation be linear and homogeneous in the differentials, it is an extremal. If it be homogeneous and of degree n, a category of covariant curves results, and indeed the given ϕ defined by a partial differential equation. The concept can be generalized in various directions.

29. Professor Müller emphasized that descriptive geometry should be taught in the technical schools from the standpoint of its practical applications, and that elementary theoretical instruction in it should be provided in all secondary schools. The speaker then explained his own method, and illustrated it with over 300 drawings. He further urged that all candidates for positions as teachers should not only hear courses in descrip-

tive geometry, but also take part in extensive constructive exercises.

30. Professor Czuber first pointed out that from the result of tl e investigations of the statistical reports furnished by 60 English and by 28 Austrian insurance companies it is clear that the probability of death is not only a function of the age, but also of the length of the term of insurance. Moreover, other factors must be considered, such as sex, occupation, and personal equations in medical certificates. The problem of the measure of mortality is not so much to determine the functional relation between these numbers, which now seems impossible, as to determine individual cases empirically. The details of this procedure were then illustrated by some numerical examples.

In the general sessions, mention should be made of the paper by Professor Einstein, Bern, "The recent changes which our views of the nature of light have undergone."

In the business meeting of the Vereinigung, which was held on Thursday, September 23, reports of the various committees and officers were read, and appointments for the following year were made. Professors Krause and Schoenflies retired from the executive committee, and their places were filled by the election of Professors E. Czuber and R. Müller. The library and bibliography committee was continued. Professor Rudio reported on the status of the publication of the works of Euler.

Ample provisions were made for social intercourse, two or more entertainments being held every evening. All the participants felt that it was a most successful occasion. The next meeting will be held in Königsberg.

E. Dintzl.

GERGONNE'S PILE PROBLEM.

BY DR. H. ONNEN, SR.

In volume I (1895), page 184, of the Bulletin, Professor L. E. Dickson has treated Dr. C. T. Hudson's solution of the problem :* To deal a pack of ab cards into a piles of b cards each and so stack the piles after each deal that after the nth deal any selected card may be the rth in the whole pack.

* *Educational Times* Reprints, 1868, vol. 9, pp. 89–91. I have tried in vain to lay my hand on Dr. Hudson's article.

Professor Dickson gives the following result. Let $p_1, p_2, \cdots, p_n$ be the places the pile of the selected card is to hold after the first, second, $\cdots$, nth stacking of the piles ; then these numbers will be the successive remainders on dividing n times by any integer a lying between the limits

$$\frac{a^{n-1}r-b}{b}+\frac{a^n-1}{a-1} \quad \text{and} \quad \frac{a^{n-1}(r-1)}{b}+\frac{a^n-1}{a-1}$$

or being equal to one of them.

Moreover Professor Dickson points out that in the particular case

$$p_1=p_2=\cdots=p_n=\frac{a+1}{2} \quad \text{and} \quad r=\frac{ab+1}{2}$$

n stackings are necessary and sufficient, where n is the least integer for which $a^{n-1}\geqq b$, and so proves the incorrectness of Dr. Hudson's condition for that case, viz., $a^{n+1}+2\geqq b$.

As to the general problem the number n of stackings that are necessary and sufficient may be computed in the following manner :

Since the difference v between the two limits is

$$v=\frac{a^{n-1}}{b}-1,$$

the number of stackings n must be at least so great as to make

$$a^{n-1}\geqq b.$$

The least value of all is consequently that which satisfies

$$a^{n-1}\geqq b > a^{n-2}.$$

But there does not always exist an integral value for n between the two stated limits inclusive.

Suppose the lower limit

$$\frac{a^{n-1}(r-1)}{b}+\frac{a^n-1}{a-1}=\delta+\frac{\rho}{b},$$

δ being an integer and $\rho < b$. Then the upper limit will be

$$\delta+\frac{\rho+a^{n-1}-b}{b},$$

and there will be no integer between these limits inclusive if

$$\rho > 0 \quad \text{and} \quad \rho + a^{n-1} - b < b$$

or if

$$0 < \rho < 2b - a^{n-1}.$$

But, $(a^n - 1)/(a - 1)$ being always an integer, ρ is the remainder on dividing $a^{n-1}(r - 1)$ by b.

Hence we may lay down the following rule for computing the smallest number of stackings that may bring the selected card on the rth place in the whole pack:

Take n so that

$$a^{n-1} \geqq b > a^{n-2}.$$

Divide $a^{n-1}(r - 1)$ by b. If the remainder ρ be zero or $\geqq 2b - a^{n-1}$, n stackings suffice. But if

$$0 < \rho < 2b - a^{n-1},$$

$n + 1$ stackings are necessary and always sufficient (since $v = a^n/b - 1 \geqq a - 1$ and a is at least 2).

In the following I propose to give a further generalization of Hudson's problem.

I suppose the number of piles formed by dealing the cards not to be at every turn the same, but say the first time A_1, the second time A_2, etc., the nth time A_n. So let the total number of cards N be

$$N = A_1 b_1 = A_2 b_2 = \cdots = A_i b_i \cdots = A_n b_n = \cdots. \qquad (1)$$

Again let $a_1, a_2, \cdots, a_i, \cdots, a_n$ denote the number of piles placed *beneath* the pile containing the selected card after the first, second, $\cdots$, ith, $\cdots$, nth stacking.

After the first deal the selected card, which for convenience sake may be called X, is among the b_1 cards of the pile containing it.

At the second deal these b_1 cards are dealt over A_2 piles, so that the pile with X now contains a smaller group of cards, among which X must needs be found. And so on. Let $x_1, x_2, \cdots, x_i, \cdots, x_n$ denote the group of cards in the pile with X after the consecutive deals, in which X must needs be found. Then $x_1 = b_1$.

After the ith deal the pile with X, counting from the bottom, consists of a group of cards, which certainly does not contain X and may be denoted by m_i; above that the group x_i, containing X to a certainty; and finally a group of cards, again certainly without X.

After the ith stacking the whole pack, counting from the bottom, consists of $a_ib_i + m_i$ cards without X, x_i cards containing X, and again a certain number of cards without X.

At the following, the $(i+1)$th, deal we have

$$a_ib_i + m_i = m_{i+1}A_{i+1} + r_{i+1}; \quad r_{i+1} + x_i = x_{i+1}A_{i+1} - s_{i+1},$$

$r_{i+1} < A_{i+1}$ and $s_{i+1} < A_{i+1}$ being the numbers of cards completing respectively the first and the last of the layers formed by the group x_i.

Putting $i = 1, 2, \cdots, n$ and taking into account that $m_1 = 0$ and $x_1 = b_1$, we get the following two series of equations

After the		*Series I.*	*Series II.*
2d deal	a_1b_1	$= m_2A_2 + r_2;$	$r_2 + b_1 = x_2A_2 - s_2,$
3d deal	$a_2b_2 + m_2$	$= m_3A_3 + r_3;$	$r_3 + x_2 = x_3A_3 - s_3,$
. . .	. . .	. . .	. . .
nth deal	$a_{n-1}b_{n-1} + m_{n-1}$	$= m_nA_n + r_n;$	$r_n + x_{n-1} = x_nA_n - s_n,$
nth stacking	$a_nb_n + m_n$	$= z,$	

z standing for the number of cards beneath the group x_n after the nth stacking.

Multiply the two equations representing the situation after the second deal by A_1, those after the third deal by A_1A_2, etc., those after the nth deal by $A_1A_2 \cdots A_{n-1}$, and the equation after the nth stacking by $A_1A_2 \cdots A_{n-1}A_n$. Then adding separately the equations of series I and II and for the sake of brevity putting

$$A_1 = P_1, \quad A_1A_2 = P_2, \cdots, \quad A_1A_2 \cdots A_n = P_n,$$

we get, paying attention to (1),

from I $$N\left(a_1 + \sum_{i=2}^{n} [a_iP_{i-1}]\right) = zP_n + \sum_{i=2}^{n} [r_iP_{i-1}], \tag{z}$$

from II $$N = x_nP_n - \sum_{i=2}^{n} [(r_i + s_i)P_{i-1}]. \tag{x}$$

If $x_n = 1$, the selected card's place in the whole pack is exactly fixed after the nth stacking.

Now the greatest number of cards with which $x_n = 1$ may be obtained amounts to $P_n = A_1 A_2 \cdots A_n$, r_i and s_i being both constantly zero. In this case the equation (z) gives

$$z = a_1 + \sum_{i=2}^{n} [a_i P_{i-1}]$$

and the place of X after the nth stacking becomes

$$1 + z = 1 + a_1 + A_1 a_2 + A_1 A_2 a_3 + \cdots + (A_1 A_2 \cdots A_{n-1}) a_n.$$

This result affords the following variation of the popular trick, which, I believe, is not generally known :

Take a number of cards, being the product of several numbers, for instance $48 = 4 \times 3 \times 4$, and deal them successively into as many piles as indicated by the factors, taken in any definite succession, in our example the first time into 4, the second time into 3, and the third time again into 4 piles. After every deal the pile containing the card pitched upon is indicated, and one may stack the piles in any way one pleases, provided the number of piles beneath the pile with the chosen card be at every turn remembered. Suppose this number at the three successive stackings to be 3, 1, and 2; then the place of the selected card will be the

$$1 + 3 + 4 \times 1 + 4 \times 3 \times 2 = 32\text{nd}.$$

The calculation may be easily made during the manipulations.

Since $a_1, a_2, \cdots, a_n$ are the remainders on dividing z successively by $A_1, A_2, \cdots, A_n$, one may also easily compute the mode of stacking to be performed for shuffling the selected card to any place $(1 + z)$ fixed beforehand.

If

$$P_n > N > P_{n-1},$$

the n intended deals into $A_1, A_2, \cdots, A_n$ piles are not always sufficient to fix the place of X in the pack. This appears from the following three postulates, the correctness of which will be evident by paying close attention to the equations of the series I and II :

Postulate 1. If the quantities a_i are all zero, any x_i coming from a number $N < P_n$ can never be greater than the corresponding x_i originating in the number $N = P_n$. Hence for $a_i = 0$ any number $N < P_n$ always gives $x_n = 1$, since the number $N = P_n$ furnishes at all events $x_n = 1$.

Postulate 2. If the quantities a_i are not all zero, any x_i may be at most one unit greater than it would be if $a_i = 0$. Hence, N being $< P_n$, x_n in this case may be 1 or 2.

Postulate 3. If $x_n = 2$ and a $(n+1)$th deal and stacking are to be performed, it may happen that also $x_{n+1} = 2$. In this case we have

$$m_{n+1} = m_n, \quad r_{n+1} = s_{n+1} = A_{n+1} - 1,$$

which means that one of the two cards of which x_n consists is the last of a layer and the other the first of the next layer.

Now considering the particular case

$$A_1 = A_2 \cdots = A_n = A,$$

and consequently

$$b_1 = b_2 = \cdots = b_n = b,$$

as in Dr. Hudson's problem, the equations (z) and (x) become

$$N \sum_{i=1}^{n} [a_i A^{i-1}] = zA^n + \sum_{i=2}^{n} [r_i A^{i-1}], \tag{z_1}$$

$$N = x_n A^n - \sum_{i=2}^{n} [(r_i + s_i) A^{i-1}]. \tag{x_1}$$

Since the limits of r_i are $A - 1$ and 0 we have

$$A^n - A \geqq \sum_{i=2}^{n} [r_i A^{i-1}] \geqq 0.$$

Carrying these limits into (z_1) we get

$$\frac{(1+z)A^n - A}{N} \geqq \sum_{i=}^{n} [a_i A^{i-1}] \geqq \frac{zA^n}{N}, \tag{2}$$

i. e., any number

$$\sum_{i=1}^{n} [a_i A^{i-1}]$$

satisfying these conditions will give such values for $a_1, a_2, \cdots, a_n$ that after the nth stacking z cards lie beneath the group x_n containing the selected card. As however, N being $< A^n$, x_n may be 1 or 2 (postulate 2), it will sometimes be uncertain whether X be the $(1+z)$th or the $(2+z)$th card.

Putting in Dr. Hudson's limits $a = A$, $r = 1 + z$ and considering that his quantities $p_1, p_2, \cdots, p_n$ correspond to $a_1 + 1$, $a_2 + 1, \cdots, a_n + 1$ in my notation, the formula for computing $a_1, a_2, \cdots, a_n$ as proposed by Dr. Hudson may be written thus:

$$\frac{(1+z)A^n - N}{N} \geqq \sum_{i=1}^{n} [a_i A^{i-1}] \geqq \frac{zA^n}{N}. \tag{3}$$

Now it appears on further examination that any integer

$$\sum_{i=1}^{n} [a_i A^{i-1}]$$

satisfying Dr. Hudson's formula (3) gives a mode of stacking the piles, so as to make $x_n = 1$. If however such an integer do not exist, there is yet an integer satisfying the somewhat wider limits (2) and affording a proper mode of stacking as to the value of z; but with $x_n = 2$, so that it remains doubtful whether X becomes the $(1 + z)$th or the $(2 + z)$th card, and $n + 1$ stackings are required to bring it surely to the $(1 + z)$th place.

The condition (3) may be written in this manner:

$$\frac{N \sum_{i=1}^{n} [a_i A^{i-1}]}{A^n} \geqq z \geqq \frac{N \sum_{i=1}^{n} [a_i A^{i-1}]}{A^n} - \frac{A^n - N}{A^n},$$

giving one — and only one — value of z, the mode of stacking being given, if the remainder σ on dividing

$$N \sum_{i=1}^{n} [a_i A^{i-1}]$$

by A^n be $\leqq A^n - N$; in this case $x_n = 1$. If however $\sigma > A^n - N$, no integer z can be found, x_n being 2.

Taking

$$A_1 = A_2 = \cdots = A_n = A$$

and moreover

$$a_1 = a_2 = \cdots = a_n = a,$$

so as to put the pile with X at every turn in the same place between the other piles, the equations (z) and (x) become

$$aN\frac{A^n - 1}{A - 1} = zA^n + \sum_{i=2}^{n} [r_i A^{i-1}], \qquad (z_2)$$

$$N = x_n A^n - \sum_{i=2}^{n} [(r_i + s_i) A^{i-1}], \qquad (x_2)$$

whereas

$$\frac{aN}{A^n} \cdot \frac{A^n - 1}{A - 1} \geqq z \geqq \frac{aN}{A^n} \cdot \frac{A^n - 1}{A - 1} - \frac{A^n - N}{A^n}.$$

Now an integer may be found for z if the remainder σ on dividing $aN \cdot (A^n - 1)/(A - 1)$ by A^n be $\leqq A^n - N$. If $\sigma > A^n - N$, such an integer does not exist, x_n being 2.

In the second case one may deal and stack once more, and we have to examine the question whether x_{n+1} becomes 1 or 2.

In the present case, viz., $A_i = A$ and $a_i = a$, the numbers $m_2, m_3, \cdots, m_n$ in the equations of series I are increasing till a permanent maximum is attained. This maximum amounts to

$$m_n = m_{n+1} = \frac{ab - r_{n+1}}{A - 1} \quad \text{if } x_n = x_{n+1} = 1$$

or to

$$m_n = m_{n+1} = \frac{ab}{A - 1} - 1 \quad \text{if } x_n = x_{n+1} = 2.$$

Since m_n and m_{n+1} are integers, ab must be a multiple of $A - 1$ if $x_n = x_{n+1} = 2$. Conversely $x_n = x_{n+1} = 2$ if ab be a multiple of $A - 1$, *except* if $a = 0$ or $A - 1$.

Hence we have the following rule (α) for computing whether it is possible or not to fix the place of X in the pack after any number of stackings, and (β) if so, whether n or $n + 1$ stackings are necessary:

(α) If $A - 1 > a > 0$ and ab a multiple of $A - 1$, it is impossible to fix the place of X precisely. After n or more stackings its place will be the

$$\left(ab + \frac{ab}{A - 1}\right)\text{th} = \left(\frac{aN}{A - 1}\right)\text{th} \quad \text{or the} \quad \left(1 + \frac{aN}{A - 1}\right)\text{th}.$$

In all other cases the place of X is determined after n or $n + 1$ stackings.

(β) If $a = 0$, X is the first card after n stackings.—If $a = A - 1$, X is the Nth card after n stackings.—If $A - 1 > a > 0$ and ab not a multiple of $A - 1$, divide $aN(A^n - 1)/(A - 1)$ by A^n. The remainder σ being $\leqq A^n - N$, the place of X is determined after n stackings. If $\sigma > A^n - N$, $n + 1$ stackings are required. In both cases X becomes the

$$\left(1 + \frac{aN - r}{A - 1}\right)\text{th card} \qquad (r < A - 1).$$

This number is obviously $1 +$ the integer of the quotient $aN/(A - 1)$.

Example 1. Take a pack of 52 cards and make in every deal 4 piles,

$$\therefore A = 4, \quad b = 13.$$

Since

$$4^3 > 52 > 4^2,$$

$n = 3$. Neither $a = 1$ nor $a = 2$ makes ab a multiple of $A - 1$. On dividing $aN(A^n - 1)/(A - 1)$ by A^n the remainder is 4 or 8, which is $< A^n - N = 12$. After three stackings X will be:

for	$a = 0$	the	1st	card.
"	$a = 1$	"	18th	"
"	$a = 2$	"	35th	"
"	$a = 3$	"	52d	"

Example 2. Take a pack of 48 cards. Making at every turn 4 piles, we have

$$4^3 > 48 > 4^2, \quad \therefore n = 3.$$

Now $ab = 12a$ is always a multiple of $A - 1 = 3$, and for $a = 1$ or $a = 2$ it is impossible to determine the place of X exactly, whatever may be the number of stackings. After 3 or more stackings one can declare only that the selected card will be

the 16th or 17th	if $a = 1$
" 32d " 33d	" $a = 2$.

Example 3. Suppose $N = 231$, $A = 7$, $b = 33$, $\therefore n = 3$. $ab = 33a$ is divisible by $A - 1 = 6$ if $a = 2$ or 4. If $a = 1$, 3, or 5, the remainder σ on dividing $aN(A^n - 1)/(A - 1)$ by A^n is respectively 133, 56, and 222, the second being $< A^n - N = 112$, the two others $> A^n - N$. Hence we get

$a = 0$:	after 3 or more stackings	X	is the	1st card.
$a = 1$:	" 4 " " "	X	" "	39th card.
$a = 2$:	" 3 " " "	X	" "	77th or 78th card.
$a = 3$:	" 3 " " "	X	" "	116th card.
$a = 4$:	" 3 " " "	X	" "	154th or 155th card.
$a = 5$:	" 4 " " "	X	" "	193d card.
$a = 6$:	" 3 " " "	X	" "	231st card.

If A be odd and $a = \frac{1}{2}(A - 1)$, $ab = \frac{1}{2}(A - 1)b$ will be a multiple of $A - 1$ if b be even. Hence putting after every deal the pile with X in the middle, it is impossible to deter-

mine exactly the place of X if N be even. With an odd number of cards this mode of stacking always brings X to the middle of the pack, since $1 + z = \frac{1}{2}(N + 1)$.

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THE INTEGRAL EQUATION OF THE SECOND KIND, OF VOLTERRA, WITH SINGULAR KERNEL.

BY MR. G. C. EVANS.

(Read before the American Mathematical Society, September 13, 1909.)

I.

THE integral equation of the second kind, of Volterra, is written

$$(1) \qquad u(x) = \phi(x) + \int_a^x K(x, \xi)u(\xi)d\xi.$$

If the function $K(x, \xi)$ is continuous, $a \leqq \xi \leqq x \leqq b$, and the function $\phi(x)$ is continuous, $a \leqq x \leqq b$, there is one and only one continuous solution of the equation. But if $K(x, \xi)$ is not continuous in its triangular region, the case is more complicated. In I. we consider finite solutions of integral equations of which the kernel $K(x, \xi)$ is absolutely integrable, and after obtaining a theorem for that case apply it to some others where the kernel is no longer absolutely integrable. For this theorem the following conditions limit the given functions of the equation: $K(x, \xi)$ shall satisfy (A) and $\phi(x)$ shall satisfy (B).

(A) A real function of the two real variables x, ξ is to be continuous in the triangle $T: a \leqq \xi \leqq x \leqq b$, $b > a > 0$, except on a finite number of curves each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number, $x = \beta_1$, $x = \beta_2, \cdots, x = \beta_r$. On the other portions of the system of curves there are to be only a finite number of vertical tangents.

(B) In the region $t: a \leqq x \leqq b$ a real function of a single real variable x is to be continuous except at a finite number of points $\gamma_1, \gamma_2, \cdots, \gamma_s$, and is to remain finite.

Let us define the linear region t_δ, formed from t by removing the small portions $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$; and the two dimensional region T_δ, formed from T by removing the small strips $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$, where the δ is an arbitrarily small magnitude, and the α's, finite in number, are yet to be defined.

Under these conditions we have the

THEOREM. *There is one and only one finite solution of the integral equation* (1), *continuous in t except for a finite number of points, and these points will be among the points $\gamma_1, \cdots, \gamma_s, \alpha_1, \cdots, \alpha_l$ (the α's to be defined below); provided conditions (A) and (B) and the following further conditions are fulfilled:*

(a) $\int_a^x |K(x, \xi)|\, d\xi$ *converges in t except for a finite number of points $\lambda_1, \cdots, \lambda_t$, and remains finite.*

(b) *There is a finite number of points $\alpha_1, \cdots, \alpha_l$ [including the points β of (A) and λ of (a)] such that when ϵ and δ are chosen at pleasure there is a length η_δ for which*

$$\int_y^{y+\eta_\delta} |K(x, \xi)|\, d\xi < \epsilon, \qquad (x, y) \text{ and } (x, y + \eta_\delta) \text{ in } T_\delta.$$

(c) *t can be divided into k parts, bounded by points $a = a_0, a_1, \cdots, a_{k-1}, b = a_k$ such that*

$$\int_{a_i}^x |K(x, \xi)|\, d\xi \leqq H < 1 \quad \begin{cases} a_i \leqq x \leqq a_{i+1}, \\ x \neq \alpha_1, \cdots, \alpha_l. \end{cases}$$

In the proof of this theorem (a) and (c) are used in showing the convergence of the expansion of the solution, and (b) in developing what continuity exists.

The condition (A) can be replaced by conditions on the integral of the kernel; for instance (A) and (b) can together be replaced by the condition which follows:

The integral

$$\int_a^x K(x, \xi) r(\xi) d\xi,$$

where $r(x)$ is finite in t and continuous except for a finite number of points, shall converge except at most for a finite number of values of x, and the function of x thus defined shall remain finite; furthermore it shall be continuous except at most for a finite number of values of x, denoted by $\alpha_1, \cdots, \alpha_l$, which are independent of the choice of $r(x)$.

A special case of this theorem has been treated by Mr. W. A. Hurwitz.* The hypotheses for this case were

(B') $\phi(x)$ is continuous in t,

(a') $\int_a^x |K(x, \xi)| d\xi$ converges in t,

(b') $\int_a^x |K(x, \xi)| d\xi$ represents a continuous function in t,

(c') $|K(x_1, \xi)| \geqq |K(x_2, \xi)|$ when $x_1 > x_2$.

Here (a') implies (a), and (b') and (c') together imply (c) and the condition just mentioned that replaces (A) and (b).

By application of the theorem of page 131, with a change of dependent variable, equations of a still more extended type may be solved. In the equations

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi, \tag{2}$$

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi) d\xi, \tag{3}$$

and

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)\prod\limits_{i=1}^{p} \{[\xi - \psi_i(x)]^{\lambda_i}\}} u(\xi) d\xi, \tag{4}$$
$$\sum_{i=1}^{p} \lambda_i = \lambda < 1,$$

any one of which includes the previous ones as special cases, $K(x, \xi)$ shall satisfy (A) and be finite, $\phi(x)$ shall satisfy (B), $f(x)$ and $g(x)$ shall be continuous in t and unequal to zero except at the point a where they may vanish in any way, and the various ψ's shall be continuous functions of x. Then,

In (2), if $\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$ remains finite as x approaches a, there is a solution that vanishes at a as sharply as const. $e^{-\int_x^b \frac{dx}{|f(x)|}}$.

In (3), if $\phi(x)g(x)e^{\int_x^b \frac{dx}{|f(x)g(x)|}}$ remains finite as x approaches a, there is a solution that vanishes at a as sharply as

* As a problem in Professor Bôcher's course in Integral Equations, Harvard University, in 1907–1908.

$$\text{const.}\ \frac{1}{g(x)} e^{-\int_x^b \frac{dx}{|f(x)g(x)|}}.$$

In (4), if $\phi(x)g(x)e^{a\int_x^b \frac{dx}{[|f(x)g(x)|]^{(\nu+\lambda)/\nu}}}$ remains finite as x approaches a, where α is a certain constant, and ν any constant such that $1 - \lambda > \nu > 0$, there is a solution that vanishes at a as sharply as

$$\text{const.}\ \frac{1}{g(x)} e^{-a\int_x^b \frac{dx}{[|f(x)g(x)|]^{(\nu+\lambda)/\nu}}}.$$

There may however be more than one finite solution of these equations.

II.

In this section we consider kernels that are not absolutely integrable. We have the following introductory theorem:

Let the kernel of the integral equation (1) be in the form

$$\frac{K(x, \xi)}{G(x, \xi)},$$

where

$G(x, \xi)$ is analytic in T;*

$K(x, \xi)$ is continuous in T, and $\phi(x)$ continuous in t;

$K(x, \xi)$ vanishes at most at a finite number of points in T at which $G(x, \xi)$ also vanishes.

Then there is no solution of (1), continuous in t except for a finite number of points and not identically vanishing through any subinterval of t, unless the kernel $K(x, \xi)/G(x, \xi)$ can be written in the form

$$\frac{\bar{K}(x, \xi)}{g(x)f(\xi)},$$

where $\bar{K}(x, \xi)$ is continuous in T, and $f(x)$ and $g(x)$ are analytic in t.

If $K(x, \xi)/G(x, \xi)$ cannot be rewritten as $\bar{K}(x, \xi)/g(x)f(x)$ for values of ξ, $x_1 < \xi < x_1'$, $x_2 < \xi < x_2'$, $\cdots$, $x_p < \xi < x_p'$, it is necessary for all such values of ξ, if the integral is to converge, that $u(\xi) = 0$. Hence, in general, under such conditions, there

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*If we replace the triangle T by the square $S: a \leqq x \leqq b$, $a \leqq \xi \leqq b$, this theorem holds also for the equation with constant limits

$$u(x) = \phi(x) + \int_a^b \frac{K(x, \xi)}{G(x, \xi)} u(\xi) d\xi.$$

will be no solution of the integral equation (1). For that there be a solution under such conditions it is necessary, as is obvious from the form of the equation (1), that the given function $\phi(x)$ satisfy the equations

$$\phi(x) = \int_a^{x_j} \frac{K(x, \xi)}{G(x, \xi)} u(\xi) d\xi, \quad x_j < x < x_j' \quad (j = 1, 2, \cdots, p);$$

wherefore the $\phi(x)$ cannot be chosen arbitrarily in those subintervals of t. The solution when it exists is independent of the value of the kernel in the strips for which $x_j < x < x_j'$, or $x_j < \xi < x_j'$ $(j = 1, 2, \cdots, p)$.

This prepares us to state the

THEOREM. *Let the kernel of* (1) *be in the form*

$$\frac{K(x, \xi)}{f(\xi) g(x)},$$

where

1° (*a*) *$K(x, \xi)$ is continuous in T, and $f(x)$, $g(x)$ and their first derivatives are continuous in t;*
(*b*) *$\partial K(x, \xi)/\partial x$ satisfies* (*A*), *page 130, and is finite in T;*
(*c*) *$\phi(x)$ is continuous in t except at $x = a$ and is such that the function $\phi(x) g(x)$ and its first derivative satisfy* (*B*), *page 130.*

2° *The function $f(x) g(x)$ is greater than zero in the neighborhood of a, and at a vanishes in such a way that*
$$\int_a^x \frac{dx}{f(x) g(x)} \text{ is not convergent;}$$

3° *$\lim_{x=a} [K(x, x) - K(a, a)]/(x - a)^\nu$ exists, where $K(a, a) \neq 0$, and where ν is some number lying between 0 and 1 $(1 > \nu > 0)$ and is also greater than $1 - 1/\{d[f(x)g(x)]/dx\}_{x=a}$;*

4° *$\lim_{x=a} \phi(x) g(x) = 0$.*

Then, under the foregoing conditions,

(*i*) *if $K(a, a) < 0$, there exists one solution of* (1) *continuous in the neighborhood of a and at a, and*

(*ii*) *if $K(a, a) > 0$, there exists a one-parameter family of solutions of* (1) *continuous in the neighborhood of a except perhaps at a itself. As x approaches a, each solution remains less in absolute value than some constant times $f(x)/(x - a)^\nu$.*

If $K(a, a) < 0$ we may take $\nu = 0$ without change in the

theorem. Also if $K(a, a) > 0$ and $[df(x)g(x)/dx]_{x=a} \dot{<} 1$, we may take $\nu = 0$.

A slightly more special theorem, equivalent to taking $\nu = 1$, is obtained by inserting in 1° : *$\partial K(x, \xi)/\partial\xi$ satisfies (A) and is finite in T*, and dropping all of 3° except $K(a, a) \neq 0$.

If we write $K(x, \xi) = K(a, a) + \lambda[K(x, \xi) - K(a, a)]$, the solutions specified in the theorem of page 134 are analytic in the parameter λ, and are the only solutions continuous in the neighborhood of a, except possibly at a, that are analytic in λ. They are also the only solutions continuous in the neighborhood of a, except possibly at a, that satisfy the conditions

$$(a) \qquad \lim_{x=a} \Phi(x) = 0,$$

$$(b) \qquad \Phi'(x) \text{ remains finite,}$$

where

$$\Phi(x) = \int_a^x \frac{K(x, \xi) - K(a, a)}{f(\xi)} u(\xi) d\xi.$$

There are no solutions that satisfy these conditions if 1°, 2°, and 3° of page 134 hold, but not 4°.

III.

So far we have considered only finite solutions, or at most solutions that become infinite at $x = a$ to an order not greater than the first. It is possible, however, to limit the totality of solutions as to character.

THEOREM. *Let the kernel of* (1) *be in the form $K(x, \xi)/f(\xi)g(x)$, where*

1° (*a*) *$K(x, \xi)$ is continuous in T,* and $f(x)$, $g(x)$ and their first derivatives are continuous in t;*

(*b*) *$\partial K(x, \xi)/\partial x$ and $\partial K(x, \xi)/\partial\xi$ satisfy A, and are finite in T;*

(*c*) *$\phi(x)$ is continuous in t except perhaps at a, and is such that the function $\phi(x)g(x)$ and its first derivative satisfy (B);*

2° *The function $f(x)g(x)$ vanishes at most a finite number of times in t;*

3° *On any horizontal line $\xi = \xi_0$ cutting T there is at least one point in T for which $K(x, \xi) \neq 0$.*

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* If we replace T by S this theorem holds for the equation with constant limits.

Then all the solutions of (1) *continuous in* t *except for a finite number of points are such that the function* $u(x)g(x)$ *remains continuous in* t.

This theorem has several applications. If neither $f(x)$ nor $g(x)$ vanishes in t, there can be no solution becoming infinite at any point in t; therefore the continuous solution is the only solution of the equation continuous except at a finite number of points. If $K(a, a) \neq 0$, and if $\lim_{x=a} \phi(x)g(x) = 0$, the theorem of page 134 holds, as we have already noticed. The solutions there given are the only ones continuous except at a finite number of points, such that $d[u(x)g(x)]/dx$ remains finite in t; they are also the only solutions possible, continuous except at a finite number of points, provided that $K(x, \xi) - K(a, a)$ vanishes identically when $\xi = x$.

If the kernel of the integral equation (1)* is analytic in T, and if $\phi(x)$ is continuous in t, a proof similar to that of the above theorem shows that the continuous solution is the only solution of (1) continuous in t except for a finite number of points.

HARVARD UNIVERSITY,
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DESCRIPTIVE GEOMETRY.

Lehrbuch der darstellenden Geometrie für technische Hochschulen, Volume I. By Professor EMIL MÜLLER, of the Imperial Technical School at Vienna. Leipzig and Berlin, Teubner, 1908. xiv + 368 pages, 273 figures, and three plates.

Vorlesungen über darstellende Geometrie. By GINO LORIA. Volume I : *die Darstellungsmethoden.* Authorized German translation from the Italian manuscript, by FRITZ SCHÜTTE. Teubner's Sammlung, volume XXV_1. Leipzig and Berlin, Teubner, 1907. xi + 218 pages and 163 figures.

Descriptive Geometry, a treatise from a mathematical standpoint, together with a collection of exercises and practical applications. By VICTOR T. WILSON, Professor of drawing and design in the Michigan Agricultural College. New York, John Wiley and Sons, 1909. 8vo, viii + 237 pages and 149 figures.

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* If the kernel of the integral equation with constant coefficients is analytic in S and if $\phi(x)$ is continuous in t, the continuous solutions are the only solutions continuous except for a finite number of points.

During the last few years Teubner has published twenty-four treatises, several of them of two or more volumes, on descriptive geometry, and his latest circulars announce that ten more are in an advanced state of preparation. And this is only one instance ; a number of other publishers announce the early appearance of one or more books on the same subject, and the periodic literature of Europe (particularly German and Italian) contains during the last ten years some five hundred memoirs, essays, notes, solutions, etc., pertaining to this science. Under these circumstances the question may be asked, why still another book in a language represented by so many ? But an examination of Professor Müller's work will dispel any doubts as to the wisdom of its publication. Descriptive geometry was taught to most of us in America who had any instruction at all in the subject, not as a science, but rather as a clever device for producing certain graphical representations. Courses in it are given only in our technical schools, conducted by engineers for purely practical purposes, and largely without proofs. The students soon think of the procedure as empirical and frequently wonder why the drawings come out so well, when they do not really know just what they are doing. For this reason, many of our American graduates who have extensive responsibilities in draughting operations have had to learn most of the elaborate processes in the offices, and begin years later really to understand the geometric principles upon which the constructions were based.

The present book has a very different character ; a student who masters it will not doubt that the various devices will accomplish the desired ends. The author has taught the subject for years, has had extensive experience with graphical methods, and by his achievements in other fields of mathematics has proved that he can speak with authority. While on every page emphasis is laid on the fact that the entire subject is to be regarded as an auxiliary science for engineers and architects, and hence the practical applicability is the principal aim, yet nothing is taken for granted, every step being carefully analyzed. One striking feature is the early insistence on the interpretation of a figure as a whole, rather than simply the sum of a number of individual points and lines. This is an excellent means for retaining the active interest of the student. Thus, a general parallel projection of a hexagonal pyramid is discussed as early as page 25, and shades and shadows are systematically intro-

duced on page 48. Within the next few pages the ground, front, and end elevations of complicated frames are constructed, and their shadows found.

A concise system of abbreviations is used throughout. This greatly aids in reading demonstrations, the only objection being that it is different from that of other writers on this and related subjects.

In the first part of the book, which treats of the ordinary properties of rectilinear figures, two principles are successfully employed, both of which are unusual at this stage, but are always met with in practice. The first is the frequent use of the general profile, that is, projection on a plane perpendicular to but one of the fundamental planes; the other is the early omission of the ground line. These principles are used throughout the volume.

Another feature is the prominence of the discussion of projective properties, which always precede the metrical ones. Incidentally, the student is gradually gaining a comprehensive knowledge of projective geometry, but always in organic relation to graphical problems. Affinity is indeed given a separate consideration, the first part of which is quite independent of graphical notions.

The second part of the book is devoted to the general theory of curves and surfaces. The treatment is partly algebraic and partly differential. Here the author seems rather too ambitious; he acknowledges in the preface that the presentation may be too brief, but hopes by presupposing considerable knowledge of analytics and the calculus that the student may see his way through. In thirty pages we find such varied concepts as Plücker's numbers for algebraic plane curves, the circular points and absolute circle, the imaginary generators of ellipsoids, and a number of properties of space curves, including lines of curvature, asymptotic, and geodesic lines on a given surface. Either the student must have learned these things before, or he will not sufficiently know them even after this discussion. This is the only part of the book that does not seem to be carried out in the best way.

On the other hand, the applications of these principles to the curves and surfaces of the second order, which is found in the succeeding chapters, is exceptionally well done. A large number of theorems usually found in books on analytics are established, including a direct graphical determination of the center

of curvature of an arbitrary point on an ellipse, by means of three orthogonal projections. Cones, cylinders, and developables are considered before the general quadric is taken up. Much of this matter is far more extensive than is usual in books on descriptive geometry; it compares roughly with the corresponding chapters of Reye's Geometrie der Lage.

The excellent chapters on surfaces of revolution and helicoids contain a rich fund of information, which is of value for purposes other than graphical representation. The theory of illumination is developed, curves of equal illumination determined, and the results compared with true and apparent contour.

Among the commendable details of the book we may mention the extensive index, the excellent figures, the free-hand lettering, and the frequent instructions and admonitions as to procedure, including the preparation of washes, tints, inks, etc. — making it a serviceable hand-book as well as a thorough theoretical treatise.

The above will indicate why the publisher should undertake the production of another book on descriptive geometry. A second volume is to consider central perspective, axonometry, and related subjects.

The present first volume of Professor Loria's treatise is concerned entirely with methods of graphically representing configurations composed of points, straight lines, and planes. It presupposes some knowledge of projective geometry of the plane, including self-corresponding and double elements, Pascal's theorem, Desargues's theorem, and poles and polars. The work is divided into five books; the first (88 pages) discusses the ordinary problems of descriptive geometry, but the proofs are unusually systematic, and all particular and exceptional cases are treated. The constructions are all reduced to depend upon four fundamental ones. Results are expressed in bold-faced type and a number of unsolved exercises follow each article.

The profile and certain oblique planes are frequently employed, and frequent use is made of rotation. Among the themes treated in this short space are the determination of the two transversals of four skew lines and the construction of the regulus defined by three lines.

The second book considers most of the same problems from

the standpoint of central perspective. Given a center C and a plane π. Let a line pierce π in T. Through C draw a line parallel to the given one, cutting π in I. The projection of the line is then defined by TI. Similarly, a plane is expressed in terms of its trace t and the parallel line i. The third book has for subject the projective equivalent of the preceding one, the point C now being at infinity in the direction normal to π. The other defining element is a plane parallel to π at a known distance from it. The chapter closes with an instructive comparison of central and double orthogonal projection.

The discussion of axonometry in the fourth book (45 pages) is an excellent theoretic presentation, but rather too brief to be of greatest use. While all the necessary steps are given, they are so brief, and treated each alone, that the guidance of a teacher is necessary to the average reader. In the preceding parts one could easily read the text and solve the assigned problems without such assistance. The chapter on orthogonal axonometry is followed by one on oblique parallel, and one on central, but these are hardly more than outlines. Finally, there is added as a fifth book a brief treatment of photogrammetry, *i. e.*, the science of constructing a third perspective of a space figure when two are completely given, or of finding either a central or orthogonal projection when several are partially given, e. g., the drawings given, but the position of the corresponding centers not specified. The discussion of these problems requires many more theorems from projective geometry, the proofs of which are supplied. The results are very interesting; while considerable knowledge and higher maturity are required on the part of the student, this discussion cannot help being a valuable incentive to the more advanced student.

The book treats the problems it proposes with thoroughness and skill, but a reader will after all probably not get a proper idea of the beauty or of the usefulness of descriptive geometry by reading it, because it treats them in an isolated matter, and does not develop the most useful power of being able to interpret a complicated drawing as a whole. The figures of a few building fronts or plans treated entirely in straight lines would add greatly to the usefulness of this useful book. Probably such matters will be adequately treated in the second volume.

In the preface of Professor Wilson's book we find the following statement: "Descriptive geometry is essentially a math-

ematical subject. The applications of its principles to the making of working drawings, however, and the modifications which are made to meet the contingencies of practice have had a tendency to obscure this fact and like other theoretical subjects it has suffered mutilation in the interest of short cuts to immediate practical uses. But does not technical education after all consist chiefly in an equipment of sound theory?" In the treatment of the configurations consisting of straight lines and planes the author has well justified this statement, and has made a real improvement on the presentation found in most American texts. The presentation is didactic, the explanations being full, and theorems are followed by a number of numerical exercises. The idea of rotation of one or both axes is brought in on page 17 and extensively employed in subsequent problems.

The discussion of curves and surfaces is much less satisfactory, and only partly fulfils the promise in the second part of the title. Thus, in defining parallel lines (page 2) we meet the statement: "Hence parallel lines are said to have two points in common at infinity"; on page 89 we find: "Double curved surfaces are generated only by the motion of curves and have no straight line elements." The words consecutive and coincident on page 93 are confusing; the rectification of a curve, defined on page 95, is obtained by rolling the curve out on its tangent. The word touch, as applied to space curves, is everywhere incorrectly used. According to the definition of a warped surface on page 139 no ruled surface can have a double curve. A number of theorems are worded much too broadly, as they apply only to particular cases; thus, on page 149, after discussing a method for drawing a tangent plane at a point on a general quadric, we are assured that it is applicable to any surface having two systems of rectilinear generators. The theorem on page 151 refers only to rectangular hyperboloids. Unfortunately, the book is full of similar statements; many of them are found in most of the American text-books on the subject, but a few are new contributions. However, the author should not be too severely criticized; he has at least felt the need of more light, and has made decided improvements in the early part of the book.

In view of the sharp contrast between this, one of the best American text-books on descriptive geometry, and those mentioned above we must conclude that there is still room for a good treatise on the subject in English.

VIRGIL SNYDER.

ELEMENTARY MECHANICS.

A First Statics. By C. S. JACKSON, M.A., R. M. Academy, Woolwich; and R. M. MILNE, M.A., R. N. College, Dartmouth. [Dent's Mathematical and Scientific Text-books, edited by W. J. Greenstreet.] London, J. M. Dent & Co., 1907. Crown 8vo. vii + 380 pp. 4 shillings.

A Text-book of Mechanics. By LOUIS A. MARTIN, JR., M.E., A.M., Stevens Institute of Technology. Vol. I, *Statics*, 1906. 12mo. xii + 142 pp. $1.25. Vol. II, *Kinematics and Kinetics*, 1907. 12mo. xiv + 214 pp. $1.50. New York, John Wiley and Sons.

PERHAPS it is immaterial to our colleges whether their average graduate knows even the simplest principles of physics, — so as to be able, for example, to explain the advantages of a simple set of pulleys, or the physical basis of time measurements, or the easy balancing of a moving bicycle, or when a leaning ladder could be safely ascended if slipping were opposed only by friction. At least, such an inference seems warranted so long as educators suppose that familiarity with records of human activities of various sorts is a compensation for ignorance of natural laws, and until they cease placing their labels of education upon persons who have not thoroughly studied even the broader sciences. The common and dense ignorance of one fundamental science could be removed by even a first course in mechanics (developed from the physical side), whether or not the average student should later elect the more mathematical study of this science, whose problems have so greatly stimulated the growth (and still defy the powers) of mathematical analysis. These two kinds of courses in mechanics are somewhat typified by the books under review.

The First Statics is not purely mathematical, as it employs numerous simple experiments; but it is far from being a laboratory manual. Experiments are suggested merely to introduce the laws — and the first ideas of mechanics should always come in that way — while deductive analysis is used throughout, though with no stress on rigor. Trigonometry is employed, but not calculus, even in finding centers of gravity.

The authors treat the lever and the equilibrium of paral-

lel forces before introducing the parallelogram of forces, which follows next with graphical methods for three concurrent forces, and leads to analytical methods of composition and resolution, and the properties of couples. Several simple machines are then explained: weighing-bridge, "penny-in-the-slot" machines, tackle, differential pulley, Chinese wheel and axle, lifting crab, worm and wheel, and screw. There is a good treatment of friction and an extended discussion of centers of gravity — of a few bodies geometrically, of some others by applying the theorems of Pappus, and of any number of particles analytically by taking moments. Graphical analysis of framed structures, the use of the link polygon, and the general analytical conditions for the equilibrium of coplanar forces are treated at some length, followed by an exposition of the relative advantages of graphical and of analytical methods, and of the usefulness of the principle of work. The volume concludes with a chapter on forces in space, and a good set of miscellaneous problems.

Features of this very worthy text which deserve special commendation are the presence of occasional questions as to the probable extent and cause of error; the carefully graded treatment, with simplest beginnings; the excellent set of some 700 problems (especially good on taking moments); and the admirable treatment of the parallelogram of forces, not offering a pretended proof of it, but assuming it as the result of experience, after the student has been convinced by experiments. Indeed, important throughout is the inductive presentation of new ideas, whereby the student is prepared by experiment or special example for the statement of a principle. Of course, in the teaching of pure mathematics, this office of the experiment is not dispensed with, but filled by the numerical example. To illustrate: the average student of analytical geometry is puzzled if directly confronted with $S_1 + kS_2 = 0$ as the equation of a straight line through the intersection of the lines $S_1 = 0$ and $S_2 = 0$. But if he has first seen several numerical cases, and has observed how the new line changes for different values of the parameter, he can readily grasp the facts concerning the general equation, which should not even be written until the examples have been understood. Never should any new concept be introduced save through special cases. The soundness of this pedagogic principle is so incontestible that one marvels at the small part it plays in our texts.

But to return: there are parts of the First Statics which are open to criticism. In proving the theorems of Pappus for curves, would it not be better to employ exact (though elementary) limit statements than to say (page 224): "since the proposition holds for the sum of all the rectangles, it will hold for the area of the curve itself?" Or (page 228): "the solid figure formed . . . will closely coincide with the sphere. We therefore infer that the surface of the sphere itself is equal to $4\pi R^2$." It might be well in some proofs to point out what statements are not based on earlier principles but are merely assumed as true; thus it would be interesting to question whether step (1) in Stevin's solution (page 62, example 5) need be admitted unless perpetual motion is supposed impossible. It is somewhat careless to equate a pure number to a number of dimensional units, as (page 238) "$225/80 = 2.81$ feet;" or (page 302) "$125 \cos 40°/\sin 65° = 106$ lbs. nearly." * And such experiments as that on page 75 seem too clumsy to be useful.

The following misprints have been noted: page 76, example 33, for 30 read 31; page 114, top, for orignial read original; page 123, bottom, for enses read senses; page 187, bottom, for revolving read resolving; page 207, figure, for interior B read B'; page 285, frame diagram, for one b read b'; page 338, line 4, for *e* read *be*. The reviewer would also suggest more paragraph headings. American students should be informed: whether (page 144) a "piece of cotton" is a thread; (page 201) that the given weight of a penny is not that of a cent; and that an English "trapezium" (page 204 et seq.) is an American "trapezoid."

Professor Martin's Text-book of Mechanics presents, on the other hand, a purely mathematical treatment, using no calculus in the first volume but introducing it gradually in the second.

The Statics begins with the parallelogram of forces, assumed as an experimental datum, and treats the composition and resolution of forces, couples, conditions for equilibrium in two dimensions, and centers of gravity (rather briefly, without the theorems of Pappus). A summary of methods is followed by the consideration of various machines: lever, wheel and axle, systems of pulleys, inclined plane and wedge, bent lever balance, differential wheel and axle, and platform scales.

* The same criticism applies to the other text under review; e. g., Vol. I, p. 2; Vol. II, p. 19.

Except for an appendix on three-dimensional structures, and problems for review, the rest of the volume is given to the graphical methods used for centroids, resultants, and framed structures.

In kinematics the composition and resolution of velocities and accelerations in the motion of a particle are followed by space-time and velocity-time curves, the composition of simple harmonic motions, and a brief treatment of the translation and rotation of rigid bodies. The author begins kinetics with some explanatory comments on Newton's laws of motion, follows with a commendable digression on the theory of dimensions; and discusses the free translation of a particle or mass center for various forces, constrained motion, and the rotation of a rigid body, moments of inertia being introduced very naturally. He then treats work and energy, and the application of the principle of work to machines; and concludes with a chapter on impact, and a good set of review problems.

Evidences that the text is the work of a teacher appear throughout both volumes; as, for example, in the lucid explanation of the angle of repose, and in the facility with which practical problems are introduced early in the first volume; or, again, in the excellent method of analyzing very fully many typical problems (although one might wish to see more of these examples used to introduce the discussion of principles). With such a text the student must, in order to solve problems, gain an understanding of the subject by study of the solved problems and general theory; while some mathematical texts give him such explicit working rules that he can mechanically follow the rules without understanding the theory, with the result that, when the memorized rule is forgotten, he has nothing left. Still worse in the reviewer's opinion is the tendency of rules to deprive the student of the necessity for formulating methods independently; for one glaring fault of our educational system is that it cultivates so little originality and ability to think. Trained men are good, but educated men are better! Working rules have their place (along with tables) in practice, but in teaching they should be used with moderation.

There is of course, no protest against collecting the results of discussions, as the author has done in several places, thereby adding further to the success of his effort (mentioned in the preface) to make the book thoroughly teachable. In fact, the reviewer believes that as a text this work ranks with the best

previous books, despite a number of criticisms which must be stated.

For one thing, the desirability of presenting at the outset all the material of the introduction, which precedes statics, may be questioned; but more serious is a logical incompleteness similar to that which pervaded our elementary geometries before the modern revision. For example, just as we used to "prove" theorems concerning the area of a circle, which had never been defined (unless by a meaningless phrase), so here (pages 36, 43), although neither the acceleration nor any component has been defined for curvilinear motion, the author proves that $\alpha = \sqrt{\alpha_x^2 + \alpha_y^2}$ and not d^2s/dt^2. Also (page 7) we find assertions concerning velocity when $\Delta s/\Delta t$ varies, although the word has been given a meaning only when that ratio is constant; and similar remarks apply to acceleration (page 11). It may easily confuse a student to read (page 38): "In curvilinear motion the velocity can even be constant and still there must be an acceleration," when his only definition of acceleration is (page 10): "the time-rate of change of velocity;" or (page 11), "acceleration is always the first derivative of velocity with respect to time."

Again from the mathematical standpoint it is unsatisfactory to have a body treated as consisting of a finite number of particles, without an adequate transition from the sign of summation to that of integration. Also unfortunate though less important is the use of three steps (page 13) to conclude that if $v = ds/dt$ and $\alpha = dv/dt$, then $\alpha = d^2s/dt^2$; and do the remarks (page 8) about t being "an equicrescent variable so that dt is constant" mean anything more than that we choose t as the independent variable and can therefore take dt constant if we wish? Will the remarks concerning a rolling wheel (page 62) be clear before the student has heard of the instantaneous center; and would it not be shorter and more natural (page 103) to let s denote the distance from the center of the earth? In the solved problem on page 18 (Volume I) the use of similar triangles could be replaced by trigonometric methods by merely lettering an angle. It is inaccurate to say (Volume I, page 46): "CA_1 can only equal CA_2 as $CA_2 \doteq \infty$;" or write as equal the alternative values for t_1 and for t_2 (Volume II, page 152). It might be well to state that F is regarded as constant (Volume II, page 71) in defining impulse, and (page 167) in defining work; and to insert (page 140, line 3) the words "to be proved" after are.

There are some typographical errors: Volume I, page 59, middle, for $\Sigma(Fd)$ read $-\Sigma(Fd)$; Volume II, page 33, omit to in sixth line from bottom; page 88, bottom, for ΣF read ΣF_x; page 101, line 2, insert a comma before e; page 165, figure, for ax and ay read a_x and a_y; page 189, line 7, for are read is.

Nevertheless, it must be said that each of the texts under review is well printed on the whole, and presents a very attractive appearance. And despite the extent of the criticisms here offered, both books should prove thoroughly practical, often suggestive and highly satisfactory. Neither text is indexed, but the deficiency is not serious in such a subject.

As these pages go to press, a second edition of Professor Martin's Mechanics is announced by the publishers. The early appearance of a second edition is perhaps the most substantial evidence of the success of the book.

F. L. Griffin.

SHORTER NOTICES.

Opere Matematiche di Eugenio Beltrami. Pubblicate per cura della Facoltà di Scienze della R. Università di Roma. Milano, T. I, con ritratto e biografia dell' autore, 1902, vii + 437 pp.; T. II, 1904, 468 pp. 4to.

These two volumes contain 43 memoirs of Beltrami's, ranging from 1861 to 1873, and an epilogue, by Cremona which serves as an introduction. Among the papers included are those by which Beltrami earned his world-wide fame and established once for all his claim to be regarded as one of the founders of the modern subject of differential geometry. There are the memoirs on differential parameters, on the complex variable spread over an arbitrary surface, the memoirs on non-euclidean geometry, referring to ordinary and hyperspaces, the theory of geodesic lines, including the famous solution of the problem of geodesic representation of a given surface on the plane. The second volume contains also an elaborate study of the kinematics of a fluid.

If we except, perhaps, the last-named voluminous research, it may be said that the main results arrived at by Beltrami are now generally known. They have found their way into the more extensive text-books; so that we need not repeat what is, or may be, familiar to every student. Nevertheless the present

edition is highly to be welcomed. Through the necessary process of condensation an author's results often lose much of their freshness, so that the careful student will always be glad to revert to the original work. Thus the beautiful theorem concerning the deformation of an arbitrary surface to which a normal system of rays is rigidly connected comes out quite naturally where it stands (page 121 of the present edition, volume I), while the much shorter proof by which Beltrami's development has since been replaced has a somewhat artificial aspect, and fails to suggest how one may arrive at such a statement. Numerous theorems, to be found in text-books without any references, were probably discovered by Beltrami. At least Beltrami himself (who had a vast knowledge of the literature, foreign as well as Italian, and was most careful in bestowing due credit on others) apparently considered them as new. And there are many other interesting theorems still as fresh and new as they were when published for the first time.

A peculiar charm of these writings, not often to be found in work of high originality, lies in the simplicity of Beltrami's exposition. He was evidently anxious to meet the needs of his readers, and not to presuppose unnecessarily an amount of knowledge on the part of the student that in most cases, unfortunately, is not at hand. It is to be regretted that the mass of material by which nowadays all editors of mathematical journals are overwhelmed tends more and more to make this pleasant manner of writing unfeasible.

Of minor details we may point out one that has already been emphasized by Herr von Mangoldt. In one of Beltrami's earlier papers (volume I, page 75) — where, judging from its title, nobody would expect such a thing — the chief properties of a figure were studied which, fourteen years later (viz., 1879), was rediscovered and fully discussed by C. Stéphanos. The figure is now generally known under the name of desmic configuration, given to it by the latter eminent geometer. Beltrami seems also to have been the first to determine the apparent size of a surface of the second degree. His paper on this subject (the fourth in the present edition) is dated 1863, and so precedes by twenty years the corresponding publication of H. A. Schwarz (Gesammelte Werke II, page 312).

In Beltrami's work geometrical ideas are prominent throughout. His method, however, was exclusively that of analysis. It would seem that the so-called pure geometry did not attract

him much. He certainly saw the traps into which contemporaneous writers had fallen and was fully aware of the limited range of the synthetic method.

For further information we refer the reader to a sketch of Beltrami's life-work by E. Pascal, published in the *Mathematische Annalen* (volume 57, 1903, pages 65–107), and thereby made generally accessible.

The editors, who modestly disappear behind their work, have taken great pains that the edition of the writings of their illustrious compatriot should appear in a dignified form. It will, for example, be difficult to find misprints. The paper is excellent, as also is the printing, which was done in the Tipografia Matematica di Palermo.

E. STUDY.

Études sur les Angles imaginaires. Par GEORGES DE LAPLANCHE. Paris, A. Hermann, 1908. 8vo. 135 pp. 3 francs.

THIS volume devotes considerable space to the development of the ordinary formulas, the computations, and the construction of graphs of trigonometric functions of complex numbers. The imaginary angle spoken of is merely the complex argument of these trigonometric functions, and in no way is it connected with the geometric conception of angle. De Moivre's formula is made the basis of all the developments, with no attempt at rigorous proof, while the graphical treatment rests on the usual Argand diagram. A gross misstatement is made on page 35 and repeated on page 36, to the effect that in sinh x and cosh x (written sh x, ch x), defined thus

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}), \quad \cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

the argument x is the length of the arc of an equilateral hyperbola, measured from its vertex. The equations of this hyperbola may be written

$$X = \cosh x, \quad Y = \sinh x,$$

whence

$$s = \int_0^x \sqrt{1 + 2\sinh^2 x}\, dx.$$

This is obviously not equal to x. The author seems to have

been misled in trying to extend the circular argument of cos x, sin x. This, it is true, may be taken as an arc, but it has long been known that in order to extend the analogy to the hyperbolic functions it is necessary to take as argument the ratio of the *sector* to one half the square of the radius. The baneful influence of considering trigonometric functions as lines depending on a linear argument is evidently not yet extinguished.

However, the author makes no use of his erroneous statement, but depends solely on series and De Moivre's formula, so that his formulas are correct enough.

The purpose of the work, scarcely realized in its treatment, is stated thus:

"Nous nous proposons de rechercher si le nombre imaginaire a des lignes trigonométriques : sinus, cosinus, tangentes, circulaires ou hyperboliques. . . . Nous en établissons la trigonométrie. Nous montrons de nouveaux moyens pour résoudre certains problèmes."

JAMES BYRNIE SHAW.

Vorlesungen über bestimmte Integrale und die Fourierschen Reihen. Von J. THOMAE. Leipzig, Teubner, 1908. 8vo. vi + 182 pp. 7.80 Marks.

This book undertakes to give a rather general view of the subjects mentioned in its title. It is somewhat more on the order of a descriptive course than either a systematic development or a practical handbook. Thus, in the first fifty-seven pages the student will see unfolded before him a view of the main theorems with some regard to the dangerous places near them. He will learn that there are such functions as Euler's $E(x)$, Dirichlet's function, Riemann's classic function (x), written here $r(x)$, Riemann's convergent series with an infinity of discontinuities

$$f(x) = r(x) + \frac{r(2x)}{2^2} + \frac{r(3x)}{3^2} + \cdots.$$

He will find that there is a definition of integral as the limit of a sum, which indeed is suggested by the inversion of a differentiation, and that under this definition many functions become integrable, for example those of Riemann mentioned above. The first mean value theorem he finds to hold equally for such functions, and the second mean value theorem is developed. He also finds that sometimes the variable may be changed,

sometimes not. Differentiation as to a parameter may be possible, but an example is given showing that this is not always the case. No indication is given of the conditions under which such differentiation is permissible.

The reader will be disappointed if he looks for a systematic treatment, even an elementary one. The classes of integrable functions are not clearly indicated, even if the two important ones are given. The tests for integrability of improper integrals are rather left to be surmised than clearly set forth; no really practical exemplification is given. The classes of functions which have improper integrals are not distinguished. In any particular case the student would feel that in a country known to have precipices he has neither guide nor map.

A systematic development of the properties of definite integrals, with the modifications necessary in improper integrals, is nowhere given; nor are the methods of evaluating forms more than hinted at by a few standard examples. The whole question of inversion of differentiation and integration, and of inversion of iterated integration is barely touched on. This is the more disappointing as the author says in his preface: "Aber als Hauptziel der Vorlesungen ist die wirkliche Auswertung bestimmter Integrale anzusehen."

As applications Fourier's series and Euler's integral are considered. Thirty-eight pages are devoted to Fourier's series, including twelve pages on vibrating cords. Illustrations are given of the determination of the coefficients for certain developments, among them the Riemann series mentioned above. The student is shown by examples that a series cannot always be integrated term by term, but is integrable term by term when uniformly convergent. It is shown that the sine series is differentiable term by term. What happens in the case of the cosine series is not mentioned. The conditions of convergence of Dirichlet are proved, and Schwarz's example is given to show that continuity merely is not a sufficient condition. Other questions of convergence, particularly uniform convergence, are barely hinted at, so that the matter of integrability is left open. No attempt is made to set forth the modern widening of Dirichlet's conditions.

The only example given is the solution of the equation of a vibrating string

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

The usual particular cases are stated and rather fully discussed, in connection with musical strings.

Again the student desiring more than a cursory view of the subject would be disappointed. The development of the cosine series is awkward. The essential difference between representing x, say by a sine series, or a cosine series, or a mixed series, is not clearly exhibited. This is always a point of confusion to the student. The manner in which the various approximation curves give the graph of the function is not intimated. The classes of developable functions are left under the narrow Dirichlet conditions. No mention is made of Fourier's classic developments in the theory of heat, where indeed the really practical value of Fourier series shows at its best.

Twenty-eight pages are devoted to double integrals, followed by ten pages on Fourier's integrals. The practical value of the latter is barely mentioned. Eighteen pages are given to the elementary formulas of the B and the Γ functions.

Thirty-two pages are given to the integration of expressions in two independent variables,

$$dw = pdx + qdy.$$

This introduces naturally the functions of a complex variable. It is shown that these lead to solutions of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

It is regrettable that no modern text-book exists which presents the subject of definite integrals and their applications in a complete and practical way. Of course treatises on the theory of functions of a real variable usually contain the general theory, but they also contain much extraneous matter. With the growing use and importance of harmonic analysis particularly, such a text becomes more necessary. The present volume, interesting and quite smoothly carried out as it is, does not meet this demand. It is for the general reader rather than the student who desires to become skilful with these useful tools. Students of mathematical physics must look elsewhere for what they want.

JAMES BYRNIE SHAW.

NOTES.

THE annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held in the Rogers Building of the Massachusetts Institute of Technology, 491 Boylston Street, Boston, opening on Tuesday afternoon, December 28. Wednesday morning will be devoted to a joint session with Section A of the American Association. Arrangements have been made for a joint smoker with the Association of Mathematical Teachers in New England on Tuesday evening, and for the usual dinner of the Society on Wednesday evening.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary. The Register is widely circulated and it is desirable that the information which it contains should be accurate and reliable.

THE twenty-fifth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday, December 31, 1909, and Saturday, January 1, 1910. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section on or before December 10, 1909. It is desired that all abstracts be printed in advance of the meeting.

THE seventeenth regular meeting of the San Francisco Section will be held at Stanford University, February 26, 1910.

THE concluding (October) number of volume 31 of the *American Journal of Mathematics* contains the following papers: "The asymptotic representation of the elliptic cylinder functions," by W. MARSHALL; "A theory of invariants," by L. E. DICKSON; "Symmetric binary forms and involutions (continued)," by A. B. COBLE; "A theory of geometrical relations," by A. R. SCHWEITZER.

UNIVERSITY OF PARIS. The following mathematical courses are announced for the semester beginning November 3, 1909: —By Professor G. DARBOUX: Infinitesimal geometry, particu-

larly triply orthogonal systems, two hours. — By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours. — By Professor L. RAFFY: Twisted curves, properties of curves traced on surfaces, one hour. — By Professor E. BOREL: Definite integrals, with certain applications, two hours. — By Professor P. PAINLEVÉ: General laws of equilibrium and motion, two hours. — By Professor P. APPELL and Mr. E. BLUTEL: General mathematics, first part, one hour. — By Professor H. POINCARÉ: Movements of the heavenly bodies around their center of gravity, two hours. — By Professor J. BOUSSINESQ: Mechanical theory of light, two hours.

In the Ecole Normale. By Professor J. TANNERY: Differential and integral calculus. — By Professor L. RAFFY: Applications of analysis to geometry. — By Professor E. BOREL: Mathematics. — By Professor J. HADAMARD: Mathematics.

Weekly conferences will be held by Professor RAFFY and Messrs. CARTAN, BLUTEL, and SERVANT.

For the second semester the following courses are announced: — By Professor E. PICARD: The principal developments in series occurring in mathematical physics. — By Professor E. GOURSAT: Differential equations, partial differential equations. — By Professor P. PAINLEVÉ: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics. — By Professor P. APPELL: Analysis and mechanics. — By Professor J. BOUSSINESQ: Theory of light. — By Professor G. KOENIGS: General theory of mechanisms.

In the Ecole Normale. By Professor J. TANNERY: Differential and integral calculus. — By Professor E. BOREL: Mechanics. — By Professor J. HADAMARD: Mathematics.

THE president of the Euler commission, Professor F. Rudio, reported at the Salzburg meeting of the Deutsche Mathematiker-Vereinigung that the sixth international congress for the science of insurance and the association of German engineers had each contributed five thousand francs toward the publication of the works of Euler. Subscriptions have now been received for more than 275 sets, and in addition cash contributions of more than 130,000 fr. have been acknowledged. This sum has been conservatively invested, and it is estimated that it will earn at least 30,000 fr. before it need be paid out. The total assets are now over 450,000 fr., and a larger edition than was originally planned (400 sets) will probably be undertaken.

THE index of Euler's writings, prepared by Dr. G. ENESTRÖM, is now in press, and will appear in the spring of 1910 as a supplementary volume of the *Jahresbericht der deutschen Mathematiker-Vereinigung*. An appendix to volume 10 of the *Jahresbericht*, containing additions and an index to Professor BURKHARDT's report on oscillating functions, is also in preparation.

THE first Scandinavian congress of mathematicians was held at Stockholm, September 22–25, 1909, under the presidency of Professor G. MITTAG-LEFFLER. The following papers were read: By A. C. BJERKNES, "Mathematical investigation of meteorological problems"; by I. FREDHOLM, "On integral equations"; by HJELMSLER, "Some geometric principles"; by H. v. KOCH, "Systems of equations with an infinite number of variables"; by E. LINDELÖF, "Picard's theorem in the theory of functions"; by G. MITTAG-LEFFLER, "Arithmetic formulation of the theory of functions"; by E. PHRAGMÉN, "Theory of integral functions of the second order"; by C. STÖRMER, "Mathematical treatment of polar and magnetic storms"; by SUNDMAN, "Real singularities in the problem of three bodies"; by H. G. ZEUTHEN, "Some methods in the texts on geometry"; by K. BIRKELAND, "On irregular integrals of linear differential equations"; by A. C. BJERKNES, "On a modified form of the Maxwell lines"; by BOHR, "On Dirichlet's summation formula, particularly on the boundary of the region of convergence"; by F. BRODÉN, On the so-called Richard paradoxes"; by BUCHT, "Reduction of cyclic to metacyclic equations"; by CHARLIER, "Mathematical desiderata in the theory of probabilities"; by EKMAN, "Questions of stability in hydrodynamics"; by P. HANSEN, "On Taylor's series"; by HESSELBERG, "An application of the lines of hydrodynamic continuity to the determination of the vertical motion of the atmosphere"; by E. A. HOLMGREN, "On systems of linear partial differential equations with real characteristics"; by C. JUEL, "Note on a non-analytic surface of revolution"; by LINDEBERG, "On the proper name for the so-called Weierstrass functions in the calculus of variations"; by MELLIN, "A uniform theory for gamma functions and hypergeometric functions"; by J. MOLLERUP, "Regions of convergence for orthogonal functions"; by NÖRLUND, "Linear difference equations"; by C. W. OSEEN, "A partial differential equation in mathematical physics"; by A. PALMSTROM, "On cyclic numbers"; by STRIDSBERG, "Some arithmetic properties of certain

transcendental functions," and "A study of the arithmetic properties of the integrals of algebraic differential equations"; by A. THUE, "Some new arithmetic properties of algebraic numbers with an application to Diophantine equations"; by WICKSELL, "Mathematical deduction of investments"; by VON ZEIPEL, "On secular perturbations of comets."

THE association of Swiss mathematical teachers held its eleventh meeting at Solothurn, October 10, 1909. Four formal papers were read, reports were heard and plans adopted regarding the activity of the international committee on mathematical instruction, and a committee of fifteen was appointed to consider and report on a general plan for vacation courses in mathematics.

AT the meeting of the New York section of the Association of teachers of mathematics in the middle states and Maryland held on November 12 the following papers were read: By E. R. von NARDROFF, "Mathematics for service"; by W. H. METZLER, "Mathematics for training and culture." Two meetings will be held during the first half of the year 1910.

THE committee of fifteen appointed by the Central association to consider the teaching of geometry has nearly completed its report, and expects to present it during the winter.

DR. A. ADLER, of the technical school at Vienna, has been promoted to an associate professorship of mathematics.

PROFESSOR P. STÄCKEL, of the technical school at Hanover, and Dr. G. ENESTRÖM, librarian of the University of Stockholm, have been elected honorary members of the society of Swiss naturalists.

DR. G. FABER, of the technical school at Karlsruhe, has been appointed associate professor of mathematics at the University of Tübingen.

AN honorary doctorate of philosophy has been conferred by the University of Leipzig upon Professor I. FREDHOLM, of the University of Stockholm.

DR. E. HILB, of the University of Erlangen, has been appointed associate professor of mathematics at the University of Würzburg.

PROFESSOR F. KLEIN, of the University of Göttingen, has been elected chairman of the board of directors of the German museum at Munich.

PROFESSOR R. LEHMANN-FILHÉS, of the University of Berlin, has been appointed honorary professor of mathematics.

PROFESSOR C. NEUMANN, of the University of Leipzig, has received the title of Geheimrat.

THE University of Heidelberg recently publicly celebrated the eightieth birthday of Professor MORITZ CANTOR, who was born at Mannheim, August 23, 1829.

THE Royal Society has awarded its Copley medal to Dr. G. W. HILL.

PROFESSOR G. E. FISHER, of the University of Pennsylvania, has been appointed Dean of the College of Arts and Sciences.

AT the inauguration of President Bryan, of Colgate University, the degree of doctor of science was conferred upon Professor F. C. FERRY, of Williams College.

PROFESSOR R. J. ALEY, of the University of Indiana, has been appointed Superintendent of Public Instruction of the State of Indiana.

AT the University of Pennsylvania Professor I. J. SCHWATT has been promoted to a full professorship of mathematics.

AT Purdue University Professor C. H. BECKETT has been promoted to an associate professorship of mathematics.

AT Oberlin College Miss M. E. SINCLAIR has been promoted to an associate professorship of mathematics.

DR. C. A. MOORE, of the University of Cincinnati, has been promoted to an assistant professorship of mathematics.

AT Syracuse University Professor W. G. BULLARD has been promoted to a full professorship of mathematics.

AT the University of Michigan Dr. J. W. BRADSHAW has been promoted to an assistant professorship of mathematics.

At the University of Nebraska Dr. W. C. Brenke has been promoted to an associate professorship of mathematics.

At Wesleyan University Mr. B. H. Camp has been promoted to an associate professorship of mathematics.

Mr. A. F. Carpenter has been appointed instructor in mathematics at the University of Washington.

At Stevens Institute Mr. R. F. Deimel has been promoted to an assistant professorship of mathematics.

At Indiana University Dr. U. S. Hanna has been promoted to an associate professorship of mathematics.

At the University of Pennsylvania Professor I. J. Schwatt has been promoted to a full professorship of mathematics.

Dr. Clara E. Smith has been appointed instructor in mathematics at Wellesley College.

Dr. Ruth G. Wood, of Smith College, has been promoted to an associate professorship of mathematics.

Professor J. C. Stone, of the Michigan State Normal College, has been appointed head of the department of mathematics at the New Jersey State Normal School, Montclair, N. J.

Mr. F. C. Moore has been appointed assistant professor of mathematics at the New Hampshire College of Agriculture and Mechanic Arts.

Dr. H. L. Slobin has been appointed assistant in mathematics at the University of Minnesota.

Dr. O. Dunkel, of the University of Missouri, has been granted leave of absence to study in Europe.

Dr. M. O. Tripp, of the College of the City of New York, has been granted leave of absence for the present academic year, which he will spend abroad.

Professor R. D. Carmichael, of the Alabama Presbyterian College for Men, is spending a year's leave of absence in study at Princeton University.

Professor B. F. Yanney, of Mount Union College, is spending a year's leave of absence in study at the University of Chicago.

Professor J. H. Van Amringe, head of the department of mathematics in Columbia University, and dean of the college, will retire from active service at the end of the present academic year, when he will have completed fifty years of service in the university and reached his seventy-fifth birthday.

Professor Hugh Blackburn, emeritus professor of mathematics at the University of Glasgow, died October 9, 1909, at the age of 86 years. He held the chair of mathematics from 1849 to 1879.

Major General O. O. Howard, U. S. A., who died October 26, 1909, at the age of 78 years, was associate professor of mathematics at the Military Academy at West Point from 1857 to 1861.

Catalogues of second-hand mathematical books: Gustav Fock, Schlossgasse 7, Leipzig, catalogue no. 361, 2,600 titles in pure and applied mathematics. — Koehler's Antiquarium, Kurprinzstrasse 6, Leipzig, catalogue no. 581, about 1,100 titles in history, encyclopedia, and bibliography of exact sciences; catalogue no. 582, about 3,800 titles in pure and applied mathematics. — A. Hermann, rue de la Sorbonne 6, Paris, catalogue no. 99, 1,767 titles in pure and applied mathematics. — Basler Buch- und Antiquariatshandlung, catalogue no. 330, 460 titles in pure and applied mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMSON (E.). Eine zeichnerisch wohl verwertbare Konstruktionsweise des allgemeinen Kegelschnittes. (Progr.) München, 1909. 8vo. 19 pp.

AQUINO (L. D'). Nota sulla teoria degli integratori. Napoli, D'Auria, 1909. 8vo. 16 pp.

BEUTEL (E.). Algebraische Kurven. Teil I: Kurvendiskussion. Leipzig, 1909. 8vo. 147 pp. Cloth. M. 0.80

BEZLOJA (A.). Das Schliessungsproblem. (Progr.) M.—Weisskirchen, 1908. 8vo. 16 pp.

BIANCHI (L.). Lezioni di geometria differenziale. Vol. III: Teoria delle trasformazioni delle superficie applicabili sulle quadriche. Pisa, Spoerri, 1909. 8vo. 5 + 350 pp. L. 15.00

BLAINE (R. G.). The calculus and its applications. A practical treatise. London, Constable, 1909. 8vo. 332 pp. Cloth. 4s. 6d.

BÜCHER, neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet.) Mitgeteilt Sommer 1909. Leipzig, Hinrich, 1909. 8vo. Pp. 25–46. M. 0.30

BURALI-FORTI (C.) e MARCOLONGO (R.). Elementi di calcolo vettoriale, con numerose applicazioni alla geometria, alla meccanica e alla fisica matematica. Bologna, Zanichelli, 1909. 8vo. 5 + 174 pp. L. 5.00

——. Omografie vettoriali, con applicazioni alle derivate rispetto ad un punto ed alla fisica-mathematica. Torino, Petrini, 1909. 8vo. 11 + 115 pp. L. 4.00

CAIN (W.). A brief course in the calculus. 2nd edition. London, Blackie, 1909. 8vo. Cloth. 6s.

CHATELAIN (E.). Relations entre les nombres de classes dans les différents ordres de formes binaires quadratiques d'un déterminant donné. (Diss.) Zurich, 1908. 8vo. 82 pp.

ENGEL (F.). Hermann Grassmann. Leipzig, 1909. 8vo. 15 pp. M. 0.80

EISENHART (L. P.). A treatise on the differential geometry of curves and surfaces. Boston, Ginn & Co., 1909. 8vo. 11 + 474 pp. Cloth. $4.50.

FOUËT (E. A.). Leçons élémentaires sur la théorie des fonctions analytiques. 2e édition, entièrement refondue. Vol. 2: les fonctions algébriques; les séries simples et multiples; les intégrales. Paris, Gauthier-Villars, 1910. 8vo. 11 + 267 pp. Fr. 9.00

FRANKENBACH (F. W.). Lineare Erzeugung der Kegelschnitte und auf ihr beruhende Ableitung der Kegelschnittsgleichungen. Beitrag zur Lehre von den Kurven 2ter Ordnung. Liegnitz, 1909. 8vo. 49 pp. M. 1.00

FRICKE (R.). Hauptsätze der Differential- und Integral-Rechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. 5te Auflage. Braunschweig, Vieweg, 1909. 8vo. 15 + 219 pp. M. 5.80

GOEDE (G. DE). Vollständige Lösung des grossen Fermatschen Satzes. Antwort auf das Preisausschreiben der königlichen Gesellschaft der Wissenschaften zu Göttingen von 1908. Deventer (Niederlande), Dixon, 1909. 8vo. 11 pp. M. 0.40

GÜNTZEL (F.). Ueber Gruppierungen und Realitätsverhältnisse gewisser Punkte bei Raumkurven vierter Ordnung erster Species. (Diss.) Jena, 1909. 8vo. 62 pp.

HAUSSNER (R.). See SCHERING (E.).

HEIDELBERG (P.). Allgemeiner Beweis des grossen Fermatschen Satzes. Wiesbaden, Staadt, 1909. 8vo. 3 pp. M. 0.80

KARNASCH. Beweis für den Fermatschen Satz, dass die Gleichung $x^n + y^n = z^n$ in ganzen Zahlen nicht lösbar ist, wenn $n > 2$. Berlin, Mayer, 1909. 8vo. 7 pp. M. 0.80

KLEINSCHMIDT (M.). Elementarer Beweis des Fermatschen Satzes. Rostock, Boldt, 1909. 8vo. 6 pp. M. 1.50

KLOBASA (C.). Ueber parallelepipedische Zahlen. (Progr.) Troppau, 1909. 8vo. 6 pp.

LEBON (E.). Henri Poincaré. Biographie, bibliographie analytique des écrits. Paris, Gauthier-Villars, 1909. 8vo. 8 + 80 pp. Fr. 7.00

LICHTENSTEIN (L.). Zur Theorie der gewöhnlichen Differentialgleichungen und der partiellen Differentialgleichungen zweiter Ordnung. Die Lösungen als Funktionen der Randwerte und der Parameter. (Diss.) Berlin, 1909. 8vo. 40 pp.

LOVE (A. E. H.). Elements of the differential and integral calculus. Cambridge, University Press, 1909. 8vo. 222 pp. Cloth. 5 s.

MARCOLONGO (R.). See BURALI-FORTI (C.).

MANTZ (O.). Ebene Inversionsgeometrie. (Progr.) Basel, 1909. 4to. 62 pp.

MEYER (C.). Zur Theorie des logarithmischen Potentials. Berlin, 1909. 8vo. 68 pp. M. 2.00

NOETHER (F.). Ueber rollende Bewegung einer Kugel auf Rotationsflächen. (Diss., Munich.) Leipzig, Teubner, 1909.

PAULY (J.). Notions élémentaires du calcul différentiel et du calcul intégral. 2e edition. Paris, 1909. 8vo. Fr. 6.80

PICKFORD (A. G.). Elementary projective geometry. Cambridge, University Press, 1909. 8vo. 268 pp. Cloth. 4s.

PUCCINI (A.). Le configurazioni piane regolari d'indice 3. Padova, Prosperini, 1909. 8vo. 65 pp.

SCHEFFERS (G.). See SERRET (J. A.).

SCHERING (E.). Gesammelte mathematische Werke. Herausgegeben von R. Haussner und K. Schering. 2ter (Schluss-) Band. Berlin, Mayer, 1909. 8vo. 8 + 472 pp. M. 25.00

SCHERING (K.). See SCHERING (E.).

SCHMALL (C. U.). A first course in analytic geometry. London, Blackie, 1909. 8vo. Cloth. 6s.

SCHMIDT (J.). Der Infinitesimalkalkül. (Progr.) Wien, 1909. 8vo. 15 pp.

SCHUR (F.). Grundlagen der Geometrie. Leipzig, Teubner, 1909. 8vo. 10 + 192 pp. M. 7.00

SERRET (J. A.). Lehrbuch der Differential- und Integralrechnung. Nach Axel Harnacks Uebersetzung. 3te Auflage. Neu bearbeitet von G. Scheffers. 3ter (Schluss-) Band. Differentialgleichungen und Variationsrechnung. Leipzig, Teubner, 1909. 8vo. 12 + 658 pp. Cloth. M. 13.00

SMITH (D. E.). Rara Arithmetica. Second edition. Boston, Ginn & Co., 1909. Large 8vo. 16 + 507 pp. Cloth. $4.50

TEIXEIRA (J. G.). Obras sobre matematica publicadas por ordem do governo Portuguès. Vol. V : Traité des courbes spéciales remarquables planes et gauches. Traduit de l'Espagnol, revu et très augmenté. Coimbra, 1909. 4to. 497 pp. M. 16.00

THAER (C.). Eine Ausdehnung der Galoisschen Theorie auf algebraische Gleichungen mit mehrfachen Wurzeln. (Habilitationsschrift.) Jena, 1909. 8vo. 36 pp.

VAERTING (M.). Die hyperbolischen Funktionen und das Dreieck. Bonn, 1909. 8vo. 22 pp. M. 1.00

VILLAFANE Y VIÑALS (J. M.). Tratado de análisis matemático (algebra superior). 3a parte ; Teoría general de ecuaciones. Barcelona, Caridad, 1909. 284 pp. P. 10.00

VOGT (W.). Synthetische Theorie der Cliffordschen Parallelen und der linearen Linienörter des elliptischen Raumes. (Habilitationsschrift.) Karlsruhe, 1909. 8vo. 8 + 58 pp.

II. ELEMENTARY MATHEMATICS.

AMODEO (F.). Complementi di analisi algebrica elementare, con appendice sulle sezioni coniche. Parte II del Vol. II degli *Elementi di matematica*, ad uso del 20 biennio degli istituti tecnici. Napoli, Pierro, 1909. 8vo. 11 + 284 + 28 pp. L. 3.00

AUBRY. See BALL (W. R.).

BALL (W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. 2e édition française, traduite d'après la 4e édition anglaise et enrichie de nombreuses additions par J. Fitz-Patrick. 3e partie, avec addition de MM. Margossian, Reinhart, Fitz-Patrick et Aubry. Paris, Hermann, 1909. 16mo. 370 pp. Fr. 5.00

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Ausgabe für höhere Mädchenlehranstalten, zugleich Unterstufe für Lyzeen und Studienanstalten. Leipzig, Teubner, 1909. 8vo. 7 + 310 pp. M. 3.00

BELIKOW (A.) und NATHING (A.). Lehrbuch der Algebra nebst einer Sammlung von Uebungsaufgaben. Für Gymnasien und Realschulen bearbeitet. 3te Auflage. 1ter Teil. St. Petersburg, Eggers, 1909. 8vo. 7 + 166 pp. M. 3.00

BOURLET (C.). Eléments d'algèbre, contenant 631 exercices et problèmes, rédigés conformément aux programmes de l'enseignement secondaire 1er et 2e cycles, classes de troisième A, seconde et première A et B de l'enseignement secondaire et aux programmes de l'enseignement primaire supérieur. 6e édition, revue. Paris, Hachette, 1909. 16mo. 314 pp. Fr. 2.00

BOUVART (C.) et RATINET (A.). Règles et formules usuelles servant de supplément aux tables de logarithmes. A l'usage des candidats au baccalauréat et aux Ecoles polytechniques et de Saint-Cyr. Paris, Hachette, 1909. 8vo. 47 pp. Fr. 0.60

BUSH (W. N.) and CLARKE (J. B.). The elements of geometry. New York, Silver, Burdette & Co., 1909. 12mo. 12 + 355 pp. Cloth. $1.25

CARSLAW (H. S.). Plane trigonometry: an elementary text-book for the higher classes of secondary schools and for colleges. London, Macmillan, 1909. 8vo. 324 pp. Cloth. 4s. 6d.

CLARKE (J. B.). See BUSH (W. N.).

COHN (B.). Tafeln der Additions- und Subtraktions-Logarithmen auf sechs Dezimalen. Leipzig, Engelmann, 1909. 8vo. 4 + 63 pp. M. 4.00

CRACKNELL (A. G.). See WORKMAN (W. P.).

CRANTZ (P.). Lehrbuch der Mathematik für höhere Mädchenschulen und Lyzeen. Auf Grund der neuen Lehrpläne bearbeitet. 1ter Teil: Für höhere Mädchenschulen. 2te Auflage. Leipzig, Teubner, 1909. 8vo. 6 + 177 pp. M. 2.40

CROOK (C. W.). Notes of lessons on arithmetic, mensuration, and practical geometry. London, Pitman, 1909. 8vo. 176 pp. 3 s.

DUPUIS (J.). Table de logarithmes à sept décimales. 14e tirage. Paris, Hachette, 1909. 8vo. 11 + 581 pp. Fr. 8.50

FAWDRY (R. C.). Problem papers in mathematics. On the lines of the examination by the Civil Service Commission. With revision papers in trigonometry, etc. London, Macmillan, 1909. 8vo. 248 pp. 4s. 6d.

FITZ-PATRICK (J.). See BALL (W. R.).

FOURREY (E.). Curiosités géométriques. 2e édition. Paris, Vuibert, 1909. 8vo. 8 + 431 pp.

FREDIANI ZAVARDI (D.). Lezioni di algebra elementare, fatte alle alunne della prima classe normale. Firenze, Landi, 1909. 16mo. 87 pp. L. 1.20

FRENZEL (C.). See MEHLER (F. G.).

GÖTTING (E.). See BEHRENDSEN (O.).

HENRICI (J.) und TREUTLEIN (P.). Lehrbuch der Elementar-Geometrie 1ter Teil. Gleichheit der Gebilde in einer Ebene und deren Abbildung ohne Massänderung (nebst einer Aufgabensammlung). 4te Auflage. Leipzig, Teubner, 1909. 8vo. 8 + 139 pp. M. 2.40

HOČEVAR (F.). Lehr- und Uebungsbuch der Arithmetik für Gymnasien, Realgymnasien und Realschulen. Unterstufe (I, II und III Klasse). 7te nach den neuen Lehrplänen umgearbeitete Auflage. Vienna, Tempsky, 1909. Kr. 2.10

JADANZA (N.). Trattato di geometria pratica. Dispensa 31–50 (fine). Torino, Bona, 1909. 8vo. Pp. 481-796.

LESSER (O.). Lehr- und Uebungsbuch für Unterricht in der Arithmetik und Algebra. Teil II. Ausgabe A: Für die oberen Klassen der Realanstalten. Vienna, Tempsky, 1909. M. 3.00

——. See SCHWAB (K.).

MAHLERT (A.). See MÜLLER (H.).

MANUEL d'algèbre et de trigonométrie. Paris, Poussielgue, 1909. 16mo. 8 + 216 pp.

MARGOSSIAN. See BALL (W. R.).

MATRICULATION model answers: mathematics. Being the London University matriculation papers in mathematics from June 1906 to June 1909. London, Clive, 1909. 8vo. 160 pp. 2s.

MATTINA (P.). Aritmetica goniometrica: introduzione alla trigonometria. Palermo, Virzi, 1909. 8vo. 15 + 273 pp. L. 5.00

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Bearbeitet von A. Schulte-Tigges. Ausgabe B. Oberstufe, 2ter Teil: Arithmetik mit Einschluss der niederen Analysis, Trigonometrie und Stereometrie. Für die oberen Klassen höherer Lehranstalten bearbeitet unter Mitwirkung von C. Frenzel. Berlin, Reimer, 1909. 8vo. 7 + 169 pp. M. 1.50

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. (H. Müller's mathematisches Unterrichtswerk.) 2ter Teil: Die Oberstufe. Ausgabe A: Für Gymnasien. 3te umgearbeitete Auflage. Leipzig, Teubner, 1909. 8vo. 14 + 306 pp. M. 3.40

MÜLLER (H.) und MAHLERT (A.). Lehr- und Uebungsbuch der Arithmetik und Algebra für Studienanstalten. Ausgabe B: Für Oberrealschul- und realgymnasiale Kurse. 1ter Teil: Bis zur Lehraufgabe der Klasse IV. Leipzig, Teubner, 1909. 8vo. 6 + 192 pp. M. 2.40

MÜLLER. See PLATH (J.).

MURRAY (D. A.). Essentials of trigonometry and mensuration, with four-place tables. New edition. New York, Longmans, 1909. 8vo. 10 + 157 pp. Cloth. $0.80

NATHING (A.). See BELIKOW (A.).

OXFORD local examinations. Papers set in arithmetic and mathematics at the senior, junior and preliminary examinations held in March and July, 1909. London, Parker, 1909. 8vo. 8d.

PLATH (J.). Lehrbuch der Mathematik zur Vorbereitung auf die Mittelschullehrer-Prüfung und auf das Abiturientenexamen am Realgymnasium. Im Anschluss an die Baltin-Maiwaldsche Seminarausgabe des Müllerschen Lehrbuches und in Verbindung mit Prof. Müller für den Selbstunterricht bearbeitet. 2te Auflage. Leipzig, Teubner, 1909. 8vo. 8 + 272 pp. M. 4.00

PREMIÈRES NOTIONS d'algèbre avec de nombreux exercices, à l'usage des écoles primaires. Solutions des exercices et problèmes. Livre du maître. Paris, Poussielgue, 1909. 12mo. 88 pp.

RADFORD (E. M.). Exercise papers in elementary algebra. London, Dent, 1909. 12mo. 120 pp. 2 s.

RATINET (A.). See BOUVART (C.).

REINHART. See BALL (W. R.).

SCHRÖDER (R.). Das Dreieck und seine Berührungskreise. Ein Uebungsgebiet aus der rechnenden Geometrie. Teil I. (Progr.) Gross-Lichterfelde, 1909. 8vo. 41 pp.

SCHULTE-TIGGES (A.). See MEHLER (F. G.).

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauche an höheren Lehranstalten. Im Sinne der Meraner Lehrpläne bearbeitet. Vol. I: Lehr- und Uebungsbuch für den Unterricht in der Arithmetik und Algebra. Wien, Tempsky, 1909. 8vo.
1ter Teil, 203 pp. M. 2.80
2ter Teil, 238 pp. M. 3.00

SCHWERING (K.). Stereometrie für höhere Lehranstalten. 3te Auflage. Freiburg, Herder, 1909.

SMITH (D. E.). The Teaching of Arithmetic. New York, Columbia University Press, 1909. 120 pp. $0.75

SUPPANTSCHITSCH (R.). Grundriss der Geometrie für Gymnasien und Realgymnasien. Heft II, für die IIIte Klasse. Wien, Tempsky. 1909. Kr. 1.70

THIEME (H.). Leitfaden der Mathematik für Realanstalten. 1ter Teil: Die Unterstufe. 4te Auflage. Leipzig, Freytag, 1909. 8vo. 135 pp. M. 1.80

TREUTLEIN (P.). See HENRICI (J.).

WIENECKE (E.). Ebene Trigonometrie mit reichem Aufgabenmaterial nebst Lösungen zum Gebrauche an Seminaren mit gewerblichen Anstalten. 2te Auflage. Berlin, Oehmigke, 1909. 8vo. 3 + 72 pp. M. 1.40

WORKMAN (W. P.) and CRACKNELL (A. G.). Introduction to the school geometry. (University Tutorial Series.) London, Clive, 1909. 8vo. 91 pp. 1 s.

—— School geometry. (University Tutorial Series.) London, Clive, 1909. 8vo. 403 pp. 3s. 6d.

III. APPLIED MATHEMATICS.

APPELL (P.). Traité de mécanique rationnelle. 3e édition, entièrement refondue. Vol. I: Statique. Dynamique du point. Paris, Gauthier-Villars, 1909. 8vo. 10 + 616 pp. Fr. 20.00

ATTWOOD (E. L.). Textbook of theoretical naval architecture. 5th edition, revised and enlarged. New York, Longmans, 1909. 12mo. 9 + 458 pp. Cloth. $2.50

BENTELI (A.). Die konstruktive Kreisperspektive. (Progr.) Bern, 1909. 8vo. 13 pp.

BROWN (H. T.). Tratado práctico de mecánica con la descripción de más de 500 movimientos mecánicos, et cétera. Versión española de E. Lozano. Barcelona, Feliú, 1909. 336 pp. P. 5.00

CAROL (J.). Résistance des matériaux appliquée à la construction des machines. Vol. I. Paris, Béranger, 1909. 8vo. 6 + 477 pp.

COURS de physique. Pesanteur et mécanique. Energie. Equilibres physiques. Unités. Mouvements vibratoires et phénomènes périodiques. Ouvrage suivi d'un recueil de 615 problèmes. Paris, Poussielgue, 1909. 8vo. 7 + 472 pp.

CULLER (J. A.). A textbook of general physics for colleges. Mechanics and heat. Philadelphia, Lippincott, 1909. 8vo. 311 pp. Cloth. $1.80

ESPURZ Y CAMPODARBE (D.). Lecciones de física general. Seguidas de apéndice sobre diversos puntos. Vol. I: Mecánica, acústica y electricidad. Avila, Jiménez, 1909. 591 pp. P. 14.00

EXERCICES de géométrie descriptive. 4e édition. Paris, Poussielgue, 1909. 8vo. 10 + 1099 pp.

FERGUSON (R. B.). Aids to the mathematics of hygiene. 4th edition. London, Bailliere, 1909. 12mo. 2s. 6d.

GIL BALLESTER (F.). Trazado de curvas circulares. Tablas trigonométricas para el trazado de curvas circulares sobre el terreno, aplicables al estudio y construcción de carreteras, ferrocarriles y canales. Barcelona, Martin, 1909. 228 pp. P. 7.00

GOLDING (H. A.). See LARARD (C. E.).

GONZALEZ GÓMEZ (R.). Elementos de geometría descriptiva para la reso lución de problemas de rectas y planos. Toledo, 1909. 70 pp.

HAMMER (M.). Untersuchungen über Hertz'sche stehende Schwingungen in Luft. (Diss.) Halle, 1909. 8vo. 54 pp.

HARDY (E.). The elementary principles of graphical statics. 2nd edition, revised and enlarged. London, Batsford, 1909. 8vo. 206 pp. Cloth. 3 s.

HARDY (G. F.). Theory of the construction of tables of mortality and of similar statistical tables in use by the actuary. London, 1909. 8vo. Cloth. M. 8.50

HAUHART (H.) und WALDNER (A.). Tracirungs-Handbuch für die Ingenieurarbeiten im Felde bei der Projektierung und dem Bau von Eisenbahnen und Wegen. 3te unveränderte Auflage. Berlin, Ernst, 1909. 8vo. 7 + 379 pp. M. 4.00

HOUGHTON (C. E.). The elements of mechanics of materials: a text for students in engineering courses. New York, Van Nostrand, 1909 8vo. 8 + 186 pp. Cloth. $2.00

HUMMEL (L.). See REBBER (W.).

JAMIESON (A.). A textbook of applied mechanics and mechanical engineering. Vol. I, 7th edition. London, Griffin, 1909. 8vo. Cloth. 6 s.

J. (F.). Eléments de géométrie descriptive, avec de nombreux exercices. Paris, Poussielgue, 1910. 16mo. 462 pp.

KRÖHNKE (G. H. A.). Manuale pel tracciamento delle curve delle ferrovie e strade carrettiere, calcolato nel modo più accurato per tutti gli angoli e raggi, tradotto dal tedesco da L. Loria. 3a edizione, riveduta. Milano, Hoepli, 1909. 16mo. 167 pp.

LARARD (C. E.) and GOLDING (H. A.). Practical calculations for engineers. 2nd edition, enlarged. London, Griffin, 1909. 8vo. 478 pp. Cloth. 6s.

LEA (F. C.). Hydraulics for engineers and engineering students. New York, Longmans, 1909. 8vo. 12 + 536 pp. Cloth. $4.25

LOHSE (O.). Tafeln für numerisches Rechnen mit Maschinen. Leipzig, Engelmann, 1909. 8vo. 6 + 123 pp. Cloth. M. 13.50

LORIA (G.). Geometria descrittiva. Parte I. Milano, Hoepli, 1909. 16mo. 16 + 325 pp.

LORIA (L.). See KRÖHNKE (G. H. A.).

LOZANO (E.). See BROWN (H. T.).

MARTIN (L. A., Jr.). Text-book of Mechanics, second edition. New York, Wiley and Sons, 1909. 12mo. 12 +142 pp. 167 fig. Cloth. $1.25 net.

MASTRODOMENICO (F.). Il vero meccanismo dell' universo, ossia la scoperta della causa da cui risulta la pesantezza e la gravitazione universale della materia. Napoli, Granito, 1908. 8vo. 105 pp.

——. La gravitazione universale, ossia il mondo materiale e il giuoco delle forze che ne animano la macchina. Napoli, De Rosa, 1909. 16mo. 36 pp.

MATRICULATION mechanics papers, being the papers set at the matriculation examination of the University of London, with model answers to the papers of June, 1909. London, Clive, 1909. 8vo. 96 pp. 1s. 6d.

MÜLLER (K.). Die graphische Darstellung der Pendelbewegung. (Progr.) Teplitz-Schönau, 1909. 8vo. 6 pp.

MURANI (O.). Onde hertziane e telegrafo senza fili. 2a edizione, riveduta ed accresiuta dall' autore. Milano, Hoepli, 1909. 16mo. 15 + 397 pp.

NAGEL (G.). Ueber die Bildung fester Oberflächen auf Flüssigkeiten. (Diss.) Heidelberg, 1909. 8vo. 44 pp.

PANEBIANCO (H.). Facilitazione del calcolo numerico di formule usuali in fisica. Padova, Salmin, 1909. 8vo. 8 pp.

RAMSAY (J.). Engineering units of measurement. Glasgow, Smith, 1909. 8vo. 36 pp. 1s.

RAUSENBERGER (F.). The theory of the recoil of guns with recoil cylinders. Translated by A. Slater. New York, Van Nostrand, 1909. 8vo. 124 pp. Cloth. $4.50

REBBER (W.). Die Festigkeitslehre und ihre Anwendung auf den Maschinenbau. Elementar behandelt zum Gebrauche für Studierende und in der Praxis. 5te, nach den neuesten Erfahrungen neu bearbeitete und vermehrte Auflage. Herausgegeben von L. Hummel. Mittweida, Polytechnische Buchhandlung, 1909. 8vo. 7 + 623 pp. Cloth. M. 12.00

ROHRBECK (E.). Die Berechnung elektrischer Leitungen, insbesondere der Gleichstrom-Verteilungsnetze. 2te Auflage. Leipzig, Leiner, 1909. 8vo. 4 + 75 pp. M. 2.50

SÁNCHEZ TIRADO (P.). Elementos de topografía escritos por lecciones con arreglo al programa oficial para el Cuerpo de Topógrafos. 3a edición. Madrid, Santos, 1909. 197 pp. P. 7.50

SCHMIDT (F.). Die gebräuchlichsten Kanalprofile mit ihren Leistungs- und Geschwindigkeits-Kurven. Duisburg, Leistner, 1909. 8vo. 6 pp. M. 8.00

SLATER (A.). See RAUSENBERGER (F.).

SLOANE (T. O.). Elementary electrical calculations. London, Lockwood, 1909. 8vo. 314 pp. Cloth. 9 s.

SMITH (H. E.). Strength of material: an elementary study prepared for the use of midshipmen at the U. S. Naval Academy. 2nd edition, revised. New York, Van Nostrand, 1909. 12mo. 9 + 170 pp. Cloth. $1.25

WALDNER (A.). See HAUHART (H.).

WEYRAUCH (R.). Hydraulisches Rechnen. Formeln und Zahlenwerte aus dem Gebeite des Wasserbaues, für die Praxis bearbeitet. Stuttgart, Wittwer, 1909. 8vo. 8 + 88 pp. Cloth. M. 3.00

WOODS (R. J.). The theory of structures. New York, Longmans, 1909. 8vo. 12 + 276 pp. Cloth. $3.00

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and forty-fifth regular meeting of the Society was held in New York City on Saturday, October 30, 1909, extending through a morning and an afternoon session. About forty persons were in attendance, including the following twenty-seven members of the Society:

Professor G. D. Birkhoff, Professor E. W. Brown, Professor A. S. Chessin, Professor F. N. Cole, Miss L. D. Cummings, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor C. C. Grove, Professor C. N. Haskins, Professor W. H. Jackson, Mr. S. A. Joffe, Dr. L. C. Karpinski, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Professor W. W. Landis, Professor J. H. Maclagan-Wedderburn, Mr. H. H. Mitchell, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Dr. W. M. Strong, Professor Oswald Veblen, Mr. C. B. Walsh, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams.

Vice-President Edward Kasner occupied the chair at the morning session, being relieved at the afternoon session by Ex-Presidents Osgood and White. The Council announced the election of the following persons to membership in the Society: Dr. H. T. Burgess, University of Wisconsin; Professor H. H. Dalaker, University of Minnesota; Mr. G. C. Evans, Harvard University; Mr. Louis Gottschall, New York City; Dr. J. V. McKelvey, Cornell University; Miss H. H. MacGregor, Yankton College; Mr. H. H. Mitchell, Princeton University; Mr. U. G. Mitchell, Princeton University; Mr. R. R. Shumway, University of Minnesota; Dr. H. L. Slobin, University of Minnesota; Mr. I. W. Smith, University of North Dakota. Four applications for admission to membership in the Society were received.

The list of nominations for officers and other members of the Council to be placed on the official ballot for the annual meeting was reported. Mr. C. B. Upton was appointed Assistant Librarian of the Society.

In memory of Ex-President Simon Newcomb the following resolutions, presented by the Council, were adopted by the Society:

Resolved: That the American Mathematical Society record its great regret at the loss to the Society and to Science occasioned by the death of Professor Simon Newcomb.

As a member of the Society from the time when its scope became national in character and as one of its early Presidents, he showed his interest in its development. His advice and influence greatly contributed to the success which has attended its efforts toward the organization and progress of mathematical science throughout the American continent.

His achievements in many branches of thought and particularly in astronomical science will entitle him to high rank amongst those who have labored to advance knowledge and to place it within the reach of every student.

Resolved: That these resolutions be recorded in the minutes of the Society and that a copy of them be sent by the Secretary to the family of the late Professor Newcomb.

The following papers were read at this meeting:

(1) Professor C. N. HASKINS: "Note on the extremes of functions."

(2) Professor P. A. LAMBERT: "On the solution of linear differential equations."

(3) Professor FLORIAN CAJORI: "A note on the history of the slide rule."

(4) Professor CARL RUNGE: "A hydrodynamic problem treated graphically."

(5) Professor EDWARD KASNER: "The motion of particles starting from rest."

(6) Professor G. A. MILLER: "Note on the groups generated by two operators whose squares are invariant."

(7) Professor C. N. MOORE: "On the uniform convergence of the developments in Bessel functions of order zero."

Professor Runge was introduced by the Secretary. In the absence of the authors, the papers of Professor Lambert, Professor Cajori, Professor Miller, and Professor Moore were read by title.

Professor Miller's paper appears in full in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Haskins discusses a representation of the maximum maximorum and minimum minimorum of a function of a real variable as the limit of a definite integral. If in the interval $0 \leqq x \leqq 1, f(x)$ is continuous and m and M respectively its greatest lower and least upper bounds, and

$$J(z) = \tfrac{1}{z} \log \int_0^1 e^{zf(x)}\, dx,$$

then

$$\lim_{z=+\infty} J(z) = M, \quad \lim_{z=-\infty} J(z) = m,$$

and

$$\lim_{z=0} J(z) = \int_0^1 f(x) dx.$$

Moreover $J(z)$ is a monotonic increasing function of z, a fact which is proved by repeated use of the integral inequality of Schwarz. The theorems are closely related to those recently published by Dunkel,* but the proofs proceed by methods of the integral calculus.

2. The object of Professor Lambert's paper is to present a method of building up the solution of ordinary differential equations with algebraic coefficients. The method consists of the following steps: (a) Break up the given differential equation

$$(1) \qquad f\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \cdots,\ \frac{d^ny}{dx^n}\right) = 0$$

into two parts, one of which equated to zero gives a differential equation which is readily solved by the usual methods, and introduce a parameter t as a factor of the other part, so that the given equation becomes

$$(2) \quad f_1\left(x,\ y,\ \frac{dy}{dx},\ \cdots,\ \frac{d^ny}{dx^n}\right) + f_2\left(x,\ y,\ \frac{dy}{dx},\ \cdots,\ \frac{d^ny}{dx^n}\right) t = 0.$$

(b) Assume that

$$(3) \qquad y = A + Bt + Ct^2 + Dt^3 + \cdots + Nt^n + \cdots,$$

where $A, B, C, D, \cdots, N, \cdots$ are undetermined functions of x, makes equation (2) an identity. Determine $A, B, C, D, \cdots, N, \cdots$ by solving the differential equations found by equating

* *Annals of Mathematics*, sec. 2, vol. 11 (1909), p. 26.

to zero the coefficients of the successive powers of t in this identity.

(c) Substitute these values of $A, B, C, D, \cdots, N, \cdots$ in (3) and place t equal to unity; then

$$(4) \qquad y = A + B + C + D + \cdots + N + \cdots,$$

provided this series terminates or is convergent, is a solution of the differential equation.

3. It is shown by Professor Cajori that the earliest German description of the rectilinear slide rule, found in Leupold's Theatrum arithmetico-geometricum, Leipzig, 1727, and copied by Leupold from an old manuscript of hitherto unknown authorship, is a translation of passages in Seth Partridge's Description and use of an instrument called the double scale of proportion, which appeared at London in several editions. The earliest edition bears the date 1662, and not 1671, as hitherto supposed.

4. The amount of water flowing from a reservoir of infinite depth over a weir of given section may be approximately found by graphical methods, while there do not exist any methods of treating the problem analytically. In Professor Runge's paper the viscosity is neglected and the problem is considered in two dimensions only. The velocity potential is deduced by starting with a plausible assumption of the free surface. On this curve the velocity of the water, i. e., the gradient of the velocity potential, is known. From these data the stream lines and lines of constant potential may be constructed and any stream line may serve as a possible section of a weir. For practical purposes it is only necessary to approximate the given weir at the top, because the rest of it will have little influence on the amount of flow.

5. Professor Kasner showed that when a particle, starting from rest, is acted on by any positional field of force, it describes a path whose curvature is one third the curvature of the line of force through the given point. The case where the curvature is zero is studied separately, a simple result being obtained concerning the lowest non-vanishing derivatives. The discussion is extended to constrained motion and to systems of particles.

7. In Professor Moore's paper the following theorem is established: If in the interval $0 \leqq x \leqq 1$ the function $f(x)$ is continuous, save at a finite number of points at which it has a finite jump, and if in each interval of continuity it has a second derivative that is finite and integrable, then the development of $f(x)$ in Bessel functions of order zero will converge uniformly to the value $f(x)$ throughout the interval $0 \leqq x \leqq x_0 < c$, where c is the smallest value of x for which $f(x)$ is discontinuous.

It is a well known fact that, under much wider conditions than those imposed here, the development will converge uniformly to $f(x)$ throughout any closed interval that does not include nor reach up to a point of discontinuity of $f(x)$, and does not reach up to the origin. The uniform convergence of the series in the neighborhood of the origin, however, does not seem to have been discussed.

F. N. COLE,
Secretary.

NOTE ON THE GROUPS GENERATED BY TWO OPERATORS WHOSE SQUARES ARE INVARIANT.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, October 30, 1909.)

LET s_1, s_2 be any two operators which satisfy the conditions

$$s_1^{-1}s_2^2s_1 = s_2^2, \quad s_2^{-1}s_1^2s_2 = s_1^2.$$

From these equations it results directly that both s_1^2 and s_2^2 are invariant under the non-abelian group G generated by s_1, s_2. On the other hand the equations

$$s_1^{-1} \cdot s_1s_2^{-1} \cdot s_1 = s_2^{-1}s_1 = (s_1s_2^{-1})^{-1} \cdot s_1^2s_2^{-2} = s_2^{-1} \cdot s_1s_2^{-1} \cdot s_2,$$

$$s_2^{-1} \cdot s_2s_1^{-1} \cdot s_2 = s_1^{-1}s_2 = (s_2s_1^{-1})^{-1} \cdot s_1^{-2}s_2^2 = s_1^{-1} \cdot s_2s_1^{-1} \cdot s_1$$

show that the abelian group generated by the three operators $s_1^2, s_2^2, s_1s_2^{-1}$ is invariant under G, and also that the quotient group of G with respect to the group H generated by s_1^2, s_2^2 is dihedral. Hence it results also that H is the central of G whenever $s_1s_2^{-1}$ is of odd order. When $s_1s_2^{-1}$ is of even order, the central of G is either H or it includes H as a subgroup of half its order. Since the quotient group of G with respect to an abelian invariant subgroup is dihedral G must be solvable.

The object of the present note is to show clearly that G may be regarded as a generalization of the dihedral group which includes the earlier generalization obtained by considering the groups generated by two operators which have a common square.* If two operators have a common square, this square is clearly invariant under the group generated by these operators; but if the squares are invariant under this group they evidently are not necessarily the same. From this it follows directly that the present generalization includes the earlier one, and it gives rise to an almost equally elementary category of groups as a result of the equations established in the preceding paragraph. If two operators have a common square, it is known that the product of either one into the inverse of the other is transformed into its inverse by each of the operators. The analogous theorem as regards the operators under consideration may be expressed as follows:

When each of two operators is commutative with the square of the other, the product of one into the inverse of the other is transformed by each of the two operators into its inverse multiplied by an invariant operator under the group generated by the two operators.

UNIVERSITY OF ILLINOIS.

THE SOLUTION OF THE EQUATION IN TWO REAL VARIABLES AT A POINT WHERE BOTH THE PARTIAL DERIVATIVES VANISH.

BY DR. L. S. DEDERICK.

(Read before the American Mathematical Society, September 14, 1909.)

IF $F(x, y)$ is a real function of the real variables x and y which is continuous at and near the point (x_0, y_0), and vanishes at this point, but has one first order partial derivative at the point not equal to zero, there are a number of well-known theorems about the existence of other values of x and y satisfying the equation

$$(1) \qquad F(x, y) = 0,$$

* *Archiv der Mathematik und Physik*, vol. 9 (1905), p. 6.

and about the properties of y as a function of x, or x as a function of y, thus defined. The case, however, where both the first partial derivatives of $F(x, y)$ vanish at (x_0, y_0), seems to have received very little attention, although analogous theorems for this case are tacitly used in most treatments of the singularities of plane curves. In this subject, of course, it is frequently assumed that $F(x, y)$ is analytic at the point in question; but even where much less than this is demanded, it frequently happens that more is required than is at all necessary,* that the exact conditions are not stated, and that existence proofs are omitted.

The object of the present paper is to examine the possible cases that may arise when the first partial derivatives both vanish, and to state theorems analogous to Dini's theorem in these cases. No attempt has been made to reduce the sufficient conditions to an absolute minimum, but in most cases it seems unlikely that any simple statement could be made of conditions materially less restrictive. The general statement may be made concerning the requirements of continuity, that there must be one order of partial derivatives of $F(x, y)$ continuous at and near (x_0, y_0) and not all vanishing at that point, and derivatives of any higher orders must be continuous at and near (x_0, y_0) whose values at (x_0, y_0) are required in order to determine the character of the solution or the value of a derivative of y with respect to x whose existence is asserted; i. e., we may perform the formal differentiations for ascertaining any desired facts, and know that they are justified if all the partial derivatives of $F(x, y)$ that we use are continuous at and near (x_0, y_0).

In all the following theorems we shall assume that $F(x, y)$, together with all its partial derivatives of order n or lower, is continuous at and near (x_0, y_0), and that $F(x, y)$ and all its derivatives of order less than n, vanish at this point, but that not all the derivatives of order n are equal to zero there. The following notation will be used throughout:

$$h = x - x_0, \quad k = y - y_0, \quad \alpha = \frac{k}{h} = \frac{y - y_0}{x - x_0},$$

$$F_{ij}(x, y) = \frac{\partial^{i+j} F(x, y)}{\partial x^i \partial y^j}, \quad F_{ij} = F_{ij}(x_0, y_0),$$

* E. g., de la Vallée-Poussin, Cours d'analyse, vol. 2, chap. 8.

$$b_{ij} = \frac{1}{i\,!\,j\,!}, \quad bF_{ij} = b_{ij} F_{ij},$$

$$\overline{F}_{ij} = F_{ij}(x_0 + \theta h,\ y_0 + \theta_1 k).$$

where $0 \leqq \theta \leqq 1$, $0 \leqq \theta_1 \leqq 1$. In most cases $0 < \theta = \theta_1 < 1$, but not always. Moreover θ or θ_1 need not preserve the same value throughout any one discussion.

THEOREM 1. — *There is a neighborhood of* (x_0, y_0) *in which all pairs of values of* x *and* y *satisfying equation* (1) *can be grouped into not more than* n *branches such that on each branch* α *approaches a limit (finite or infinite) equal to some real root of the nth degree equation in* α

$$\sum_{i=0}^{n} bF_{n-i,\,i}\alpha^i = 0. \tag{2}$$

By Taylor's theorem for two variables there is a neighborhood of (x_0, y_0) in which we can write

$$F(x, y) = h^n \sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha^i. \tag{3}$$

If this vanishes when $x \neq x_0$, then

$$\sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha^i = 0. \tag{4}$$

From the continuity of the partial derivatives each coefficient in (4) approaches the corresponding coefficient in (2) as (x, y) approaches (x_0, y_0), and hence each root of (4) approaches some root of (2). There is, therefore, a neighborhood of (x_0, y_0) in which the possible solutions of (1) are separated into not more than n branches according to the root of (2) whose neighborhood contains the corresponding value of α. If $F_{0,n} = 0$, equation (2) must be regarded, of course, as having an infinite root; and accordingly, there may be a branch on which α becomes infinite. This includes the case where $F(x_0, y) = 0$ throughout some interval including y_0. Here (3) is satisfied by the vanishing of h. In the subsequent theorems, however, we shall always understand by a root of equation (2) a real

finite root. The question of a solution belonging to an infinite root is easily reduced to this by interchanging the variables x and y.

By a *branch* here is meant merely a certain aggregate of pairs of values. We have not shown that a branch defines y as a single valued function of x, nor yet a continuous function. We have shown, however, that the values of x and y belonging to one branch satisfy a relation

$$y - y_0 = (x - x_0)(\alpha_1 + \zeta) = h(\alpha_1 + \zeta),$$

where α_1 is a root of (2) and ζ approaches zero with h, and that in a sufficiently small neighborhood of (x_0, y_0) any solution belongs to some one such branch.*

THEOREM 2.—*Corresponding to any simple root* α_1 *of equation* (2), *there is a neighborhood* D *of* (x_0, y_0) *and a single-valued function* $f(x)$ *such that when* $y = f(x)$ *in* D, $F(x, y) \equiv 0$. *This function is continuous and has a continuous derivative in* D, *and represents all the values of* x *and* y *in* D *which satisfy* (1) *and belong to the branch corresponding to* α_1;

i. e., there is an interval $(\alpha_1 - \gamma, \alpha_1 + \gamma)$, such that if (x, y) is in D, and $\alpha_1 - \gamma < \alpha < \alpha_1 + \gamma$, then $F(x, y) = 0$ if and only if $y = f(x)$.

If (x, y) and (x_1, y_1) are any two points in the neighborhood of (x_0, y_0),

$$F(x, y) = F(x_1, y_1) + [F(x, y) - F(x_1, y_1)].$$

If $x = x_1$,

$$F(x, y) - F(x_1, y_1) = F(x, y) - F(x, y_1) = \Delta y F_{01}(x, y_1 + \theta \Delta y),$$

* NOTE.—For this theorem the conditions of continuity may be slightly reduced. It is sufficient to demand that the partial derivatives of $F(x, y)$ of order n shall exist at and near (x_0, y_0) and be continuous at that point. We can then stop the Taylor's development one term sooner and expand each partial derivative of order $(n-1)$ by the unsymmetrical form of the law of the mean,

$$F_{n-1-i,\, i}(x_0 + \theta h, y_0 + \theta k)$$
$$= \theta h F_{n-i,\, i}(x_0 + \theta_1 \theta h, y_0) + \theta k F_{n-i-1,\, i+1}(x_0 + \theta h, y_0 + \theta_2 \theta k).$$

This gives the same result as the Taylor's development except for the factor $n\theta$, which does not affect the reasoning.

where $\Delta y = y - y_1$. In general,

$$y = y_0 + \alpha(x - x_0).$$

In particular, let

$$y_1 = y_0 + \alpha_1(x - x_0).$$

Then

$$\Delta y = y - y_1 = (\alpha - \alpha_1)(x - x_0) = h\Delta\alpha,$$

where $\Delta\alpha = \alpha - \alpha_1$. Hence

$$(5) \qquad F(x, y) = F(x, y_1) + h\Delta\alpha F_{01}(x, y_1 + \theta\Delta y).$$

Now by Taylor's theorem,

$$(6) \quad F_{01}(x, y) = h^{n-1}\sum_{i=0}^{n-1} b_{n-1-i,\, i}\bar{F}_{n-1-i,\, i+1}\alpha^i = h^{n-1}\sum_{i=1}^{n} ib\bar{F}_{n-i,\, i}\alpha^{i-1}.$$

As h approaches zero and α approaches α_1, the coefficient of h^{n-1} in this expression approaches

$$\sum_{i=1}^{n} ibF_{n-i,\, i}\alpha_1^{i-1} = M,$$

where $M \neq 0$ since α_1 is a simple root of (2). If then we take any positive ϵ such that $\epsilon < |M|$, we can find a neighborhood D_1 of (x_0, y_0) and an interval $(\alpha_1 - \gamma, \alpha_1 + \gamma)$, such that when (x, y) is in D_1, and $\alpha_1 - \gamma < \alpha < \alpha_1 + \gamma$, we have

$$(7) \qquad \left|\sum_{i=1}^{n} ibF_{n-i,\, i}(x, y)\alpha^{i-1}\right| > \epsilon.$$

Again, from (3),

$$(8) \qquad F(x, y_1) = h^n \sum_{i=0}^{n} b\bar{F}_{n-i,\, i}\alpha_1^i.$$

Since α_1 is a root of (2), there is a neighborhood D_2 of (x_0, y_0) in which

$$(9) \qquad \left|\sum_{i=0}^{n} bF_{n-i,\, i}(x, y)\alpha_1^i\right| < \epsilon|\Delta\alpha|.$$

Substituting in (5) from (8) and (6), we have

$$F(x, y) = h^n \left[\sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha_1^i + \Delta\alpha \sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^{i-1} \right].$$

From (7) and (9), if (x, y) lies in D, the region common to D_1 and D_2, and $\alpha_1 - \gamma < \alpha < \alpha_1 + \gamma$, then the first term in the bracket is less in absolute value that $\epsilon|\Delta\alpha|$, and the second is greater. If then we leave h unchanged but replace $\Delta\alpha$ by its negative, we change the sign of $F(x, y)$. Therefore $F(x, y)$ must vanish between $y = y_1 - \Delta y$ and $y = y_1 + \Delta y$; and since in this interval $F_{01}(x, y)$ does not vanish, it remains of one sign, and hence $F(x, y)$ can not vanish more than once. Thus for any value of x in D, there is one and only one value of y between $y = y_0 + h(\alpha_1 - \gamma)$ and $y = y_0 + h(\alpha_1 + \gamma)$ for which $F(x, y) = 0$. This defines y as a single-valued function of x in D, and makes it the complete solution which satisfies the condition $\lim_{x=x_0} \alpha = \alpha_1$. This last condition shows that $f(x)$ is continuous at $x = x_0$, and that $f'(x_0) = \alpha_1$. When $y = f(x)$ in D and $x \neq x_0$, we have seen that $F_{01}(x, y) \neq 0$. Hence by Dini's theorem, $f(x)$ and $f'(x)$ are continuous in D when $x \neq x_0$.

It only remains now to show that $f'(x)$ is continuous at $x = x_0$. When $x \neq x_0$, we know that

$$f'(x) = -\frac{F_{10}(x, y)}{F_{01}(x, y)} = -\frac{h^{n-1}\sum_{i=0}^{n-1} b_{n-1-i,\,i}\overline{F}_{n-i,\,i}\alpha^i}{h^{n-1}\sum_{i=0}^{n-1} b_{n-1-i,\,i}\overline{F}_{n-1-i,\,i+1}\alpha^i}$$

$$= -\frac{\sum_{i=0}^{n-1}(n-i)b\overline{F}_{n-i,\,i}\alpha^i}{\sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^{i-1}}.$$

As x approaches x_0 and hence α approaches α_1, the numerator and denominator of this both approach limits, and we have seen that the limit of the latter is not zero. Therefore $\lim_{x=x_0} f'(x)$ exists. But since $f(x)$ is continuous at x_0, this is sufficient that $f'(x)$ be continuous there also.*

This same general method of proof may be extended to the case where α_1 is a multiple root. The method is essentially this: to determine an approximate solution and a strip includ-

* Dini, Funzioni di variabili reali, § 75; or E. W. Hobson, Theory of functions of a real variable, § 220.

ing it within which the actual solution must lie, and then to show that on the approximation $F(x, y)$ is so small, and throughout the strip $F_{01}(x, y)$ is so large, that for any value of x, $F(x, y)$ must vanish for one and only one value of y in the strip. In Theorem 2, the approximation and the boundaries of the strip are straight lines. For later cases the approximation must be closer and the boundaries of the strip must approach each other more rapidly.

When $n = 1$, this theorem reduces to Dini's theorem and the proof becomes that given by de la Vallée-Poussin (Cours d'analyse infinitésimale, I, ¶ 142). As in the case of Dini's theorem, we can show here that every additional order of partial derivatives of $F(x, y)$ which are continuous at and near (x_0, y_0) insures the existence and continuity of an additional derivative of $f(x)$. This is shown in the following theorem, which completes the treatment of a simple root of equation (2).

Theorem 3. — *If $f(x)$ is a solution of the sort discussed in Theorem 2, and the partial derivatives of $F(x, y)$ of order $(n + m)$ are continuous at and near (x_0, y_0), then $f^{(m+1)}(x)$ exists and is continuous at and near x_0.*

We know from Theorem 2 that this is true when $m = 0$. To show that it is true in general, we assume that $f^{(m)}(x)$ exists and is continuous at and near x_0. This we know is true when $m = 0$ or $m = 1$. It is also true if this theorem holds for all values of m less than the particular one in question. The proof, then, on the assumption that $f^{(m)}(x)$ is continuous, establishes the theorem by complete induction.

We can expand $F(x, y)$ by Taylor's theorem in powers of h from h^n to h^{n+m} with coefficients that are polynomials in α and contain no other variable except in the case of the last. From the assumption of the continuity of $f^{(m)}(x)$, we can write when $y = f(x)$

$$\alpha = f' + \tfrac{1}{2}hf'' + \cdots + \frac{1}{m!}h^{m-1}(f^{(m)} + \zeta), \tag{10}$$

where ζ approaches zero with h. If we substitute this in the expansion of $F(x, y)$ and arrange according to powers of h, the coefficients of the powers up to and including h^{n+m-2} will be constants. That of h^{n+m-1} will contain ζ linearly but no other variable. Hence, since $F(x, y) = 0$ when $y = f(x)$, we have an equation of the form,

$$A_n h^n + \cdots + A_{n+m-2}h^{n+m-2} + (B + \omega\zeta)h^{n+m-1} + h^{n+m}\phi(h) = 0,$$

where $A_n, \cdots, A_{n+m-2}$, and B are are constants, ω and ϕ approach limits as h approaches zero, and

$$\lim_{h=0} \omega = \frac{1}{m!} \sum_{i=1}^{n} ibF_{n-i,\,i}\alpha_1^i \neq 0.$$

Since ζ approaches zero with h,

$$A_n = A_{n+1} = \cdots = A_{n+m-2} = B = 0$$

and

$$\omega\zeta + h\phi(h) = 0.$$

Therefore $\zeta = h\eta$, where $\lim_{h=0} \eta$ exists. If we put this in (10), we have a development of α including the power h^m, in which the coefficient of h^{m-1} is constant. From the continuity of $f^{(m)}$, moreover, we have

$$y' = f'(x) = f' + hf'' + \cdots + \frac{h^{m-1}}{(m-1)!}(f^{(m)} + \zeta_1),$$

where ζ_1 approaches zero with h. When $y = f(x)$ we have also

$$F_{10}(x, y) + y'F_{01}(x, y) = 0.$$

If we expand the partial derivatives by Taylor's theorem and replace α and y' by their developments, we find that the powers of h in the expansions of the partial derivatives run from h^{n-1} to h^{n+m-1}, and hence after the substitution, the coefficients are constants up to and including that of h^{n+m-3}, while the coefficient of h^{n+m-2} involves ζ_1 linearly and no other variable. In the same way as for ζ, we show that $\zeta_1 = h\eta_1$ where $\lim_{h=0} \eta_1$ exists. This extends the development of y' one step further. In like manner, by substituting in the successive total derivatives of $F(x, y)$ with regard to x and equating them to zero, we add one power of h to the known developments of each of the derivatives, $y'', y''', \cdots, y^{(m)}$, finally getting $y^{(m)} = f^{(m)} + h\eta_m$, where $\lim_{h=0} \eta_m$ exists. By Dini's theorem we know that when $x \neq x_0$, $y^{(m+1)}$ exists, and hence the $(m+1)$th total derivative of $F(x, y)$ exists and is equal to zero. If we expand this in powers of h and substitute for $\alpha, y', \cdots, y^{(m)}$ their developments in powers of h, we have an equation for determining $y^{(m+1)}$, in which the coefficient of $y^{(m+1)}$ is $F_{01}(x, y)$, or

$$h^{n-1} \sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^i.$$

In the rest of the equation the coefficients are constants up to and including that of h^{n-2}; and the coefficient of h^{n-1} approaches a limit. We cannot infer here as before that the constant coefficients are all zero, because we do not know that $y^{(m+1)}$ approaches a limit as h approaches zero. We can infer, however, that when h approaches zero, $y^{(m+1)}$ either approaches a limit or else becomes infinite. But since $f^{(m+1)}(x)$ exists when $x \neq x_0$, we know that $y^{(m)} = f^{(m)} + hf^{(m+1)}(x_0 + \theta h)$; and we have seen that $y^{(m)} = f^{(m)} + h\eta_m$. Therefore $f^{(m+1)}(x_0 + \theta h) = \eta_m$, and this approaches a limit. But this is impossible if $f^{(m+1)}$ becomes infinite. Therefore $\lim_{x=x_0} f^{(m+1)}(x)$ exists and from this it follows that $f^{(m+1)}(x)$ is continuous at $x = x_0$.

There now remains to consider the case of a multiple root of equation (2). We know from the theory of plane curves that in this case the existence and character of the solution depend upon the values of derivatives of an order higher than n. The simplest case of this sort is treated in the following theorem.

THEOREM 4.—*Let $F(x, y)$ have all its partial derivatives of order $(n + 1)$ continuous at and near (x_0, y_0), and let α_1 be a multiple root of order m of equation (2); but let*

$$\sum_{i=0}^{n+1} bF_{n+1-i,\,i}\alpha_1^i \neq 0.$$

Then the values of x and y in the neighborhood of (x_0, y_0) for which $F(x, y) = 0$ and α approaches α_1 are completely given by a solution in the form $x = x_0 + et^m$, $y = y_0 + e\alpha_1 t^m + t^{m+1}f(t)$, where e is $+1$ or -1 and $f(t)$ is a single-valued function which is continuous at $t = 0$.

We shall omit the proof of this and proceed at once to a much more general discussion, from which this theorem follows as a special case. Let us assume that the partial derivatives of $F(x, y)$ of order $n + r$ are continuous at and near (x_0, y_0). Then we can expand $F(x, y)$ in the form,

$$(11) \qquad F(x, y) = h^n\phi_n + h^{n+1}\phi_{n+1} + \cdots + h^{n+r}\bar{\phi}_{n+r},$$

where ϕ_i is a polynomial of degree i in α, having for coefficients the partial derivatives of $F(x, y)$ of order i with the arguments (x_0, y_0) except where the dash as in $\bar{\phi}_{n+r}$ indicates the arguments $x_0 + \theta h$, $y_0 + \theta k$. Since the coefficients of $\bar{\phi}_{n+r}$ are continuous at and near (x_0, y_0), we may write $\bar{\phi}_{n+r} = \phi_{n+r} + \zeta$, where ζ approaches zero with h. Then at any point of a solution

$$(12) \qquad \phi_n + \phi_{n+1}h + \phi_{n+2}h^2 + \cdots + \phi_{n+r}h^r + \zeta h^r = 0.$$

We have seen in Theorem 1 that for any solution, α must approach a root of ϕ_n, which we have called α_1. In Theorems 2 and 3 we have discussed the case where α_1 is a simple root. Theorem 4 supposes that α_1 is a multiple root of ϕ_n but not a root of ϕ_{n+1}. We shall now allow α_1 to be a root of any number of ϕ's and of any possible orders, and obtain certain necessary conditions for a solution.

Let α_1 be a root of each of the polynomials, $\phi_n, \phi_{n+1}, \cdots, \phi_{n+r}$; and let its orders of multiplicity as a root of these be respectively $m_0, m_1, \cdots, m_r$. The important case that α_1 is not a root of ϕ_{n+r} is included here by making $m_r = 0$. We may assume without loss of generality that no previous m is zero, as that case is treated by taking a smaller value of r. For any solution belonging to α_1, we have $\alpha = \alpha_1 + \zeta_1$ where ζ_1 approaches zero with h. If we put this value of α in (12) and expand, the terms in ζ_1 of degree less than m_i in the expansion of ϕ_i will drop out from the vanishing of their coefficients, and we shall have

$$(13) \quad \psi_0(\zeta_1)\zeta_1^{m_0} + \psi_1(\zeta_1)h\zeta_1^{m_1} + \cdots + \psi_r(\zeta_1)h^r\zeta_1^{m_r} + \zeta h^r = 0,$$

where the ψ's are polynomials in ζ_1 with constant coefficients, and the constant term in each is not zero. In fact

$$\psi_i(0) = \frac{1}{m_i!}\left(\frac{\partial}{\partial \alpha}\right)^{m_i} \phi(\alpha_1) \neq 0.$$

Now let c and d be any two integers that are relatively prime, and let e be $+1$ or -1, its sign being at present undetermined. It is always possible, however, to choose its sign so that for any value of h there is at least one value of t such that $h = et^d$. If we define t in this way and set $\eta = \zeta_1/t^c$, we have $\zeta_1 = \eta t^c$. Replacing h and ζ_1 in (13) by these values, we get

$$(14) \qquad \sum_{i=0}^{r} e^i\psi_i(\eta t^c)\eta^{m_i}t^{cm_i+di} + \zeta e^r t^{dr} = 0.$$

From this we may cancel out the lowest power of t that appears and get an algebraic equation that η must satisfy. If now we let t approach zero and impose the condition that η shall remain finite, we see that η must approach a root of the limiting form of equation (14). This is

$$\sum_i e^i \psi_i(0)\eta^{m_i} = 0, \tag{15}$$

the sum including only such terms as had the minimum exponent of t in (14). In general this equation will consist of only one term, and hence its only solution is $\eta = 0$. If now we wish to exclude the case where η approaches zero with t, we must choose c and d so that two or more exponents of t in (14) are equal to each other and less than any others. Let $\lambda = c/d$. Then the exponent $cm_i + di$ becomes $d(\lambda m_i + i)$. As d is independent of i, the exponent is a minimum when $\lambda m_i + i$ is a minimum. For small values of λ, $\lambda m_0 < 1$, and hence the exponent of the first term is less than any other;

$$\lambda m_0 < i + \lambda m_i \qquad (i = 1, 2, \cdots, r).$$

Let λ_1 be the smallest value of λ for which this is not true, i. e., for which there is an i such that

$$\lambda m_0 = i + \lambda m_i.$$

Let this value of i be i_1, or if it happens simultaneously for two or more values of i, let i_1 be the greatest of these. Then it is easy to show that for increasing values of λ, the exponent $d(i_1 + \lambda m_{i_1})$ becomes and remains the minimum until it becomes equal to the exponent of some later term,

$$i_1 + \lambda m_{i_1} = i_2 + \lambda m_{i_2} \leqq i + \lambda m_i,$$

where $i_2 > i_1$. If there are two or more possible values of i_2, then i_2 is to indicate the largest of these. In like manner $d(i_2 + \lambda m_{i_2})$ becomes and remains the minimum exponent until it becomes equal to some exponent where $i > i_2$.

We proceed in this way, determining all possible values of λ (not more than r in number) for which η does not become zero or infinite as t approaches zero, but subject always to the condition that $i + \lambda m_i \leqq r$, as otherwise the term $\zeta e^r t^{dr}$ may become the principal infinitesimal, since we have no means of comparing the infinitesimals ζ and t. We can take account of this condition by setting m_r arbitrarily equal to zero. If $m_r = 0$, then $r + \lambda m_r (= r)$ appears in the equation for determining the last value of λ, and this value is to be treated like any previous one. If $m_r \neq 0$, then the coefficient of η^{m_r} in (15) is zero and $\eta = 0$ is a root. In this case there is a possible value of η

which approaches zero for all values of λ for which we can determine its behavior with the assumptions that we have made. We cannot say whether or not it yields a solution unless we know the existence and values of derivatives of order greater than $(n+r)$.

If η corresponds to an actual solution of (1), the identity of this solution is not affected by varying λ. Let us consider, then, the case that η belonging to this solution becomes infinite for all values of λ. This means in particular that the coefficient of the highest power of η in (14) approaches zero with h when λ has the values which make the exponent λm_0 the minimum. This, however, shows that the coefficient of the highest power of η involves a ψ which is not ψ_0. If then we set $r=0$, this power of η will not appear. But we know from Theorem 1 that setting $r=0$ gives the necessary condition for all solutions which approach (x_0, y_0). As we are concerned only with these, the case that η becomes infinite for all values of λ is disposed of. If η approaches zero for some values of λ and becomes infinite for others, it is easy to show that there is an intermediate value of λ for which it approaches a limit not zero.

Thus for any solution approaching (x_0, y_0) whose character is determined by the assumed data, we have a certain value of λ and an equation of the form (15) for determining the limiting value of η, or to write the equation more explicitly,

$$(16)\qquad e^p\psi_p(0)\eta^{m_p}+e^q\psi_q(0)\eta^{m_q}+\cdots+e^s\psi_s(0)\eta^{m_s}=0,$$

where

$$(17)\quad cm_p+dp=cm_q+dq=\cdots=cm_s+ds=\mu\leqq cm_i+di.$$

Or, since $\eta \neq 0$,

$$(18)\quad e^{p-s}\psi_p(0)\eta^{m_p-m_s}+e^{q-s}\psi_q(0)\eta^{m_q-m_s}+\cdots+\psi_s(0)=0.$$

We now have that any possible solution is expressible in the form

$$x=x_0+et^d,$$

$$y=y_0+et^d(\alpha_1+\eta t^c)=y_0+a_1t^d+a_2t^{d+c}+\zeta_2t^{d+c},$$

where

$$a_1=e\alpha_1,\quad a_2=\lim_{t=0} e\eta,\quad \text{and}\quad \lim_{t=0}\zeta_2=0;$$

and where c, d, e, and the limiting value of η, satisfy (17) and (18), and $e=\pm 1$.

These are the necessary conditions for a solution. If, however, the limit of η is a real simple root of (18), they are also sufficient. For let

$$y_1 = y_0 + a_1 t^d + a_2 t^{d+c}.$$

Then

$$F(x, y_1) = h^n \left(\sum_{i=0}^{r} e^i \psi_i(\eta t^c) \eta^{m_i} t^{cm_i + di} + \zeta e^r t^{dr} \right)$$

$$= t^{nd+\mu}\omega(t), \quad \text{where } \lim_{t=0} \omega(t) = 0.$$

Developing $F_{01}(x, y)$ in the same form as (6), we get

$$F_{01}(x, y) = h^{n-1} \left(\sum_{i=0}^{r} h^i \frac{\partial \phi_{n+i}}{\partial \alpha} + \zeta' h^r \right),$$

where ζ' approaches zero with h. Substituting $\alpha = \alpha_1 + \zeta_1$ in $\partial\phi_{n+i}/\partial\alpha$, we get

$$F_{01}(x, y) = h^{n-1} \left(\sum_{i=0}^{r} h^i \zeta_1^{m_i - 1} m_i \psi_i(\zeta_1) + \zeta' h^r \right)$$

$$= e^{n-1} t^{(n-1)d} \left(\sum_{i=0}^{r} e^i m_i \psi_i(\zeta_1) \eta^{m_i - 1} t^{id + (m_i - 1)c} + \zeta' e^r t^{rd} \right)$$

$$= t^{(n-1)d + \mu - c} \Omega(t, \eta),$$

where the limit of $\Omega(t, \eta)$ is the partial derivative of the expression in (16) and hence not equal to zero on the assumption that η approaches a simple root. Since Ω is a continuous function of t and η, we can find small intervals for t and η within which $|\Omega| > \epsilon$, where ϵ is some positive constant. If now

$$y = y_0 + a_1 t^d + e\eta t^{d+c},$$

η having any fixed value in its interval, then

$$\Delta y = y - y_1 = (e\eta - a_2) t^{d+c} = e\Delta\eta \cdot t^{d+c}.$$

$$F(x, y) - F(x, y_1) = \Delta y F_{01}(x, y_1 + \theta\Delta y) = e\Delta\eta \cdot t^{nd+\mu} \Omega(t, ea_2 + \theta\Delta\eta).$$

$$F(x, y) = t^{nd+\mu} \left[\omega(t) + e\Delta\eta \Omega(t, ea_2 + \theta\Delta\eta) \right].$$

As ω approaches zero and Ω approaches some other limit, we can take t so small that

$$|\omega| < |\Omega \cdot \Delta\eta|,$$

and hence the sign of $F(x, y)$ depends on the sign of $\Delta\eta$. As in Theorem 2, it now follows that $F(x, y)$ vanishes once and only once in the strip and hence there is a single valued function of t which yields the complete solution belonging to this limiting value of η. A short computation will show that if we change the sign of e and make no other change, we either get the same solution that we had before or else none at all.

All this is on the assumption that η approaches a simple root of (18). If, however, η approaches a multiple root, we substitute $\eta = e(\alpha_2 + \zeta_2)$ in (14) and arrange according to powers of t, up to t^{dr}. We then have an equation connecting ζ_2 and t of exactly the same form as equation (13) connecting ζ_1 and h. With this we can proceed in the same way, getting ζ_2 and t expressed in powers of a new parameter as a necessary condition, which becomes sufficient if the equation for the new coefficient has a simple root. For a multiple root we must repeat the process. With regard to the continued repetition of the process, there are three possibilities: (1) it may come to an end at some point through the appearance of an equation for the next coefficient, which has only simple roots; (2) it may end on account of the non-existence of continuous partial derivatives of the orders required to carry it on; or (3) the appearance of multiple roots may continue indefinitely. An example of this last is the case where $F(x, y)$ is such a function as $(y - \sin x)^2$. Here

$$F(x, y) = x^2 - 2xy + y^2 - \tfrac{1}{3}x^4 + \tfrac{1}{3}x^3y + \tfrac{2}{45}x^6 - \tfrac{1}{60}x^5y + \cdots,$$

and the coefficient of every term in the assumed solution will be a double root of the equation for determining it. Hence no finite number of terms from the development would enable us to determine the character of the solution, or assert that the origin is not a cusp, an isolated point, or a point common to two distinct branches.

The question of the derivatives of ζ_1 with regard to h, η with regard to t, etc., may be treated by expressing x and y in terms of these new variables, and finding the derivatives of $F(x, y)$ with regard to them, thus reducing the problem to that of Theorem 3.

HARVARD UNIVERSITY,
September 1, 1909.

TABLES OF GALOIS FIELDS OF ORDER LESS THAN 1,000.

BY PROFESSOR W. H. BUSSEY.

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EVERY field of a finite number of marks may be represented as a Galois field of order $s = p^n$, where p^n is a power of a prime. The $GF[p^n]$ is defined uniquely by its order, and is therefore independent of the particular irreducible congruence used in its construction. In each of the following tables the $GF[p^n]$ is constructed by means of a primitive irreducible congruence which appears at the top of the table. The marks of each field are arranged in two tables. In each table each mark appears as a power of a primitive root i, and also as a polynomial in i of degree $k \leqq n - 1$. The coefficients in this polynomial are integers reduced modulo p. The mark $Ai^k + Bi^{k-1} + \cdots + Di + E$, $A \neq 0$, is denoted by $AB \cdots DE$, a symbol consisting of its detached coefficients in order. Zero coefficients must not be omitted. This is the usual symbol for a positive integer in the notation of the number system whose base is p. In the first table the marks are arranged according to ascending powers of i. In the second table the marks are arranged so that the symbols $AB \cdots DE$ represent the positive integers in natural order. By means of these two tables it is possible to perform with ease the operations of addition, subtraction, multiplication and division, within the field.

The foregoing paragraph is reprinted from a paper entitled "Galois field tables for $p^n \leqq 169$," published in the BULLETIN, volume 12 (1905), pages 21–38. The present paper is an extension of that work and contains tables made according to the same plan but more compactly printed. The tables for fields of orders 2^8 and 2^9 are printed completely as were all of the tables in the paper mentioned. The other tables are abbreviated according to the following plan: The first of the two tables for a field of order p^n contains the marks i^λ for which $\lambda = 1, 2, 3, \cdots, \nu$ (where ν is the smallest value of λ such that i^λ is equal to one of the numbers $1, 2, 3, \cdots, p - 1$) and also the marks i^λ for which λ is a multiple of ν. The table expresses each of these marks as a polynomial in i of degree $k \leqq n - 1$. If λ

be any other exponent, the value of i^λ as a polynomial in i may be obtained by writing $\lambda = q\nu + r$, where $q\nu$ is the largest multiple of ν less than λ and consequently r is less than ν. Then $i^\lambda = i^{q\nu+r} = i^{q\nu}\cdot i^r$. The values of $i^{q\nu}$ and i^r may be read off directly from the table and their product obtained mentally because the value of $i^{q\nu}$ is integral. The second of the two tables for a field of order p^n contains the integral marks 1, 2, $3, \cdots, p-1$ and also the marks $\alpha i^k + \beta i^{k-1} + \gamma i^{k-2} + \cdots$ for which $\alpha = 1$. It expresses each of these marks as a power i^λ of the primitive root i. If $\alpha \neq 1$, the mark may be written

$$\alpha\left[i^k + \frac{\beta}{\alpha}i^{k-1} + \frac{\gamma}{\alpha}i^{k-2} + \cdots\right],$$

the divisions by α, modulo p, being made mentally or by means of the ordinary tables of indices for the prime p. The second table gives the value of α and the value of the expression in brackets each as a power of i. Their product is a power of i obtained by adding exponents.

It is convenient to use detached coefficients and denote the mark $\alpha i^k + \beta i^{k-1} + \gamma i^{k-2} + \cdots$ by the symbol $\alpha\beta\gamma\cdots$. These coefficients are separated by commas when any one of them is an integer of more than one digit, that is when the modulus p is an integer of more than one digit. Otherwise it is more convenient not to make such a separation.

This paper and the one published in 1905 contain tables of all Galois fields of order $p^n < 1000$, $n > 1$. This is the limit set by Jacobi in his Canon Arithmeticus, which contains tables of indices for all primes less than 1000, *i. e.*, tables of all Galois fields of prime order $p < 1000$.

Example. The first table for the $GF[17^2]$ contains the marks i^λ for which $\lambda = 1, 2, 3, \cdots, 18$ or $\lambda = 18q$. The table gives the value of each of these marks as a polynomial $\alpha i + \beta$. To express i^{243} as a polynomial of the form $\alpha i + \beta$, write $i^{243} = i^{234}\cdot i^9$. From the table, $i^{234} = 12$ and $i^9 = 11i + 3$. Consequently $i^{243} = 12(11i + 3) = 132i + 36 = 13i + 2$ because $132 \equiv 13$ and $36 \equiv 2$ modulo 17. The second table contains in order the integral marks $1, 2, 3, \cdots, 16$ and the marks $\alpha i + \beta$ for which $\alpha = 1$. It gives the value of each of these marks as a power of i. The mark $8i + 7$ does not occur in the table because $\alpha = 8$. To express it as a power of i, write $8i + 7 = 8(i + \frac{7}{8}) = 8(i + 3)$ because $\frac{7}{8} \equiv 3$, modulo 17. From the table, $8 = i^{180}$ and $i + 3 = i^{150}$. Therefore $8i + 7 = i^{180+150} = i^{330} = i^{42}$ because $i^{288} = 1$.

$GF[3^5]$, $i^5 \equiv i + 2$, modulo 3. $i^\lambda = \alpha i^4 + \beta i^3 + \gamma i^2 + \delta i + \varepsilon$.

FIRST TABLE.

λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$
1	10	21	20222	41	11121	61	1211	81	22112	101	211
2	100	22	2211	42	11222	62	12110	82	21111	102	2110
3	1000	23	22110	43	12202	63	21112	83	11101	103	21100
4	10000	24	21121	44	22002	64	11111	84	11022	104	11021
5	12	25	11201	45	20011	65	11122	85	10202	105	10222
6	120	26	12022	46	101	66	11202	86	2002	106	2202
7	1200	27	20202	47	1010	67	12002	87	20020	107	22020
8	12000	28	2011	48	10100	68	20002	88	221	108	20221
9	20012	29	20110	49	1012	69	11	89	2210	109	2201
10	111	30	1121	50	10120	70	110	90	22100	110	22010
11	1110	31	11210	51	1212	71	1100	91	21021	111	20121
12	11100	32	12112	52	12120	72	11000	92	10201	112	1201
13	11012	33	21102	53	21212	73	10012	93	2022	113	12010
14	10102	34	11011	54	12111	74	102	94	20220	114	20112
15	1002	35	10122	55	21122	75	1020	95	2221	115	1111
16	10020	36	1202	56	11211	76	10200	96	22210	116	11110
17	212	37	12020	57	12122	77	2012	97	22121	117	11112
18	2120	38	20212	58	21202	78	20120	98	21201	118	11102
19	21200	39	2111	59	12011	79	1221	99	12001	119	11002
20	12021	40	21110	60	20122	80	12210	100	20022	120	10002
										121	2

SECOND TABLE.

λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$	λ	$\alpha\beta\gamma\delta\epsilon$
242	1	198	1021	4	10000	85	10202	64	11111	37	12020
1	10	149	1022	189	10001	199	10210	117	11112	20	12021
69	11	71	1100	120	10002	181	10211	217	11120	26	12022
5	12	227	1101	208	10010	232	10212	41	11121	140	12100
2	100	230	1102	221	10011	150	10220	65	11122	179	12101
46	101	11	1110	73	10012	235	10221	211	11200	219	12102
74	102	115	1111	16	10020	105	10222	25	11201	62	12110
70	110	216	1112	130	10021	72	11000	66	11202	54	12111
10	111	210	1120	166	10022	165	11001	31	11210	32	12112
209	112	30	1121	48	10100	119	11002	56	11211	52	12120
6	120	143	1122	148	10101	228	11010	218	11212	174	12121
138	121	7	1200	14	10102	34	11011	144	11220	57	12122
222	122	112	1201	215	10110	13	11012	202	11221	224	12200
3	1000	36	1202	142	10111	231	11020	42	11222	154	12201
207	1001	139	1210	229	10112	104	11021	8	12000	43	12202
15	1002	61	1211	50	10120	84	11022	99	12001	80	12210
47	1010	51	1212	159	10121	12	11100	67	12002	176	12211
214	1011	223	1220	35	10122	83	11101	113	12010	145	12212
49	1012	79	1221	76	10200	118	11102	59	12011	161	12220
75	1020	160	1222	92	10201	116	11110	212	12012	184	12221
										203	12222

$GF[2^8]$, $i^8 \equiv i^4 + i^3 + i^2 + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \epsilon i^3 + \zeta i^2 + \eta i + \theta$.

FIRST TABLE.

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$
1	10	49	10001100	97	10101111	145	1001101
2	100	50	101	98	1000011	146	10011010
3	1000	51	1010	99	10000110	147	101001
4	10000	52	10100	100	10001	148	1010010
5	100000	53	101000	101	100010	149	10100100
6	1000000	54	1010000	102	1000100	150	1010101
7	10000000	55	10100000	103	10001000	151	10101010
8	11101	56	1011101	104	1101	152	1001001
9	111010	57	10111010	105	11010	153	10010010
10	1110100	58	1101001	106	110100	154	111001
11	11101000	59	11010010	107	1101000	155	1110010
12	11001101	60	10111001	108	11010000	156	11100100
13	10000111	61	1101111	109	10111101	157	11010101
14	10011	62	11011110	110	1100111	158	10110111
15	100110	63	10100001	111	11001110	159	1110011
16	1001100	64	1011111	112	10000001	160	11100110
17	10011000	65	10111110	113	11111	161	11010001
18	101101	66	1100001	114	111110	162	10111111
19	1011010	67	11000010	115	1111100	163	1100011
20	10110100	68	10011001	116	11111000	164	11000110
21	1110101	69	101111	117	11101101	165	10010001
22	11101010	70	1011110	118	11000111	166	111111
23	11001001	71	10111100	119	10010011	167	1111110
24	10001111	72	1100101	120	111011	168	11111100
25	11	73	11001010	121	1110110	169	11100101
26	110	74	10001001	122	11101100	170	11010111
27	1100	75	1111	123	11000101	171	10110011
28	11000	76	11110	124	10010111	172	1111011
29	110000	77	111100	125	110011	173	11110110
30	1100000	78	1111000	126	1100110	174	11110001
31	11000000	79	11110000	127	11001100	175	11111111
32	10011101	80	11111101	128	10000101	176	11100011
33	100111	81	11100111	129	10111	177	11011011
34	1001110	82	11010011	130	101110	178	10101011
35	10011100	83	10111011	131	1011100	179	1001011
36	100101	84	1101011	132	10111000	180	10010110
37	1001010	85	11010110	133	1101101	181	110001
38	10010100	86	10110001	134	11011010	182	1100010
39	110101	87	1111111	135	10101001	183	11000100
40	1101010	88	11111110	136	1001111	184	10010101
41	11010100	89	11100001	137	10011110	185	110111
42	10110101	90	11011111	138	100001	186	1101110
43	1110111	91	10100011	139	1000010	187	11011100
44	11101110	92	1011011	140	10000100	188	10100101
45	11000001	93	10110110	141	10101	189	1010111
46	10011111	94	1110001	142	101010	190	10101110
47	100011	95	11100010	143	1010100	191	1000001
48	1000110	96	11011001	144	10101000	192	10000010

$GF[2^8]$, $i^8 \equiv i^4 + i^3 + i^2 + 1$, modulo 2.

$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \epsilon i^3 + \zeta i^2 + \eta i + \theta$.

First Table.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$
193	11001	209	10100010	225	100100	241	1011000
194	110010	210	1011001	226	1001000	242	10110000
195	1100100	211	10110010	227	10010000	243	1111101
196	11001000	212	1111001	228	111101	244	11111010
197	10001101	213	11110010	229	1111010	245	11101001
198	111	214	11111001	230	11110100	246	11001111
199	1110	215	11101111	231	11110101	247	10000011
200	11100	216	11000011	232	11110111	248	11011
201	111000	217	10011011	233	11110011	249	110110
202	1110000	218	101011	234	11111011	250	1101100
203	11100000	219	1010110	235	11101011	251	11011000
204	11011101	220	10101100	236	11001011	252	10101101
205	10100111	221	1000101	237	10001011	253	1000111
206	1010011	222	10001010	238	1011	254	10001110
207	10100110	223	1001	239	10110	255	1
208	1010001	224	10010	240	101100		

Second Table.

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$
255	1	138	100001	191	1000001	66	1100001
1	10	101	100010	139	1000010	182	1100010
25	11	47	100011	98	1000011	163	1100011
2	100	225	100100	102	1000100	195	1100100
50	101	36	100101	221	1000101	72	1100101
26	110	15	100110	48	1000110	126	1100110
198	111	33	100111	253	1000111	110	1100111
3	1000	53	101000	226	1001000	107	1101000
223	1001	147	101001	152	1001001	58	1101001
51	1010	142	101010	37	1001010	40	1101010
238	1011	218	101011	179	1001011	84	1101011
27	1100	240	101100	16	1001100	250	1101100
104	1101	18	101101	145	1001101	133	1101101
199	1110	130	101110	34	1001110	186	1101110
75	1111	69	101111	136	1001111	61	1101111
4	10000	29	110000	54	1010000	202	1110000
100	10001	181	110001	208	1010001	94	1110001
224	10010	194	110010	148	1010010	155	1110010
14	10011	125	110011	206	1010011	159	1110011
52	10100	106	110100	143	1010100	10	1110100
141	10101	39	110101	150	1010101	21	1110101
239	10110	249	110110	219	1010110	121	1110110
129	10111	185	110111	189	1010111	43	1110111
28	11000	201	111000	241	1011000	78	1111000
193	11001	154	111001	210	1011001	212	1111001
105	11010	9	111010	19	1011010	229	1111010
248	11011	120	111011	92	1011011	172	1111011
200	11100	77	111100	131	1011100	115	1111100
8	11101	228	111101	56	1011101	243	1111101
76	11110	114	111110	70	1011110	167	1111110
113	11111	166	111111	64	1011111	87	1111111
5	100000	6	1000000	30	1100000	7	10000000

$GF\ [2^8]$, $i^8 \equiv i^4 + i^3 + i^2 + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \epsilon i^3 + \zeta i^2 + \eta i + \theta$.

SECOND TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$
112	10000001	63	10100001	45	11000001	89	11100001
192	10000010	209	10100010	67	11000010	95	11100010
247	10000011	91	10100011	216	11000011	176	11100011
140	10000100	149	10100100	183	11000100	156	11100100
128	10000101	188	10100101	123	11000101	169	11100101
99	10000110	207	10100110	164	11000110	160	11100110
13	10000111	205	10100111	118	11000111	81	11100111
103	10001000	144	10101000	196	11001000	11	11101000
74	10001001	135	10101001	23	11001001	245	11101001
222	10001010	151	10101010	73	11001010	22	11101010
237	10001011	178	10101011	236	11001011	235	11101011
49	10001100	220	10101100	127	11001100	122	11101100
197	10001101	252	10101101	12	11001101	117	11101101
254	10001110	190	10101110	111	11001110	44	11101110
24	10001111	97	10101111	246	11001111	215	11101111
227	10010000	242	10110000	108	11010000	79	11110000
165	10010001	86	10110001	161	11010001	174	11110001
153	10010010	211	10110010	59	11010010	213	11110010
119	10010011	171	10110011	82	11010011	233	11110011
38	10010100	20	10110100	41	11010100	230	11110100
184	10010101	42	10110101	157	11010101	231	11110101
180	10010110	93	10110110	85	11010110	173	11110110
124	10010111	158	10110111	170	11010111	232	11110111
17	10011000	132	10111000	251	11011000	116	11111000
68	10011001	60	10111001	96	11011001	214	11111001
146	10011010	57	10111010	134	11011010	244	11111010
217	10011011	83	10111011	177	11011011	234	11111011
35	10011100	71	10111100	187	11011100	168	11111100
32	10011101	109	10111101	204	11011101	80	11111101
137	10011110	65	10111110	62	11011110	88	11111110
46	10011111	162	10111111	90	11011111	175	11111111
55	10100000	31	11000000	203	11100000		

$GF\ [17^2]$, $i^2 \equiv i + 14$, modulo 17. $i^\lambda = \alpha i + \beta$.

FIRST TABLE. SECOND TABLE.

λ	α , β	λ	α , β	λ	α , β	λ	α , β	λ	α , β	λ	α , β
1	1 , 0	12	7 , 6	108	15	288	1	234	12	87	1 , 6
2	1 , 14	13	13 , 13	126	11	252	2	72	13	266	1 , 7
3	15 , 14	14	9 , 12	144	16	18	3	162	14	171	1 , 8
4	12 , 6	15	4 , 7	162	14	216	4	108	15	58	1 , 9
5	1 , 15	16	11 , 5	180	8	90	5	144	16	183	1 , 10
6	16 , 14	17	16 , 1	198	7	270	6	1	1 , 0	136	1 , 11
7	13 , 3	18	3	216	4	198	7	229	1 , 1	223	1 , 12
8	16 , 12	36	9	234	12	180	8	178	1 , 2	102	1 , 13
9	11 , 3	54	10	252	2	36	9	150	1 , 3	2	1 , 14
10	14 , 1	72	13	270	6	54	10	191	1 , 4	5	1 , 15
11	15 , 9	90	5	288	1	126	11	152	1 , 5	161	1 , 16

$GF\,[7^3]$, $i^3 \equiv i + 5$, modulo 7. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE. SECOND TABLE.

λ	$\alpha\beta\gamma$	λ	$\alpha\beta\gamma$	λ	$\alpha\beta\gamma$	λ	$\alpha\beta\gamma$	λ	$\alpha\beta\gamma$	λ	$\alpha\beta\gamma$
1	10	22	561	43	11	342	1	267	111	78	141
2	100	23	664	44	110	228	2	218	112	16	142
3	15	24	632	45	115	285	3	194	113	307	143
4	150	25	312	46	165	114	4	133	114	64	144
5	515	26	151	47	665	57	5	45	115	195	145
6	134	27	525	48	642	171	6	201	116	148	146
7	355	28	234	49	412	1	10	33	120	4	150
8	511	29	363	50	166	43	11	86	121	26	151
9	164	30	661	51	605	32	12	181	122	142	152
10	655	31	602	52	42	264	13	168	123	82	153
11	542	32	12	53	420	280	14	277	124	323	154
12	404	33	120	54	246	3	15	268	125	134	155
13	16	34	215	55	413	13	16	283	126	296	156
14	160	35	103	56	106	2	100	265	130	14	160
15	615	36	45	57	5	240	101	290	131	312	161
16	142	37	450	114	4	222	102	75	132	186	162
17	435	38	546	171	6	35	103	293	133	245	163
18	326	39	444	228	2	99	104	6	134	9	164
19	221	40	416	285	3	202	105	219	135	46	165
20	233	41	136	342	1	56	106	41	136	50	166
21	353	42	305			44	110	281	140		

$GF\,[19^2]$, $i^2 \equiv i + 17$, modulo 19. $i^\lambda = \alpha i + \beta$.

FIRST TABLE. SECOND TABLE.

λ	$\alpha\,,\,\beta$	λ	$\alpha\,,\,\beta$	λ	$\alpha\,,\,\beta$	λ	$\alpha\,,\,\beta$	λ	$\alpha\,,\,\beta$	λ	$\alpha\,,\,\beta$
1	1 , 0	14	4 , 2	160	9	360	1	140	14	46	1 , 8
2	1 , 17	15	6 , 11	180	18	20	2	220	15	310	1 , 9
3	18 , 17	16	17 , 7	200	17	260	3	80	16	334	1 , 10
4	16 , 2	17	5 , 4	220	15	40	4	200	17	288	1 , 11
5	18 , 6	18	9 , 9	240	11	320	5	180	18	284	1 , 12
6	5 , 2	19	18 , 1	260	3	280	6	1	1 , 0	185	1 , 13
7	7 , 9	20	2	280	6	120	7	218	1 , 1	193	1 , 14
8	16 , 5	40	4	300	12	60	8	183	1 , 2	331	1 , 15
9	2 , 6	60	8	320	5	160	9	349	1 , 3	57	1 , 16
10	8 , 15	80	16	340	10	340	10	247	1 , 4	2	1 , 17
11	4 , 3	100	13	360	1	240	11	95	1 , 5	199	1 , 18
12	7 , 11	120	7			300	12	176	1 , 6		
13	18 , 5	140	14			100	13	252	1 , 7		

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa$.

FIRST TABLE.

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$
1	10	49	111111111	97	1100110	145	101011011
2	100	50	11100001	98	11001100	146	110101001
3	1000	51	111000010	99	110011000	147	1001101
4	10000	52	10011011	100	101111	148	10011010
5	100000	53	100110110	101	1011110	149	100110100
6	1000000	54	101110011	102	10111100	150	101110111
7	10000000	55	111111001	103	101111000	151	111110001
8	100000000	56	11101101	104	111101111	152	11111101
9	100011111	57	111011010	105	11000001	153	111111010
10	100100001	58	10101011	106	110000010	154	11101011
11	101011101	59	101010110	107	11011	155	111010110
12	110100101	60	110110011	108	110110	156	10110011
13	1010101	61	1111001	109	1101100	157	101100110
14	10101010	62	11110010	110	11011000	158	111010011
15	101010100	63	111100100	111	110110000	159	10111001
16	110110111	64	11010111	112	1111111	160	101110010
17	1110001	65	110101110	113	11111110	161	111111011
18	11100010	66	1000011	114	111111100	162	11101001
19	111000100	67	10000110	115	11100111	163	111010010
20	10010111	68	100001100	116	111001110	164	10111011
21	100101110	69	100000111	117	10000011	165	101110110
22	101000011	70	100010001	118	100000110	166	111110011
23	110011001	71	100111101	119	100010011	167	11111001
24	101101	72	101100101	120	100111001	168	111110010
25	1011010	73	111010101	121	101101101	169	11111011
26	10110100	74	10110101	122	111000101	170	111110110
27	101101000	75	101101010	123	10010101	171	11110011
28	111001111	76	111001011	124	100101010	172	111100110
29	10000001	77	10001001	125	101001011	173	11010011
30	100000010	78	100010010	126	110001001	174	110100110
31	100011011	79	100111011	127	1101	175	1010011
32	100101001	80	101101001	128	11010	176	10100110
33	101001101	81	111001101	129	110100	177	101001100
34	110000101	82	10000101	130	1101000	178	110000111
35	10101	83	100001010	131	11010000	179	10001
36	101010	84	100001011	132	110100000	180	100010
37	1010100	85	100001001	133	1011111	181	1000100
38	10101000	86	100001101	134	10111110	182	10001000
39	101010000	87	100000101	135	101111100	183	100010000
40	110111111	88	100010101	136	111100111	184	100111111
41	1100001	89	100110101	137	11010001	185	101100001
42	11000010	90	101110101	138	110100010	186	111011101
43	110000100	91	111110101	139	1011011	187	10100101
44	10111	92	11110101	140	10110110	188	101001010
45	101110	93	111101010	141	101101100	189	110001011
46	1011100	94	11001011	142	111000111	190	1001
47	10111000	95	110010110	143	10010001	191	10010
48	101110000	96	110011	144	100100010	192	100100

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa$.

First Table.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$
193	1001000	241	10110001	289	100100100	337	100111010
194	10010000	242	101100010	290	101010111	338	101101011
195	100100000	243	111011011	291	110110001	339	111001001
196	101011111	244	10101001	292	1111101	340	10001101
197	110100001	245	101010010	293	11111010	341	100011010
198	1011101	246	110111011	294	111110100	342	100101011
199	10111010	247	1101001	295	11110111	343	101001001
200	101110100	248	11010010	296	111101110	344	110001101
201	111110111	249	110100100	297	11000011	345	101
202	11110001	250	1010111	298	110000110	346	1010
203	111100010	251	10101110	299	10011	347	10100
204	11011011	252	101011100	300	100110	348	101000
205	110110110	253	110100111	301	1001100	349	1010000
206	1110011	254	1010001	302	10011000	350	10100000
207	11100110	255	10100010	303	100110000	351	101000000
208	111001100	256	101000100	304	101111111	352	110011111
209	10000111	257	110010111	305	111100001	353	100001
210	100001110	258	110001	306	11011101	354	1000010
211	100000011	259	1100010	307	110111010	355	10000100
212	100011001	260	11000100	308	1101011	356	100001000
213	100101101	261	110001000	309	11010110	357	100001111
214	101000101	262	1111	310	110101100	358	100000001
215	110010101	263	11110	311	1000111	359	100011101
216	110101	264	111100	312	10001110	360	100100101
217	1101010	265	1111000	313	100011100	361	101010101
218	11010100	266	11110000	314	100100111	362	110110101
219	110101000	267	111100000	315	101010001	363	1110101
220	1001111	268	11011111	316	110111101	364	11101010
221	10011110	269	110111110	317	1100101	365	111010100
222	100111100	270	1100011	318	11001010	366	10110111
223	101100111	271	11000110	319	110010100	367	101101110
224	111010001	272	110001100	320	110111	368	111000011
225	10111101	273	111	321	1101110	369	10011001
226	101111010	274	1110	322	11011100	370	100110010
227	111101011	275	11100	323	110111000	371	101111011
228	11001001	276	111000	324	1101111	372	111101001
229	110010010	277	1110000	325	11011110	373	11001101
230	111011	278	11100000	326	110111100	374	110011010
231	1110110	279	111000000	327	1100111	375	101011
232	11101100	280	10011111	328	11001110	376	1010110
233	111011000	281	100111110	329	110011100	377	10101100
234	10101111	282	101100011	330	100111	378	101011000
235	101011110	283	111011001	331	1001110	379	110101111
236	110100011	284	10101101	332	10011100	380	1000001
237	1011001	285	101011010	333	100111000	381	10000010
238	10110010	286	110101011	334	101101111	382	100000100
239	101100100	287	1001001	335	111000001	383	100010111
240	111010111	288	10010010	336	10011101	384	100110001

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa$.

FIRST TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$
385	101111101	417	111010000	449	1101101	481	1000101
386	111100101	418	10111111	450	11011010	482	10001010
387	11010101	419	101111110	451	110110100	483	100010100
388	110101010	420	111100011	452	1110111	484	100110111
389	1001011	421	11011001	453	11101110	485	101110001
390	10010110	422	110110010	454	111011100	486	111111101
391	100101100	423	1111011	455	10100111	487	11100101
392	101000111	424	11110110	456	101001110	488	111001010
393	110010001	425	111101100	457	110000011	489	10001011
394	111101	426	11000111	458	11001	490	100010110
395	1111010	427	110001110	459	110010	491	100110011
396	11110100	428	11	460	1100100	492	101111001
397	111101000	429	110	461	11001000	493	111101101
398	11001111	430	1100	462	110010000	494	11000101
399	110011110	431	11000	463	111111	495	110001010
400	100011	432	110000	464	1111110	496	1011
401	1000110	433	1100000	465	11111100	497	10110
402	10001100	434	11000000	466	111111000	498	101100
403	100011000	435	110000000	467	11101111	499	1011000
404	100101111	436	11111	468	111011110	500	10110000
405	101000001	437	111110	469	10100011	501	101100000
406	110011101	438	1111100	470	101000110	502	111011111
407	100101	439	11111000	471	110010011	503	10100001
408	1001010	440	111110000	472	111001	504	101000010
409	10010100	441	11111111	473	1110010	505	110011011
410	100101000	442	111111110	474	11100100	506	101001
411	101001111	443	11100011	475	111001000	507	1010010
412	110000001	444	111000110	476	10001111	508	10100100
413	11101	445	10010011	477	100011110	509	101001000
414	111010	446	100100110	478	100100011	510	110001111
415	1110100	447	101010011	479	101011001	511	1
416	11101000	448	110111001	480	110101101		

$$GF[2^9],\ i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1, \text{ modulo } 2.$$
$$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa.$$

SECOND TABLE.

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$
511	1	353	100001	380	1000001	41	1100001
1	10	180	100010	354	1000010	259	1100010
428	11	400	100011	66	1000011	270	1100011
2	100	192	100100	181	1000100	460	1100100
345	101	407	100101	481	1000101	317	1100101
429	110	300	100110	401	1000110	97	1100110
273	111	330	100111	311	1000111	327	1100111
3	1000	348	101000	193	1001000	130	1101000
190	1001	506	101001	287	1001001	247	1101001
346	1010	36	101010	408	1001010	217	1101010
496	1011	375	101011	389	1001011	308	1101011
430	1100	498	101100	301	1001100	109	1101100
127	1101	24	101101	147	1001101	449	1101101
274	1110	45	101110	331	1001110	321	1101110
262	1111	100	101111	220	1001111	324	1101111
4	10000	432	110000	349	1010000	277	1110000
179	10001	258	110001	254	1010001	17	1110001
191	10010	459	110010	507	1010010	473	1110010
299	10011	96	110011	175	1010011	206	1110011
347	10100	129	110100	37	1010100	415	1110100
35	10101	216	110101	13	1010101	363	1110101
497	10110	108	110110	376	1010110	231	1110110
44	10111	320	110111	250	1010111	452	1110111
431	11000	276	111000	499	1011000	265	1111000
458	11001	472	111001	237	1011001	61	1111001
128	11010	414	111010	25	1011010	395	1111010
107	11011	230	111011	139	1011011	423	1111011
275	11100	264	111100	46	1011100	438	1111100
413	11101	394	111101	198	1011101	292	1111101
263	11110	437	111110	101	1011110	464	1111110
436	11111	463	111111	133	1011111	112	1111111
5	100000	6	1000000	433	1100000	7	10000000

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \epsilon i^4 + \varsigma i^3 + \eta i^2 + \theta i + \kappa$.

SECOND TABLE.—*Continued.*

λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ
29	10000001	241	10110001	50	11100001	70	100010001
381	10000010	238	10110010	18	11100010	78	100010010
117	10000011	156	10110011	443	11100011	119	100010011
355	10000100	26	10110100	474	11100100	483	100010100
82	10000101	74	10110101	487	11100101	88	100010101
67	10000110	140	10110110	207	11100110	490	100010110
209	10000111	366	10110111	115	11100111	383	100010111
182	10001000	47	10111000	416	11101000	403	100011000
77	10001001	159	10111001	162	11101001	212	100011001
482	10001010	199	10111010	364	11101010	341	100011010
489	10001011	164	10111011	154	11101011	31	100011011
402	10001100	102	10111100	232	11101100	313	100011100
340	10001101	225	10111101	56	11101101	359	100011101
312	10001110	134	10111110	453	11101110	477	100011110
476	10001111	418	10111111	467	11101111	9	100011111
194	10010000	434	11000000	266	11110000	195	100100000
143	10010001	105	11000001	202	11110001	10	100100001
288	10010010	42	11000010	62	11110010	144	100100010
445	10010011	297	11000011	171	11110011	478	100100011
409	10010100	260	11000100	396	11110100	289	100100100
123	10010101	494	11000101	92	11110101	360	100100101
390	10010110	271	11000110	424	11110110	446	100100110
20	10010111	426	11000111	295	11110111	314	100100111
302	10011000	461	11001000	439	11111000	410	100101000
369	10011001	228	11001001	167	11111001	32	100101001
148	10011010	318	11001010	293	11111010	124	100101010
52	10011011	94	11001011	169	11111011	342	100101011
332	10011100	98	11001100	465	11111100	391	100101100
336	10011101	373	11001101	152	11111101	213	100101101
221	10011110	328	11001110	113	11111110	21	100101110
280	10011111	398	11001111	441	11111111	404	100101111
350	10100000	131	11010000	8	100000000	303	100110000
503	10100001	137	11010001	358	100000001	384	100110001
255	10100010	248	11010010	30	100000010	370	100110010
469	10100011	173	11010011	211	100000011	491	100110011
508	10100100	218	11010100	382	100000100	149	100110100
187	10100101	387	11010101	87	100000101	89	100110101
176	10100110	309	11010110	118	100000110	53	100110110
455	10100111	64	11010111	69	100000111	484	100110111
38	10101000	110	11011000	356	100001000	333	100111000
244	10101001	421	11011001	85	100001001	120	100111001
14	10101010	450	11011010	83	100001010	337	100111010
58	10101011	204	11011011	84	100001011	79	100111011
377	10101100	322	11011100	68	100001100	222	100111100
284	10101101	306	11011101	86	100001101	71	100111101
251	10101110	325	11011110	210	100001110	281	100111110
234	10101111	268	11011111	357	100001111	184	100111111
500	10110000	278	11100000	183	100010000	351	101000000

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \epsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa$.

SECOND TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\kappa$
405	101000001	485	101110001	197	110100001	224	111010001
504	101000010	160	101110010	138	110100010	163	111010010
22	101000011	54	101110011	236	110100011	158	111010011
256	101000100	200	101110100	249	110100100	365	111010100
214	101000101	90	101110101	12	110100101	73	111010101
470	101000110	165	101110110	174	110100110	155	111010110
392	101000111	150	101110111	253	110100111	240	111010111
509	101001000	103	101111000	219	110101000	233	111011000
343	101001001	492	101111001	146	110101001	283	111011001
188	101001010	226	101111010	388	110101010	57	111011010
125	101001011	371	101111011	286	110101011	243	111011011
177	101001100	135	101111100	310	110101100	454	111011100
33	101001101	385	101111101	480	110101101	186	111011101
456	101001110	419	101111110	65	110101110	468	111011110
411	101001111	304	101111111	379	110101111	502	111011111
39	101010000	435	110000000	111	110110000	267	111100000
315	101010001	412	110000001	291	110110001	305	111100001
245	101010010	106	110000010	422	110110010	203	111100010
447	101010011	457	110000011	60	110110011	420	111100011
15	101010100	43	110000100	451	110110100	63	111100100
361	101010101	34	110000101	362	110110101	386	111100101
59	101010110	298	110000110	205	110110110	172	111100110
290	101010111	178	110000111	16	110110111	136	111100111
378	101011000	261	110001000	323	110111000	397	111101000
479	101011001	126	110001001	448	110111001	372	111101001
285	101011010	495	110001010	307	110111010	93	111101010
145	101011011	189	110001011	246	110111011	227	111101011
252	101011100	272	110001100	326	110111100	425	111101100
11	101011101	344	110001101	316	110111101	493	111101101
235	101011110	427	110001110	269	110111110	296	111101110
196	101011111	510	110001111	40	110111111	104	111101111
501	101100000	462	110010000	279	111000000	440	111110000
185	101100001	393	110010001	335	111000001	151	111110001
242	101100010	229	110010010	51	111000010	168	111110010
282	101100011	471	110010011	368	111000011	166	111110011
239	101100100	319	110010100	19	111000100	294	111110100
72	101100101	215	110010101	122	111000101	91	111110101
157	101100110	95	110010110	444	111000110	170	111110110
223	101100111	257	110010111	142	111000111	201	111110111
27	101101000	99	110011000	475	111001000	466	111111000
80	101101001	23	110011001	339	111001001	55	111111001
75	101101010	374	110011010	488	111001010	153	111111010
338	101101011	505	110011011	76	111001011	161	111111011
141	101101100	329	110011100	208	111001100	114	111111100
121	101101101	406	110011101	81	111001101	486	111111101
367	101101110	399	110011110	116	111001110	442	111111110
334	101101111	352	110011111	28	111001111	49	111111111
48	101110000	132	110100000	417	111010000		

$GF[23^2]$, $i^2 \equiv i + 16$, modulo 23. $i^\lambda = \alpha i + \beta$.

FIRST TABLE. SECOND TABLE.

λ	α, β	λ	α, β	λ	α, β	λ	α, β	λ	α, β	λ	α, β
1	1, 0	16	19, 19	192	12	528	1	288	16	329	1, 8
2	1, 16	17	15, 5	216	15	336	2	120	17	469	1, 9
3	17, 16	18	20, 10	240	13	48	3	432	18	366	1, 10
4	10, 19	19	7, 21	264	22	144	4	408	19	132	1, 11
5	6, 22	20	5, 20	288	16	168	5	312	20	234	1, 12
6	5, 4	21	2, 11	312	20	384	6	72	21	491	1, 13
7	9, 11	22	13, 9	336	2	24	7	264	22	439	1, 14
8	20, 6	23	22, 1	360	14	480	8	1	1, 0	423	1, 15
9	3, 21	24	7	384	6	96	9	136	1, 1	2	1, 16
10	1, 2	48	3	408	19	504	10	10	1, 2	213	1, 17
11	3, 16	72	21	432	18	456	11	523	1, 3	28	1, 18
12	19, 2	96	9	456	11	192	12	380	1, 4	149	1, 19
13	21, 5	120	17	480	8	240	13	411	1, 5	494	1, 20
14	3, 14	144	4	504	10	360	14	310	1, 6	224	1, 21
15	17, 2	168	5	528	1	216	15	489	1, 7	287	1, 22

$GF[5^4]$, $i^4 \equiv i^3 + i + 2$, modulo 5. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.

λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$
1	10	28	3311	55	323	82	3434	109	1020	136	3044
2	100	29	1141	56	3230	83	2321	110	1212	137	3421
3	1000	30	2422	57	331	84	234	111	3132	138	2241
4	1012	31	1244	58	3310	85	2340	112	4301	139	4434
5	1132	32	3402	59	1131	86	424	113	2003	140	3333
6	2332	33	2001	60	2322	87	4240	114	2004	141	1311
7	344	34	2034	61	244	88	1443	115	2014	142	4122
8	3440	35	2314	62	2440	89	442	116	2114	143	213
9	2431	36	114	63	1424	90	4420	117	3114	144	2130
10	1334	37	1140	64	202	91	3243	118	4121	145	3324
11	4302	38	2412	65	2020	92	411	119	203	146	1221
12	2013	39	1144	66	2224	93	4110	120	2030	147	3222
13	2104	40	2402	67	4214	94	143	121	2324	148	201
14	3014	41	1044	68	1133	95	1430	122	214	149	2010
15	3121	42	1402	69	2342	96	312	123	2140	150	2124
16	4241	43	32	70	444	97	3120	124	3424	151	3214
17	1403	44	320	71	4440	98	4231	125	2221	152	121
18	42	45	3200	72	3443	99	1303	126	4234	153	1210
19	420	46	31	73	2411	100	4042	127	1333	154	3112
20	4200	47	310	74	1134	101	4413	128	4342	155	4101
21	1043	48	3100	75	2302	102	3123	129	2413	156	3
22	1442	49	4031	76	44	103	4211	130	1104	312	4
23	432	50	4303	77	440	104	1103	131	2002	468	2
24	4320	51	2023	78	4400	105	2042	132	2044	624	1
25	2243	52	2204	79	3043	106	2444	133	2414		
26	4404	53	4014	80	3411	107	1414	134	1114		
27	3033	54	4133	81	2141	108	102	135	2102		

$GF[5^4]$, $i^4 \equiv i^3 + i + 2$, modulo 5. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

SECOND TABLE.

λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$
624	1	475	133	276	1040	413	1142	31	1244	231	1401
468	2	299	134	365	1041	613	1143	332	1300	42	1402
156	3	512	140	190	1042	39	1144	291	1301	17	1403
312	4	523	141	21	1043	516	1200	169	1302	467	1404
1	10	240	142	41	1044	196	1201	99	1303	524	1410
388	11	94	143	390	1100	362	1202	500	1304	216	1411
514	12	404	144	338	1101	323	1203	399	1310	277	1412
330	13	3	1000	208	1102	424	1204	141	1311	239	1413
511	14	287	1001	104	1103	153	1210	306	1312	107	1414
2	100	270	1002	130	1104	186	1211	550	1313	241	1420
220	101	189	1003	383	1110	110	1212	328	1314	225	1421
108	102	269	1004	608	1111	440	1213	279	1320	366	1422
304	103	221	1010	222	1112	579	1214	438	1321	619	1423
275	104	495	1011	281	1113	218	1220	548	1322	63	1424
389	110	4	1012	134	1114	146	1221	237	1323	95	1430
382	111	412	1013	526	1120	262	1222	410	1324	559	1431
525	112	207	1014	451	1121	585	1223	476	1330	191	1432
401	113	109	1020	496	1122	622	1224	540	1331	454	1433
36	114	261	1021	294	1123	336	1230	272	1332	430	1434
515	120	288	1022	181	1124	194	1231	127	1333	405	1440
152	121	482	1023	402	1130	289	1232	10	1334	162	1441
217	122	361	1024	59	1131	229	1233	300	1340	22	1442
335	123	305	1030	5	1132	285	1234	379	1341	88	1443
564	124	547	1031	68	1133	565	1240	605	1342	615	1444
331	130	271	1032	74	1134	570	1241	592	1343		
398	131	604	1033	37	1140	483	1242	415	1344		
278	132	168	1034	29	1141	165	1243	513	1400		

$GF[3^6]$, $i^6 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

FIRST TABLE.

λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$
1	10	14	12100	27	11121	40	120220	53	101102
2	100	15	121000	28	111210	41	202211	54	11001
3	1000	16	210011	29	112111	42	22102	55	110010
4	10000	17	100102	30	121121	43	221020	56	100111
5	100000	18	1001	31	211221	44	210222	57	1121
6	11	19	10010	32	112202	45	102212	58	11210
7	110	20	100100	33	122001	46	22101	59	112100
8	1100	21	1011	34	220021	47	221010	60	121011
9	11000	22	10110	35	200202	48	210122	61	210121
10	110000	23	101100	36	2012	49	101212	62	101202
11	100011	24	11011	37	20120	50	12101	63	12001
12	121	25	110110	38	201200	51	121010	64	120010
13	1210	26	101111	39	12022	52	210111	65	200111

$GF\ [3^6]$, $i^6 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

FIRST TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$
66	1102	114	220022	162	11122	210	2221	258	12102
67	11020	115	200212	163	111220	211	22210	259	121020
68	110200	116	2112	164	112211	212	222100	260	210211
69	102011	117	21120	165	122121	213	221022	261	102102
70	20121	118	211200	166	221221	214	210212	262	21001
71	201210	119	112022	167	212202	215	102112	263	210010
72	12122	120	120201	168	122012	216	21101	264	100122
73	121220	121	202021	169	220101	217	211010	265	1201
74	212211	122	20202	170	201002	218	110122	266	12010
75	122102	123	202020	171	10012	219	101201	267	120100
76	221001	124	20222	172	100120	220	12021	268	201011
77	210002	125	202220	173	1211	221	120210	269	10102
78	100012	126	22222	174	12110	222	202111	270	101020
79	101	127	222220	175	121100	223	21102	271	10211
80	1010	128	222222	176	211011	224	211020	272	102110
81	10100	129	222212	177	110102	225	110222	273	21111
82	101000	130	222112	178	101001	226	102201	274	211110
83	10011	131	221112	179	10021	227	22021	275	111122
84	100110	132	211112	180	100210	228	220210	276	111201
85	1111	133	111112	181	2111	229	202122	277	112021
86	11110	134	111101	182	21110	230	21212	278	120221
87	111100	135	111021	183	211100	231	212120	279	202221
88	111011	136	110221	184	111022	232	121222	280	22202
89	110121	137	102221	185	110201	233	212201	281	222020
90	101221	138	22221	186	102021	234	122002	282	220222
91	12221	139	222210	187	20221	235	220001	283	202212
92	122210	140	222122	188	202210	236	200002	284	22112
93	222111	141	221212	189	22122	237	12	285	221120
94	221102	142	212112	190	221220	238	120	286	211222
95	211012	143	121112	191	212222	239	1200	287	112212
96	110112	144	211101	192	122212	240	12000	288	122101
97	101101	145	111002	193	222101	241	120000	289	221021
98	11021	146	110001	194	221002	242	200011	290	210202
99	110210	147	100021	195	210012	243	102	291	102012
100	102111	148	221	196	100112	244	1020	292	20101
101	21121	149	2210	197	1101	245	10200	293	201010
102	211210	150	22100	198	11010	246	102000	294	10122
103	112122	151	221000	199	110100	247	20011	295	101220
104	121201	152	210022	200	101011	248	200110	296	12211
105	212021	153	100212	201	10121	249	1122	297	122110
106	120202	154	2101	202	101210	250	11220	298	221111
107	202001	155	21010	203	12111	251	112200	299	211102
108	20002	156	210100	204	121110	252	122011	300	111012
109	200020	157	101022	205	211111	253	220121	301	110101
110	222	158	10201	206	111102	254	201202	302	101021
111	2220	159	102010	207	111001	255	12012	303	10221
112	22200	160	20111	208	110021	256	120120	304	102210
113	222000	161	201110	209	100221	257	201211	305	22111

$GF\,[3^6]$, $i^6 \equiv i+1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \varepsilon i + \zeta$.

FIRST TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$
306	221110	318	121200	330	2021	342	211211	354	220110
307	211122	319	212011	331	20210	343	112102	355	201122
308	111212	320	120102	332	202100	344	121001	356	11212
309	112101	321	201001	333	21022	345	210021	357	112120
310	121021	322	10002	334	210220	346	100202	358	121211
311	210221	323	100020	335	102222	347	2001	359	212121
312	102202	324	211	336	22201	348	20010	360	121202
313	22001	325	2110	337	222010	349	200100	361	212001
314	220010	326	21100	338	220122	350	1022	362	120002
315	200122	327	211000	339	201212	351	10220	363	200001
316	1212	328	110022	340	12112	352	102200	364	2
317	12120	329	100201	341	121120	353	22011		

SECOND TABLE.

λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$
728	1	316	1212	591	11012	340	12112	153	100212
1	10	689	1220	67	11020	317	12120	612	100220
6	11	480	1221	98	11021	594	12121	209	100221
237	12	545	1222	717	11022	72	12122	429	100222
2	100	4	10000	476	11100	690	12200	82	101000
79	101	472	10001	644	11101	587	12201	178	101001
243	102	322	10002	700	11102	580	12202	471	101002
7	110	19	10010	86	11110	481	12210	487	101010
474	111	83	10011	490	11111	296	12211	200	101011
512	112	171	10012	502	11112	465	12212	485	101012
238	120	712	10020	575	11120	546	12220	270	101020
12	121	179	10021	27	11121	91	12221	302	101021
688	122	611	10022	162	11122	637	12222	157	101022
3	1000	81	10100	514	11200	5	100000	23	101100
18	1001	486	10101	406	11201	600	100001	97	101101
711	1002	269	10102	410	11202	727	100002	53	101102
80	1010	22	10110	58	11210	473	100010	489	101110
21	1011	488	10111	553	11211	11	100011	26	101111
694	1012	551	10112	356	11212	78	100012	643	101112
244	1020	695	10120	250	11220	323	100020	552	101120
400	1021	201	10121	648	11221	147	100021	647	101121
350	1022	294	10122	669	11222	606	100022	405	101122
8	1100	245	10200	240	12000	20	100100	696	101200
197	1101	158	10201	63	12001	399	100101	219	101201
66	1102	656	10202	626	12002	17	100102	62	101202
475	1110	401	10210	266	12010	84	100110	202	101210
85	1111	271	10211	697	12011	56	100111	593	101211
574	1112	434	10212	255	12012	196	100112	49	101212
513	1120	351	10220	519	12020	172	100120	295	101220
57	1121	303	10221	220	12021	479	100121	90	101221
249	1122	524	10222	39	12022	264	100122	586	101222
239	1200	9	11000	14	12100	713	100200	246	102000
265	1201	54	11001	50	12101	329	100201	534	102001
518	1202	677	11002	258	12102	346	100202	685	102002
13	1210	198	11010	174	12110	180	100210	159	102010
173	1211	24	11011	203	12111	679	100211	69	102011

$GF\ [3^6]$, $i^6 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \varepsilon i + \zeta$.

SECOND TABLE.—*Continued.*

λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ
291	102012	592	110120	494	111221	380	120022	318	121200
657	102020	89	110121	457	111222	267	120100	104	121201
186	102021	218	110122	515	112000	654	120101	360	121202
632	102022	68	110200	558	112001	320	120102	595	121210
402	102100	185	110201	440	112002	698	120110	358	121211
618	102101	533	110202	407	112010	408	120111	723	121212
261	102102	99	110210	577	112011	675	120112	73	121220
272	102110	702	110211	653	112012	256	120120	506	121221
100	102111	617	110212	411	112020	578	120121	232	121222
215	102112	718	110220	277	112021	624	120122	691	122000
435	102120	136	110221	119	112022	520	120200	33	122001
703	102121	225	110222	59	112100	120	120201	234	122002
621	102122	477	111000	309	112101	106	120202	588	122010
352	102200	207	111001	343	112102	221	120210	252	122011
226	102201	145	111002	554	112110	412	120211	168	122012
312	102202	645	111010	29	112111	425	120212	581	122020
304	102210	88	111011	530	112112	40	120220	459	122021
719	102211	300	111012	357	112120	278	120221	540	122022
45	102212	701	111020	505	112121	416	120222	482	122100
525	102220	135	111021	103	112122	15	121000	288	122101
137	102221	184	111022	251	112200	344	121001	75	122102
335	102222	87	111100	458	112201	725	121002	297	122110
10	110000	134	111101	32	112202	51	121010	650	122111
146	110001	206	111102	649	112210	60	121011	395	122112
599	110002	491	111110	164	112211	469	121012	466	122120
55	110010	492	111111	287	112212	259	121020	165	122121
478	110011	133	111112	670	112220	310	121021	706	122122
398	110012	503	111120	495	112221	683	121022	547	122200
678	110020	493	111121	662	112222	175	121100	663	122201
208	110021	275	111122	241	120000	531	121101	508	122202
328	110022	576	111200	441	120001	597	121102	92	122210
199	110100	276	111201	362	120002	204	121110	671	122211
301	110101	557	111202	64	120010	555	121111	192	122212
177	110102	28	111210	516	120011	143	121112	638	122220
25	110110	504	111211	709	120012	341	121120	496	122221
646	110111	308	111212	627	120020	30	121121	569	122222
96	110112	163	111220	559	120021	438	121122		

$GF[29^2]$, $i^2 \equiv i + 26$, modulo 29, $i^\lambda = \alpha i + \beta$.

FIRST TABLE. SECOND TABLE.

λ	α, β	λ	α, β	λ	α, β	λ	α, β	λ	α, β	λ	α, β
1	1, 0	20	22, 1	300	5	840	1	480	20	43	1, 10
2	1, 26	21	23, 21	330	15	510	2	270	21	741	1, 11
3	27, 26	22	15, 18	360	16	30	3	660	22	417	1, 12
4	24, 6	23	4, 13	390	19	180	4	120	23	565	1, 13
5	1, 15	24	17, 17	420	28	300	5	720	24	195	1, 14
6	16, 26	25	5, 7	450	26	540	6	600	25	5	1, 15
7	13, 10	26	12, 14	480	20	240	7	450	26	753	1, 16
8	23, 19	27	26, 22	510	2	690	8	90	27	69	1, 17
9	13, 18	28	19, 9	540	6	60	9	420	28	827	1, 18
10	2, 19	29	28, 1	570	18	810	10	1	1, 0	522	1, 19
11	21, 23	30	3	600	25	150	11	234	1, 1	469	1, 20
12	15, 24	60	9	630	17	210	12	478	1, 2	728	1, 21
13	10, 13	90	27	660	22	780	13	67	1, 3	124	1, 22
14	23, 28	120	23	690	8	750	14	200	1, 4	706	1, 23
15	22, 18	150	11	720	24	330	15	734	1, 5	340	1, 24
16	11, 21	180	4	750	14	360	16	656	1, 6	683	1, 25
17	3, 25	210	12	780	13	630	17	532	1, 7	2	1, 26
18	28, 20	240	7	810	10	570	18	581	1, 8	486	1, 27
19	19, 3	270	21	840	1	390	19	438	1, 9	449	1, 28

$GF[31^2]$, $i^2 \equiv i + 19$, modulo 31, $i^\lambda = \alpha i + \beta$.

FIRST TABLE. SECOND TABLE.

λ	α, β	λ	α, β	λ	α, β	λ	α, β	λ	α, β	λ	α, β
1	1, 0	21	7, 3	320	25	960	1	352	21	471	1, 10
2	1, 19	22	10, 9	352	21	192	2	736	22	542	1, 11
3	20, 19	23	19, 4	384	4	608	3	96	23	509	1, 12
4	8, 8	24	23, 20	416	17	384	4	224	24	888	1, 13
5	16, 28	25	12, 3	448	18	640	5	320	25	530	1, 14
6	13, 25	26	15, 11	480	30	800	6	160	26	912	1, 15
7	7, 30	27	26, 6	512	19	704	7	864	27	590	1, 16
8	6, 9	28	1, 29	544	11	576	8	128	28	168	1, 17
9	15, 21	29	30, 19	576	8	256	9	672	29	899	1, 18
10	5, 6	30	18, 12	608	3	832	10	480	30	2	1, 19
11	11, 2	31	30, 1	640	5	544	11	1	1, 0	681	1, 20
12	13, 23	32	12	672	29	32	12	388	1, 1	38	1, 21
13	5, 30	64	20	704	7	928	13	884	1, 2	263	1, 22
14	4, 2	96	23	736	22	896	14	427	1, 3	175	1, 23
15	6, 14	128	28	768	16	288	15	150	1, 4	243	1, 24
16	20, 21	160	26	800	6	768	16	827	1, 5	197	1, 25
17	10, 8	192	2	832	10	416	17	333	1, 6	330	1, 26
18	18, 4	224	24	864	27	448	18	145	1, 7	277	1, 27
19	22, 1	256	9	896	14	512	19	953	1, 8	44	1, 28
20	23, 15	288	15	928	13	64	20	698	1, 9	28	1, 29
				960	1					511	1, 30

INTEGRAL EQUATIONS.

An Introduction to the Study of Integral Equations. By MAXIME BÔCHER. Cambridge Tracts in Mathematics and Mathematical Physics, No. 10. Cambridge, The University Press, 1909. iv + 72 pages.

IN 1823 Abel proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the first kind, and in 1837 Liouville showed that the determination of a particular solution of a linear differential equation of the second order could be effected by solving an integral equation of a different type, called the integral equation of the second kind. The ripple of mathematical interest which had its origin in these investigations increased at first but slowly. Recently, however, stimulated by the researches of Volterra, Fredholm, and Hilbert in the period between 1896 and the present time, that which seemed at first only a ripple has grown into a formidable wave which bids fair to carry the integral equation theory into a place beside the most important of the mathematical disciplines. Notwithstanding the rapidly multiplying investigations in integral equations and the numerous applications of them which have been made, the sources of information concerning the theory have remained widely scattered and none too easily accessible to any but the specialist in the subject. It is with a hearty welcome, therefore, that the thoughtful mathematician will receive an introduction to the theory written by so clear a thinker and writer as the author of the book which is the subject of this review. As stated in the preface, the purpose of the author was to furnish the careful student with a firm foundation for further study, and at the same time so to display and arrange the principal theorems that one may with only a superficial reading obtain some idea of the subject. These objects seem to have been successfully attained. The book should furthermore be very useful as a text in an introductory course, especially if the instructor would content himself at first with the discussion of integral equations whose kernels are continuous or have discontinuities of the explicit forms which occur in the problem of Abel, treated in § 2, and other applications.

The emphasis which is placed on the historical development of the subject is an interesting feature of the book. After an introductory section in which some essential theorems concerning definite integrals are set down, the problems of Abel and Liouville, probably the earliest applications of integral equations, are discussed, and thereafter the reader's attention is constantly directed to the contributions which have been made by Volterra, Fredholm, Hilbert, Schmidt, Kneser, and other writers. In the papers of Hilbert and Schmidt the kernel of the integral equation is first assumed to be symmetric and continuous. Later they show that the theory for unsymmetric kernels can be regarded as an application of the theory for the symmetric case. Professor Bôcher, following the earlier writers, has inverted this arrangement, which seems more convenient since many of the principal results follow as easily for the unsymmetric as for the symmetric hypothesis. The author has also admitted from the start certain kinds of discontinuities in the kernels of his equations. This is perhaps disconcerting to the reader who wishes merely a survey of the theory, but the applications of equations with discontinuities are so frequent that one must feel that the admission is justified. It seems regrettable that more of the applications of the integral equation theory, for example Hilbert's unification of the theories of the expansion of an arbitrary function in terms of other functions and some of the applications to boundary value problems, could not have been introduced. The limited size of the book was evidently the preventive.

In commenting upon the theory as developed by Professor Bôcher, I shall not attempt to follow closely the order of his arrangement, but shall try to give an idea of the contents of the book as they have impressed themselves upon me. There are two kinds of integral equations which may be written in the forms

$$f(x) = \int_a^b K(x, \xi)\, u(\xi) d\xi, \tag{1}$$

$$u(x) = f(x) + \int_a^b K(x, \xi)\, u(\xi) d\xi, \tag{2}$$

where $K(x, \xi)$ and $f(x)$ are given functions, while $u(x)$ is to be determined. The problem proposed by Abel was to determine a curve $y = y(x)$ down which a heavy particle would fall

from a variable point (x, y) to the origin $(0, 0)$ in a time $T = f(x)/2g$, where $f(x)$ is an arbitrarily assigned function. The integral equation which gives the solution is one of the first kind, in which $K(x, y) \equiv 0$ for $y > x$. It has the form

$$f(x) = \int_0^x \frac{v'(\xi)d\xi}{\sqrt{(x-\xi)}},$$

where $v(x)$ is the length of arc measured from the origin and is to be determined. Liouville later made the determination of a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + [\rho^2 - \sigma(x)]y = 0,$$

where ρ is a constant, depend upon the solution of the integral equation

$$(3) \qquad u(x) = \cos \rho\,(x - a) + \frac{1}{\rho}\int_a^x \sigma(\xi) \sin \rho(x - \xi)u(\xi)d\xi,$$

which is an equation of the second kind. In §§ 3, 5 Professor Bôcher exhibits the method which Liouville applied to the solution of these two equations, and applies the same method of successive substitutions to the general equation of the second kind.

The treatment devised by Volterra for equations of the second kind is both remarkable and elegant. It depends upon the notion of the iterated functions $K_i(x, y)$ defined by the formulas

$$K_1(x, y) = K(x, y), \quad K_i(x, y) = \int_a^b K(x, \xi)K_{i-1}(\xi, y)d\xi.$$

The series

$$(4) \qquad -k(x, y) = K_1 + K_2 + \cdots,$$

when it is uniformly convergent, determines uniquely a continuous function $k(x, y)$ which with $K(x, y)$ satisfies the equations

$$(5) \qquad K(x, y) + k(x, y) = \int_a^b K(x, \xi)k(\xi, y)d\xi$$

$$= \int_a^b k(x, \xi)K(\xi, y)d\xi.$$

Any two functions K, k which have proper continuity properties and satisfy the last equations are said to be "reciprocal." By means of equations (5) it can be shown that the integral equation of the second kind has one and only one continuous solution, which is expressed by the formula

$$(6) \qquad u(x) = f(x) - \int_a^b k(x, \xi) f(\xi) d\xi.$$

The solution by Volterra which has just been discussed depends for its validity upon the convergence of the series (4). Another method suggested by Volterra, but investigated by Fredholm, and later revised and extended by Hilbert, goes deeper into the meaning of the integral equations, explains the circumstances under which the reciprocal function $k(x, y)$ will or will not surely exist, and has besides an important application to integral equations involving an arbitrary parameter λ which will be mentioned later. Professor Bôcher shows in §7, following Fredholm, how one may regard the equation (2) as a limiting case for the system of equations

$$(7) \qquad u_n(x_i) = f(x_i) + \sum_{j=1}^{n} K(x_i, x_j) u_n(x_j) \qquad (i = 1, 2, \cdots, n)$$

as n becomes infinite. Here $x_1, x_2, \cdots, x_n = b$ are supposed to divide the interval ab into n equal parts, and $u_n(x_1), u_n(x_2), \cdots, u(x_n)$ are the quantities to be determined. The determinant D_n of these equations goes over as n approaches infinity into an infinite series of integrals involving the kernel K, and if the values (x_μ, x_ν) have the limit (x, y) the corresponding cofactor $D_n(x_\mu, x_\nu)$ of D_n approaches a limiting value $D(x, y)$ which is also expansible into an infinite series. In § 8 the convergence of the series for D and the "adjoint" $D(x, y)$ is rigorously proved, and the important relations

$$-DK(x, y) + D(x, y) = \int_a^b K(x, \xi) D(\xi, y) d\xi$$
$$= \int_a^b D(x, \xi) K(\xi, y) d\xi$$

are derived. It follows at once that when $D \neq 0$ the function

$$(8 \qquad k(x, y) = - \frac{D(x, y)}{D}$$

is reciprocal to $K(x, y)$, and equation (2) has a unique solution expressed by means of formula (6). On the other hand when $D = 0$ there will be no solution unless $f(x)$ satisfies

$$\int_a^b D(x, \xi) f(\xi) d\xi = 0,$$

a relation which is suggested by the condition which must be satisfied if equations (7) have a solution when $D_n = 0$.

The integral equation (3) which Liouville studied is of the form

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi) u(\xi) d\xi, \tag{9}$$

where $\lambda = 1/\rho$ and $K(x, y) \equiv 0$ for $y > x$. In § 9 Professor Bôcher begins the study of such equations. The determinant and adjoint function, as well as the reciprocal function, are here functions of the form $D(\lambda)$, $D(x, y, \lambda)$, $k(x, y, \lambda)$ containing the parameter λ which may take either real or complex values. The roots of $D(\lambda)$ are the "Eigenwerte" of Hilbert, or the "roots for the function $K(x, y)$." It is found in §§ 9, 10 that the necessary and sufficient condition that $K(x, y)$ have a reciprocal $k(x, y, \lambda)$ corresponding to a particular value of λ is that $D(\lambda) \neq 0$. If this condition is satisfied, equation (9) has a unique solution determined by equations (8) and (6).

The situation is somewhat different for the homogeneous equation

$$u(x) = \lambda \int_a^b K(x, \xi) u(\xi) d\xi, \tag{10}$$

as is explained in § 10. The unique solution of this equation when $D(\lambda) \neq 0$ is $u = 0$. On the other hand, for any root of $D(\lambda)$ the homogeneous equation has always an infinity of continuous solutions, called "principal solutions," which do not vanish identically. When $D = 0$ it follows from these results that the non-homogeneous equation (9) has either no continuous solution or else an infinite number found by adding to any particular solution of (9) the solutions of (10).

It was mentioned above that in the papers of Hilbert and Schmidt the theory of integral equations with unsymmetric kernel K has been made to depend upon that of equations in which the kernel is symmetric. In §§ 11, 12 Professor Bôcher

develops the theorems which relate especially to equations with symmetric kernels. For any such equation the determinant $D(\lambda)$ has at least one root, all the roots are necessarily real, and to any root of $D(\lambda)$ there corresponds only a finite number of linearly independent principal solutions of the homogeneous equation (10). A system $u_i(x)$ $(i = 1, 2, \cdots)$ of principal solutions belonging to roots of $D(\lambda)$ can be so chosen that any principal solution of equation (10) is expressible linearly and with constant coefficients in terms of a finite number of the functions $u_i(x)$, and furthermore so that

$$\int_a^b u_i^2(x)dx = 1, \quad \int_a^b u_i(x)u_j(x)dx = 0 \qquad (i \neq j).$$

A system of solutions having these properties is called "a complete normalized orthogonal system of characteristic functions for the kernel K." The trigonometric functions sin x, sin $2x$, $\cdots$, are an example of such a system, in terms of which any function satisfying suitable restrictions can be expanded as an infinite series. Similar expansion theorems hold also for the system of characteristic functions belonging to any symmetric kernel. Professor Bôcher has restricted himself here, however, to the consideration of a single expansion, that for the kernel $K(x, y)$ itself, and to some of its applications.

The theory of the integral equations of the second kind having been developed, it is a comparatively simple matter to show, as Professor Bôcher does in § 13, that the solution of the integral equation

$$f(x) = \int_a^x K(x, \xi) u(\xi) d\xi, \tag{11}$$

which is one of the first kind with $K \equiv 0$ when $y > x$, are all solutions of the equation

$$f'(x) = K(x, x)\, u(x) + \int_a^x \frac{\partial K(x, \xi)}{\partial x} u(\xi) d\xi.$$

If $K(x, x)$ does not vanish in the interval ab, this is an equation of the second kind (2), and the problem of solving it is equivalent to the solution of the original equation (11) of the first kind. The case when $K(x, x)$ vanishes identically is treated, and an example illustrative of the case when $K(x, x)$ has a finite number of zeros is given. The section concludes with the study

of the more general equation of the first kind where $K(x, y)$ is assumed to have a discontinuity along a curve $y = \phi(x)$.

In the section just described the kernel was assumed to be finite. The earliest integral equation of the first kind, that of Abel, was however one in which the kernel became infinite along the line $x = y$. The concluding section of the book is devoted to equations of the type

$$f(x) = \int_a^x \frac{G(x, \xi)}{(x-\xi)^\lambda} u(\xi) d\xi,$$

which has a kernel with an infinite discontinuity including Abel's kernel as a special case for $G = 1$, and to a number of examples not falling under the previous theory. Especially interesting is the explanation of the relation of Fourier's integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \cos(x\xi) \cos(\xi\xi_1) f(\xi_1) d\xi_1 d\xi$$

to the theory of integral equations in which the limits are nfinite.

G. A. Bliss.

SHORTER NOTICES.

Grundlagen der Analysis. Von Moritz Pasch. Ausgearbeitet unter Mitwirkung von Clemens Thaer. Leipzig, Teubner, 1908. 8vo. vi + 140 pp.

This book presents an admirable attempt to develop the concept of the real number in a more exact logical fashion. There is no attempt to reduce the assumptions to a categorical set, and even their consistency is not considered; but they are everywhere clearly stated, the theorems follow by ready deductions, and the large number of definitions would seem to be put in an unusually clear way, and one especially well adapted to the purpose of the general argument.

The book opens with a consideration of the relation of things to names, of the notions of precede and follow, and of methods of mathematical proofs. This is followed by a treatment of sets, sequences, and series, leading up to integers. By subjecting the integers to the four fundamental operations, fractions, including decimal fractions, and negative numbers are intro-

duced, and the laws of calculation are shown to be still valid in the enlarged set. The author then extends the concept of the set to non-enumerably infinite aggregates, which allows him to introduce irrational numbers in the usual Dedekind fashion. Powers with both bases and exponents, arbitrary real numbers, and logarithms of arbitrary positive real numbers to arbitrary positive real bases are then treated. The book proper concludes with an exposition of some of the principal theorems of combinations, and the binomial theorem for a positive real exponent. An appendix is added to the book, giving extracts from some of the author's previous writings.

On account of the abstract nature of the subject matter, the book is not suitable for the beginner, but it should appeal strongly to every teacher of mathematics. Notwithstanding the abstractness of the subject, the matter is attractively arranged, and may be perused with profit by one who possesses general maturity, even without an extensive knowledge of the technique of mathematics. It is unfortunate that books of this type are so inaccessible to English readers.

F. W. OWENS.

Eine konforme Abbildung als zweidimensionale Logarithmentafel zur Rechnung mit komplexen Zahlen. By Dr. F. BENNECKE, Professor at the Victoria-Gymnasium in Potsdam, 1907.

THE above is the title of the Festschrift by Dr. Bennecke at the three hundredth anniversary of the establishment of the Royal Joachimsthal Gymnasium at Berlin. In the early paragraphs is given a statement of the literature of the subject. While the author makes no pretension of presenting an exhaustive bibliography, nevertheless the citations are suggestive of the historical development and of the general interest in graphic methods, particularly as applied to operations with complex numbers. The purpose of the pamphlet is to establish a method by which the logarithms of complex numbers may be calculated with sufficient accuracy for practical purposes by means of graphs. This purpose is accomplished by a conformal representation upon the (X, Y)-plane of the two systems of curves given by $x =$ constant, $y =$ constant, where

$$W = X + iY = \log z = \log (x + iy).$$

It is shown that any curve of either of the two resulting systems

in the (X, Y)-plane is congruent with any other curve of the same system, and that the curves of the two systems are likewise congruent with one another. The relation between the curves makes it possible to construct all of the necessary graphs by the aid of a single regular curve. The construction of these graphs is furthermore simplified by showing that, to find the logarithm of any number to the base ten, it is sufficient to map on the W-plane only that portion of the z-plane which lies in the first quadrant and between the circles whose radii are 100 and 1000 respectively.

The rectangular form of the functional region makes it feasible to divide that region into nine subdivisions, thus giving a more convenient arrangement for evaluating a logarithm than by means of one large chart. From the ten charts which follow the general discussion both logarithms and anti-logarithms of complex numbers may be approximately determined. Upon the assumptions made as to the accuracy in locating points of intersections of two curves and in reading the charts, the maximum error in the absolute value of the logarithm will not exceed 0.0005. This accuracy corresponds therefore to a three place logarithmic table. Of course, as in ordinary logarithms, these errors may accumulate in the process of computation, so that the final error may be larger.

Aside from a theoretical interest in the work of Dr. Bennecke as an exercise in mapping, students of physics and others will find the charts of practical value wherever computations involving the logarithms of complex numbers are necessary.

E. J. TOWNSEND.

Elementary Algebra. By J. W. A. YOUNG and LAMBERT L. JACKSON. New York, Appleton and Co., 1908. ix + 438 pp.

THE guiding principle of the authors in this book is "a minimum of mathematical theory and a fuller recognition of the utility of the subject." This does not mean that the logical value of algebra has been ignored, but rather that the proofs given are put in a form which will appeal to and satisfy the mind of an average high school pupil; when in a few places this seemed impossible, the authors have fallen back upon simple assumptions rather than upon the introduction of subtle distinctions and arguments savoring of higher mathematical methods. For example, instead of explaining the process of

multiplying quadratic surds by means of the theory of exponents as is usually done, they introduce this topic by the statement, "In multiplying expressions containing square roots, make use of the relation $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$", and no proof of this relation is given.

The book is by no means radical, as it covers the usual topics of elementary algebra in about the conventional order, and presents material more than sufficient to meet present college entrance requirements. The method is inductive throughout. It may be characterized as modern, since the terms function and variable are used, graphs are included, determinant solution of simultaneous linear equations and methods of detached coefficients are given.

An important feature is the number of oral exercises scattered throughout the book. The reviewer believes oral work has been too much neglected in the last decade; numerous simple oral exercises often fix a principle better than complicated written exercises and besides give excellent drill in thinking and oral expression. Unfortunately there is no index of terms defined, but summaries of definitions, notions, and processes at the end of each chapter partially meet this need and have the advantage of fixing operations in the minds of the pupils. Graphs are used in treating the topics variation, simultaneous equations, and the quadratic. Graphs are not presented as selected topics from analytical geometry, but are used only where they bring insight into algebraic processes. However, the terms graph of a function and graph of an equation are used interchangeably where it seems desirable to the reviewer to distinguish carefully between graph of a function and locus of an equation. Equivalent equations are defined but no statement of operations which lead to equivalent equations is given. "A proportion is an equation between two numbers" (page 173) is doubtless a misprint, for proportion is correctly defined in another place (page 162).

This is one of the good elementary algebras recently published and deserves careful examination by teachers considering a change in algebra texts.

ERNEST B. LYTLE.

NOTES.

At the annual meeting of the London mathematical society held on November 11, 1909, the following officers were elected for the present year: President, W. Niven; secretaries, A. E. H. Love and J. H. Grace; treasurer, Sir J. Larmor; three vice-presidents and nine other members of the council. The following papers were read: By G. H. Hardy, "The ordinal relations of the terms of a convergent sequence," "The applications of Dirichlet's series to Borel's exponential method of summation," and "Theorems relating to summability and non-convergence of slowly oscillating series"; by W. Esson, "Notes on synthetic geometry"; by H. Bateman, "Kummer's quartic surface as a wave surface"; by H. S. Carslaw, "Green's function in a wedge and other problems in the conduction of heat"; by J. L. S. Hatton, "The envelope of a line cut harmonically by two conics"; by F. H. Jackson, "On a case of a q-hypergeometric series."

The American commissioners of the International commission on the teaching of mathematics earnestly request all to whom questionnaires have been addressed to send in their replies at the earliest possible moment. If the country is to prepare a set of reports that shall show the work that is being done, and if these reports are to be ready when those of other countries are presented, a great deal of hard work must be done this winter. Prompt replies therefore are necessary.

The annual meeting of the Federation of teachers of the mathematical and the natural sciences will be held at Boston, December 27–28, in affiliation with the American association for the advancement of science. Besides the reports of various administrative committees, that of the committee of fifteen on a syllabus in geometry will be presented and discussed; the committees on the college entrance problem and on the bibliography of science teaching will also present their reports.

During the year six associations have joined the federation, making the total membership fourteen. The annual election of officers will be held at the annual meeting, and standing committees will be appointed.

At the meeting of the Syracuse section of the Association of teachers of mathematics in the Middle States and Maryland

held at Syracuse December 28, 1909, the following papers were read: "Questions, answers, and per cents.," by C. F. WHEELOCK; "Algebraic number and the equation," by J. M. TAYLOR; "Shortening the course in arithmetic," by S. WILLIAMS.

AT the meeting of the Paris academy of sciences on November 7 the following prizes for contributions to geometry during the last academic year were announced: Francœur prize (1,000 fr.) to DR. E. LEMOINE; Bordin prize (3,000 fr.), divided between Professor G. BAGNERA, of the University of Palermo, and Professor M. DE FRANCHIS, of the University of Catania.

THE house of Underwood and Underwood, of London, announces a series of stereopticon views for use in teaching geometry of space. The first 25, prepared by E. LANGLEY, illustrate elementary theorems, while 23 others, by Professor A. G. GREENHILL, relate to spherical curves, particularly catenaries.

THE publishing house of Gauthier-Villars, Paris, announces that the following mathematical books are in the press, and will appear in a short time: O. BLUMENTHAL, "Principes de la théorie des fonctions entières d'ordre infini;" E. BOREL, "Leçons sur la théorie de la croissance;" C. RIQUIER, "Les systèmes d'équations aux dériveés partielles."

IN accordance with the resolution adopted by the fourth international congress of mathematicians, the president of the congress has appointed the following committee on the unification of the vectorial notation: Professors Abraham, Ball, Hadamard, Langevin, Lori, Marcolongo, Prandtl, Stekeloff, Whitehead, E. B. Wilson.

PROFESSORS F. KLEIN, of the University of Göttingen, Sir J. LARMOR, of Cambridge University, and H. POINCARÉ, of the University of Paris, have been elected honorary members of the Calcutta mathematical society.

DR. J. M. MILLER has been appointed acting head of the department of mathematics at the technical college of Glasgow.

MR. J. R. WILTON has been appointed assistant lecturer on mathematics at the University of Sheffield.

MR. G. H. HERRIOT has been appointed lecturer in mathematics at the school of mining, Kingston, Ontario.

PROFESSOR A. G. WEBSTER delivered at the University of

Illinois, November 29 to December 1, a series of public lectures on the following subjects: "Great physical problems of the past, present and future," "Sound and its measurement," "The gyroscope and its practical applications," "Classification of mathematical physics with reference to mechanics," "Waves, ether and relativity."

AT the University of Michigan Dr. J. W. BRADSHAW has been promoted to an assistant professorship of mathematics; Mr. W. V. N. GARRETSON has been granted a leave of absence to study at Yale; Mr. F. M. DRYZER, of the University of Iowa, and Mr. T. H. HILDEBRANDT, of the University of Chicago, have been appointed instructors in mathematics. Professor CARL RUNGE, of Göttingen, will deliver a series of lectures at the University in February.

AT the University of North Dakota Miss F. BALCH has been appointed instructor in mathematics.

MR. J. W. RUSSELL, former lecturer of mathematics in Balliol and St. John's Colleges, Oxford, died November 4 at the age of 81 years.

CATALOGUES of second hand mathematical books: Ernst Carlebach, Hauptstrasse 136, Heidelberg, catalogue no. 313, 1283 titles in mathematics and exact sciences.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor*, PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

ANDRÉ (D.). Des notations mathématiques. Enumération, choix et usage. Paris, Gauthier-Villars, 1909. 8vo. 18 + 502 pp. Fr. 16.00

BECKER (G. F.) and VAN ORSTRAND (C. E.). Hyperbolic Functions. Washington, Smithsonian Institution, 1909. 8vo. 51 + 321 pp. Cloth. $4.00.

BOREL (E.). Die Elemente der Mathematik. Vom Verfasser genehmigte deutsche Ausgabe besorgt von P. Stäckel. 2ter Band: Geometrie. Mit 403 Textfiguren. Leipzig, Teubner, 1909. 8vo. 12 + 324 pp. Cloth. M. 6.40

ENZYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. VI. 1ter Teil. A. Geodäsie und Geophysik. Redigiert von P. Furtwängler und E. Wiechert. 3tes Heft. Leipzig, Teubner, 1909. 8vo. Pp. 245–372. M. 4.00

FURTWÄNGLER (P.). See ENZYKLOPÄDIE.

GIOVANETTI (G.). Relazioni di condizione tra i coefficienti di una equazione cubica completa con radici particolari. Abbiategrasso, Bollini, 1909. 8vo. 11 pp.

GOLDMANN (F.). Ponceletsche Polygone bei Kreisen. (Diss.) Breslau, 1909. 8vo. 69 pp.

HOFMANN (J.). Nachweis der Richtigkeit des Fermatschen Satzes. Leipzig, 1909. 8vo. M. 1.20

KRIENELKE (K.). J. H. Lamberts Philosophie der Mathematik. Halle, 1909. Berlin, Mayer. 8vo. 101 pp. M. 2.40

LICHTENSTEIN (L.). Zur Theorie der gewöhnlichen Differentialgleichungen und der partiellen Differentialgleichungen zweiter Ordnung. Berlin, 1909. 8vo. 40 pp. M. 1.50

MARSHALL (W.). The asymptotic representation of the elliptic cylinder functions. (Diss.) Zurich, 1909. (Reprint from Am. Jour. of Math., vol. 31 (1909), pp. 311–336.)

MERCER (J.). Functions of positive and negative type, and their connection with the theory of integral equations. London, Dulau, 1909. 4to. 2s.

NEUENDORFF (R.). Ueber Kreispunktpolarkurven. (Diss.) Kiel, 1908. 8vo. 30 pp.

NEUMANN (F.). Eine Verallgemeinerung der Cylinderfunktionen. (Progr.) Halberstadt, 1909. 4to. 25 pp.

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte I: Calcolo differenziale. 3a edizione, riveduta. Milano, Hoepli, 1909. 16mo. 12 + 310 pp.

PRASAD (G.). A textbook of differential calculus. With numerous examples and answers. London, Longmans, 1909. 8vo. 5s.

REYE (T.). Die Geometrie der Lage. Vorträge. 3te Abteilung. 4te, umgearbeitete und vermehrte Auflage. Leipzig, Kröner, 1910. 8vo. 8 + 253 pp. M. 8.00

RIQUIER (C.). Les systèmes d'équations aux dérivées partielles. Paris, Gauthier-Villars, 1910. 8vo. 27 + 591 pp. Fr. 20.00

SAGERET (E.). Etude sur le théorème de Fermat. Paris, Dulac, 1909. 8vo. 38 pp.

SCHLAMP. Die Entstehung der Kegelschnitte nach Maclaurin und Grassmann. (Progr.) Darmstadt, 1908. 4to. 26 pp.

SCHMID (A.). Anwendung der Cauchy-Lipschitzschen Methode auf lineare partielle Differentialgleichungen. (Progr.) Schönberg, 1909. 8vo. 30 pp.

SCHRÖDER (R.). Das Dreieck und seine Berührungskreise. Gross-Lichterfelde, 1909. 8vo. 41 pp.

SIMON (M.). Geschichte der Mathematik im Altertum in Verbindung mit antiker Kulturgeschichte. Berlin, Cassirer, 1909. 8vo. 17 + 401 pp. M. 13.00

STÄCKEL (P.). See BOREL (E.).

STAMPER (A. W.). A history of the teaching of elementary geometry. (Diss.) New York, Columbia University Press, 1909. 8vo. 10 + 163 pp. Cloth. $1.50

STURM (R.). Die Lehre von den geometrischen Verwandtschaften. Band IV: Die nicht linearen und mehrdeutigen Verwandtschaften. Leipzig, Teubner, 1909. 8vo. 6 + 476 pp. Cloth. M. 18.00

VAN ORSTRAND (C. E.). See BECKER (G. F.).

WIDDER (W.). Untersuchungen über die allgemeinste lineare Substitution mit vorgegebener p^{ter} Potenz. (Diss.) Würzburg, 1909. 4to. 53 pp.

WIECHERT (E.). See ENZYKLOPÄDIE.

ZILLIG (J.). Ueber die infinitesimale Deformation der Minimalflächen. (Diss.) Würzburg, 1908. 8vo. 39 pp.

II. ELEMENTARY MATHEMATICS.

BARONI (E.). Algebra e trigonometria, ad uso dei licei. Vol. II (algebra ed aritmetica) per il secondo e terzo anno di liceo. Firenze, Bemporad, 1909. 16mo. 213 pp. L. 1.75

BERSANO (G. B.). See GUASCO (M.).

BOURLET (C.). Précis d'algèbre, contenant 573 exercices et problèmes. Classes de troisième B et première C et D. 5e édition, revue et complétée. Paris, Hachette, 1909. 16mo. 443 pp. Fr. 2.50

BRUNO (G. M.). Algebra y trigonometría con numerosos ejercicios. Paris, 1909. 12mo. 256 pp.

CAJORI (F.). A history of the logarithmic slide rule and allied instruments. London, Constable, 1909. 8vo. 6 + 144 pp. 4s. 6d.

C. (F. I.). Eléments d'algèbre, avec de nombreux exercices. Paris, Poussielgue, 1910. 12mo. 8 + 413 pp.

CLARKE (F. C.). See STARLING (S. G.).

CONANT (L. L.). Plane and spherical trigonometry. New York, American Book Co., 1909. 8vo. 222 pp. Cloth. $0.85
With 5-place logarithmic and trigonometric tables. $1.20

DAVISSON (S. C.). College algebra. New York, Macmillan, 1909. 12mo. 9 + 191 pp. $1.50

DIA (A. DI). Elementi di algebra, con numerosi esercizi, per uso delle scuole tecniche. Milano, Signorelli, 1909. 16mo. 78 pp. L. 1.00

EUCLIDES. The 13 books of Euclid's Elements. Translated from the text of Heiberg, with introduction and commentary, by T. L. Heath. 3 vols. Cambridge, 1908–09. 8vo. 439, 436, 554 pp. £2, 2s.

FAIFOFER (A.). Elementi di algebra, ad uso della prima classe liceale 2a edizione. Venezia, Sorteni, 1909. 16mo. 195 pp. L. 2.00

——. Elementi di geometria, ad uso degli instituti tecnici (1o biennio) e dei licei. 16a edizione. Venezia, Sorteni, 1909. 8vo. 534 pp. L. 4.00

——. Elementi di geometria, ad uso dei licei. Tredicesima edizione, ricavata dalla decimosesta edizione della stessa opera, destinata ai licei ed agl' istituti tecnici. Venezia, Sorteni, 1909. 16mo. 439 pp. L. 3.00

——. Trattato di geometria intuitiva, ad uso delle scuole tecniche e normali. 40a edizione. Venezia, Sorteni, 1909. 8vo. 165 pp. L. 2.00

GÉOMÉTRIE. Cours moyen. Paris, Poussielgue, 1909. 12mo. 191 pp.

GODFREY (C.) and SIDDONS (A. W.). Geometry for beginners. Cambridge, University Press, 1909. 8vo. 90 pp. 1s.

GUASCO (M.) e BERSANO (G. B.). Aritmetica, geometria e computisteria per le tre classi delle scuole complementari femminili. Torino, 1909. 8vo. 315 pp. L. 3.60

HALL (H. S.) and STEVENS (F. H.). A school geometry. Parts 3, 4 and 5. London, Macmillan, 1909. 8vo. 232 pp. 2s. 6d.

HAUCK (G.). Lehrbuch der Stereometrie. Auf Grund von F. Kommerell's Lehrbuch neubearbeitet und erweitert. 10te (umgearbeitete) Auflage. Herausgegeben von V. Kommerell. Tübingen, Laupp, 1909. 8vo. 16 + 170 pp. M. 2.00

HEATH (T. L.). See EUCLIDES.

KOMMERELL (V.). See HAUCK (G.).

LEPOIVRE (G.) et POIRSON (A.). Première partie du cours de géométrie théorique et pratique à l'usage des élèves des écoles pratiques d'industrie, des écoles professionnelles et des écoles primaires supérieures. Lille, Janny, 1909. 8vo. 164 pp.

MARTÍ ALPERA (F.). Las primeras lecciones de geometría. Grado elemental. Burgos, Santiago Rodríguez, 1909. 104 pp. P. 0.60

MARTINI ZUCCAGNI (A.). Aritmetica pratica e nozioni di geometria pel ginnasio inferiore. Vol. I, per la prima classe. Livorno, Giusti, 1909. 16mo. 147 pp. L. 1.40

——. Aritmetica razionale ed elementi di geometria euclidea (fondata sui postulati di Hilbert) pel ginnasio superiore. Vol. I, per la quarta classe. Livorno, Giusti, 1909. 8vo. 9 + 199 pp. L. 1.75

MATRICULATION mathematics papers. Being the papers in elementary mathematics set at the matriculation examination of the University of London from Jan. 1900 to Sept. 1909, with full solutions to the papers of Sept. 1909. London, Clive, 1909. 8vo. 136 pp. 1s. 6d.

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Bearbeitet von A. Schulte-Tigges. Ausgabe B. Oberstufe. 3ter Teil. Berlin, Reimer, 1909. 8vo. 7 + 84 pp. M. 1.50

MODEL answers to intermediate pure mathematics. Being the mathematical papers set at the London University intermediate examinations in arts and science from 1900 to 1909. London, Clive, 1909. 8vo. 196 pp. 2s. 6d.

MÜLLER (H. H.). See REINHARDT (W.).

NAMPON (G.). Cent questions de théorie (arithmétique, algèbre, géométrie) des examens du brevet supérieur, avec développements et solutions. Paris, Hachette, 1909. 16mo. 111 pp. Fr. 1.00

NOCIONES elementales de geometría aplicadas al dibujo lineal; por los hermanos de las escuelas cristianas, 14a edición. Paris, Procuraduría general, 1909. 12mo. 112 pp.

OTTO (J.). Ueber die Anwendung der Lehre von den Permutationen in der Algebra. (Progr.) Mährisch — Ostrau, 1909. 8vo. 22 pp.

POIRSON (A.). See LEPOIVRE (G.).

REINHARDT (W.) und MÜLLER (H. H.). Lehrbuch für den mathematischen Unterricht auf Grund der neuen Lehrpläne. In zwei Teilen im

Anschluss an die Aufgaben für den Rechenunterricht in höheren und mittleren Lehranstalten von J. C. Becker und K. Paul herausgegeben. Frankfurt a./M., Auffarth, 1909. 8vo.
1ter Teil. Arithmetik und Algebra. 7 + 120 pp. M. 1 60
2ter Teil. Geometrie. 7 + 136 pp. M. 1.80

Ropert (J.). Cours d'algèbre élémentaire (deuxième degré) et propriétés usuelles des trois coniques et de l'hélice. Cours de troisième division, professé à l'école nationale d'arts et métiers de Lille. Angers, Lenormand, 1909. 8vo. 241 pp.

Siddons (A. W.). See Godfrey (C.).

Starling (S. G.) and Clarke (F. C.). Preliminary practical mathematics. London, Arnold, 1909. 8vo. 192 pp. 2s.

Stevens (F. H.). See Hall (H. S.).

Traverso (G.). Costruzioni geometriche, ad uso delle scuole secondarie. Parte I. 2a edizione. Savona, Prudente, 1909. 16mo. 81 pp.

Woodward (C. J.). ABC five-figure logarithms for general use. New edition. London, 1909. 8vo. 3s. 6d.

III. APPLIED MATHEMATICS.

Berget (A.). See Chappuis (J.).

Bernhard (M.). Darstellende Geometrie mit Einschluss der Schattenkonstruktionen und der Perspektive. Als Leitfaden für den Unterricht an technischen Lehranstalten, Oberrealschulen und Realgymnasien, sowie zum Selbststudium herausgegeben. 3te, verbesserte und vermehrte Auflage. Stuttgart, Enderlen, 1909. 8vo. 8 + 294 pp. Cloth. M. 5.80

Borchardt (H.). Beiträge zur Kenntnis der stationären elektrischen Flächenströmung und ihres magnetischen Feldes. Nebst einer Anwendung auf die Variation der erdmagnetischen Vertikalintensität. (Diss.) Kiel, 1909. 8vo. 48 pp.

Cain (W.). Theory of solid and braced elastic arches. 2d edition. New York, Van Nostrand, 1909. 16mo. 7 + 190 pp. Boards. $0.50

—— Theory of steel-concrete arches and of vaulted structures. 5th edition, revised. New York, Van Nostrand, 1909. 16mo. 215 pp. Boards. $0.50

Chappuis (J.) et Berget (A.). Leçons de physique générale. 2e édition, entièrement refondue. Vol. 3: Acoustique et optique. Paris, Gauthier-Villars, 1909. 8vo. 7 + 503 pp. Fr. 14.00

Fritsch (H.). Die gegenseitige Massenanziehung bei Newton und bei seinen Nachfolgern. (Progr.) Königsberg, 1909. 8vo. 36 pp.

Fuchs (O.). Theoretische und kinematographische Untersuchung von Dampfhämmern mit selbsttätiger Schiebersteuerung. Berlin, Springer, 1909. 8vo. 20 pp. M. 1.20.

Grogan (E. S.). The economic calculus and its application to tariff. Newcastle, 1909. 8vo. 184 pp. 2s.

Guillaume (C. E.). Initiation à la mécanique. Ouvrage étranger à tout programme, dédié aux amis de l'enfance. Paris, Hachette, 1909. 16mo. 16 + 214 pp. Fr. 2.00.

Jecke (R. H.). Beiträge zur Geometrie der Bewegung. Rostock, 1909. 8vo. 42 pp. M. 1.80.

LIMASSET (L.). Théorie rationnelle du mouvement mélodique des sons. Reims, Matot, 1909. 4to. 317 pp.

LOW (D. A.). Applied mechanics. Embracing strength and elasticity of materials. London, Longmans, 1909. 8vo. 560 pp. 7s. 6d.

MANEUVRIER (G.). Traité élémentaire de mécanique. Nouvelle édition, entièrement refondue. Paris, Hachette, 1909. 16mo. 394 pp. Fr. 4.00.

MANNES (H.). Die Berechnung von Rohrnetzen städtischer Wasserleitungen. München, Oldenbourg, 1909. 8vo. 59 pp. M. 1.60

MARCHESINI (G.). Elementi di calcolo attuariale, ad uso degli istituti tecnici, delle scuole medie di commercio e degli istituti di previdenza. Udine, Del Bianco, 1910. 8vo. 6 + 126 pp. L. 3.00

PEABODY (C. H.) Thermodynamics of the steam-engine and other heat-engines. 6th edition, rewritten. New York, Wiley, 1909. 7 + 543 pp. Cloth. $5.00

PEARSON (K.) and others. On a practical theory of elliptic and pseudo-elliptic arches, with special reference to the ideal masonry arch. London, Dulau, 1909. 4to. 4s.

RISSER (R.). Etude sur l'établissement des tables de mortalité de population. Mortalité professionnelle; mortalité dans le cas de l'invalidité. Paris, Dulac, 1909. 8vo. 112 pp.

SCHUSTER (A.). An introduction to the theory of optics. 2nd edition, revised. London, Arnold, 1909. 8vo. 368 pp. 15s.

SMITH (H. E.). Strength of material. An elementary study prepared for the use of midshipmen at the U. S. Naval Academy. 2nd edition, revised. New York, Wiley, 1909. 12mo. 9 + 170 pp. Cloth. $1.25

STEINMETZ (C. P.). Theoretical elements of electrical engineering. 3d edition, thoroughly revised and greatly corrected. New York, McGraw-Hill, 1909. 8vo. 9 + 455 pp. Cloth. $4.00

TAPLA (J.). Vademekum der darstellenden Geometrie. Für Schüler gewerblicher Lehranstalten, für Schüler und Absolventen des Gymnasiums sowie für Praktiker. 2te, durchgesehene Auflage. Wien, Fromme, 1909. 8vo. 12 + 182 pp. M. 5.00

TOLLE (M.). Die Regelung der Kraftmaschinen. Berechnung und Konstruktion der Schwungräder, des Massenausgleichs und der Kraftmaschinenregler in elementarer Behandlung. 2te, verbesserte und vermehrte Auflage. Berlin, Springer, 1909. 8vo. 11 + 699 pp. Cloth. M. 26.00

WELLNER (G.). Die Flugmaschinen. Theorie und Praxis. Berechnung der Drachenflieger und Schraubenflieger. Wien, Hartleben, 1910. 8vo. 8 + 152 pp. M. 10.00

WOLFF (O.). Folgerungen aus dem dritten Kepler'schen Gesetze. Eine Studie für Naturfreunde. Dux, 1909. Teplitz-Schönau, Becker. 8vo. 29 pp. M. 1.50

THE THIRD REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE third regular meeting of the Southwestern Section of the Society was held at the University of Missouri, Columbia, Missouri, on Saturday, November 27, 1909. The following members of the Society were present:

Professor L. D. Ames, Professor W. C. Brenke, Professor E. W. Davis, Professor G. R. Dean, Professor L. M. Defoe, Mr. A. B. Frizell, Mr. E. S. Haynes, Professor E. R. Hedrick, Dr. Louis Ingold, Professor O. D. Kellogg, Professor W. H. Roever, Dr. Mary S. Walker, Dr. Paul Wernicke, and Professor W. D. A. Westfall.

The morning session was opened at 10 A.M. and the afternoon session at 2 P.M., Professor Davis presiding. Lincoln, Nebraska, was decided upon for the next meeting, and the following program committee was elected: Professor Davis (chairman), Dr. Wernicke, Professor Kellogg (secretary).

The following papers were presented:

(1) Professor G. O. JAMES: "On the reduction of time observations in vertical circle through Polaris."

(2) Professor L. D. AMES: "Some theorems of Lie on ordinary differential equations."

(3) Professor J. B. SHAW: "Scalars of lineal products."

(4) Professor M. B. PORTER: "On Fourier sequences."

(5) Dr. PAUL WERNICKE: "Note on sine theorems in hyperspace."

(6) Professor W. H. ROEVER: "The southerly deviation of falling bodies."

(7) Professor G. R. DEAN: "Stresses in an isotropic plate."

(8) Professor E. W. DAVIS: "A paradox relating to the imaginary line."

(9) Professor W. D. A. WESTFALL: "The continuity of integral rational functions of infinitely many variables."

(10) Professor H. B. NEWSON: "On linear groups in two variables."

(11) Professor W. C. BRENKE: "Summation of a series of Bessel's functions by means of an integral."

(12) Professor E. R. HEDRICK: "On a property of assemblages whose derivatives are closed."

(13) Mr. A. B. FRIZELL: "Natural numbers defined by the principles of abstract groups."

(14) Dr. LOUIS INGOLD: "Outline of a vector theory of curves."

In the absence of the authors, the papers of Professors James, Shaw, and Newson were read by title. Abstracts of the papers follow below.

1. From the rigorous formulas for the reduction of time observations in the vertical circle through Polaris Professor James develops the chronometer correction in a power series in p_0, the polar distance of Polaris, neglecting terms of the third and higher powers in p_0. By the introduction of the reduction to the pole, published each year in the *American Ephemeris and Nautical Almanac*, this series is transformed into one in which the term in p_0^2 is missing, thus reducing the approximate formula to a single term. The error of this simple formula is investigated and latitudes for which it is admissible determined. An analogous development and transformation is made for azimuth, and the resulting reduction formulas are in both cases adapted to the engineer's transit.

2. Consider an ordinary differential equation $Xy' - Y = 0$ from the standpoint of the Lie theory. Consider the three following statements: (A) Its integral curves admit the group

$$Uf = \xi \frac{\partial t}{\partial x} + \eta \frac{\partial t}{\partial y}.$$

(B) The equation itself admits the extended group. (C) An integrating factor is $1/(X\eta - Y\xi)$. Professor Ames's paper states three well-known theorems: (1) that (B) and (C) are equivalent, (2) that (A) and (B) are equivalent, (3) that (A) and (C) are equivalent. Theorem (1) was proved in a previous paper. The other two are proved in the present paper. The statements and proofs are given in analytic terms without making any use of the group theory, and the three theorems are proved independently of one another. Of course any one follows from the other two.

3. A lineal product as defined by Grassmann is one whose properties are independent of the ground or system of units in

terms of which the numbers are expressed. Scalars of such products would have the same property. Thus for quaternions $S \cdot \alpha\beta\gamma = - S \cdot \beta\alpha\gamma$, whatever numbers α, β, γ may be. Scalars of lineal products must therefore be expressible in terms of more elementary scalars of lineal products, or else be themselves elementary scalars.

Professor Shaw's paper discusses the scalar defined by the recurrence formula $I \cdot \alpha_1\alpha_2 \cdots \alpha_{2m} = I \cdot \alpha_1\alpha_2 I \cdot \alpha_3 \cdots \alpha_{2m} + I \cdot \alpha_1\alpha_3 I \cdot \alpha_4 \cdots \alpha_{2m}\alpha_2 + I \cdot \alpha_1\alpha_4 I . \alpha_5 \cdots \alpha_{2m}\alpha_2\alpha_3 + \cdots + I \cdot \alpha_1\alpha_{2m} I \cdot \alpha_2 \cdots \alpha_{2m-1}$. This scalar is a Pfaffian and is of much use. Scalars of corresponding types of the forms $I \cdot \alpha_1 \cdots \alpha_{3m} = \Sigma \cdot I \cdot \alpha_1\alpha_2\alpha_3 \cdot I \cdot \alpha_4 \cdots \alpha_{3m}$, etc., are slightly examined.

4. If

$$\sum_1^n (a_n \sin nx + b_n \cos nx) = S_n$$

be the Fourier sequence of order n of the function $f(x)$ when it is supposed that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx$$

exists, Professor Porter shows that from any sequence S_{n_i} $(i = 1, 2, \cdots)$ a sequence can be picked out which converges over a point set of measure 2π to the value $f(x)$.

5. The formula used for the volume belonging to the n-dimensional "simplex" on page 212 of the February, 1909, BULLETIN $V = p_{01}p_{12} \sin (P_{01}, P_{12}) \cdot p_{23} \sin (P_{12}, P_{23}) \cdot \sin (P_{012}, P_{123}) \cdots$ is capable of extension. It remains correct

1) if we replace the p_{ij} by the n joins p_{0j} of one point P_0 to the remaining n; and simultaneously $P_{ijk}, \cdots$ by P_{0jk};

2) if, after permuting the p_{ij} (and P_{ij}), we replace $P_{0123 \cdots \nu}$ by any ν-plane parallel to the first, second, $\cdots$, νth P_{ij} (in the new order);

3) if we permute the vertices, hence the indices $0, 1, \cdots, n$, throughout.

Equating expressions thus recognized to be equal, Dr. Wernicke obtains theorems connecting the sines. For example, in 3-space,

$$p_{12} \sin (P_{01}, P_{12}) \sin (P_{012}, P_{123}) = p_{13} \sin (P_{01}, P_{13}) \sin (P_{013}, P_{132})$$

which, when $p_{12} = p_{13}$, becomes the sine theorem of spherical trigonometry.

As a further example, in 4-space, the angles made by the 2-planes P_{012} and P_{034} are connected with those made by P_{014} and P_{023} by the sine theorem

$$\sin(P_{01}, P_{02}) \sin(P_{03}, P_{04}) \sin_1(P_{012}, P_{034}) \sin_2(P_{012}, P_{034}) = \sin(P_{01}, P_{04}) \sin(P_{02}, P_{03}) \sin_1(P_{013}, P_{024}) \sin_2(P_{013}, P_{024}),$$

$\sin_1$, $\sin_2$ denoting the sines of their "first" and "second" angles, respectively. The paper will be offered to the *Annals of Mathematics* for publication.

6. A body dropped from a point P_0, attached to the rotating earth, falls in a certain curve c, in a field of force F_1, which is fixed in space. It receives an initial velocity which is the same as the velocity of P_0 at the instant it is dropped. A plumb-line supported at P_0 hangs in a field of force F_2 which rotates with the earth. If $U_1 = f_1(r, z)$ and $U_2 = f_2(r, z)$ (where r and z denote the distances of a general point from the axis of rotation OZ and the plane of the equator of the earth) are force functions of the fields F_1 and F_2 respectively,

$$U_1 + \tfrac{1}{2}\omega^2 r^2 = U_2,$$

in which ω denotes the angular velocity of the earth's rotation.

Professor Roever reduces the problem to one in two dimensions by the following device: The curve c, when rotated about the earth's axis, generates a surface of revolution of axis OZ. The meridian curve of this surface which lies in the plane of OZ and P_0 we call curve (1). Plumb-lines of different lengths supported at P_0 do not coincide. The locus of the plumb-bobs of all plumb-lines supported at P_0, which lies in the plane of OZ and P_0, we call curve (2). Curves (1) and (2) are tangent at P_0 to the line of force of the field F_2 which passes through P_0. In order to establish the existence of a southerly deviation (in the northern hemisphere) it is enough to show that curve (1) lies south of curve (2). The magnitude of the deviation is the distance between the points in which these curves pierce the equipotential surface $U_2 = K$ which represents the earth's surface.

In particular, the form of the function U_1 was assumed to be M/ρ, where M is the mass of the earth and ρ is the distance of a general point from the earth's center O. For this function the equipotential surface $K = U_2 = M/\rho + \frac{1}{2}\omega^2 r^2$, which is taken

to represent the earth's surface, is approximately a spheroid of ellipticity .0017. (That of the earth is about .0034). The common tangent and the common normal at P_0 of the curves (1) and (2) being chosen as the axes of ξ and y, the difference between the ordinates of the curves (1) and (2) is $A\xi^2 + B\xi^3 + \cdots$, where

$$A = \frac{1}{2}\frac{\sin 2\phi_0}{\rho_0}\left\{ 8\frac{g_1^2}{g_2^3}\rho_0\omega^2 - \frac{g_1}{g_2^3}\rho_0^2\omega^4(8\cos^2\phi_0 + 3\sin^2\phi_0)\right\},$$

in which ϕ_0 is geocentric latitude of P_0, ρ_0 is ρ for P_0, g_1 and g_2 are accelerations at P_0 due to the fields F_1 and F_2 respectively. Approximately $g_1^2/g_2^3\rho_0\omega^2 = 1/(17)^2$ and $g_1/g_2^3\rho_0^2\omega^4 = 1/(17)^4$. The expression gives the southerly deviation when ξ is the height of P_0.

7. Professor Dean considers a special case of the problem in elasticity known as the problem of Clebsch, the forces being parallel to the faces of the plate and distributed continuously across the edges. The assumption of Clebsch, that certain shearing stresses are zero, restricts the solution to the case of a plate of infinitesimal thickness and is not made here. It is shown in the standard treatises that the strain perpendicular to the planes of the faces is a linear function of the coordinates parallel to the faces. From this fact and the generalized form of Hooke's law it is shown that the cubical expansion is a linear function of two variables, and that the shears are proportional to the distance from the mid-section of the plate parallel to the faces. This function substituted in the displacement equations gives two Poisson equations by means of which the displacements are expressed as logarithmic potential functions. The stresses being functions of the derivatives of the displacements are easily determined in terms of the coordinates and three arbitrary constants, which may be determined from the boundary conditions of stress. The paper will be offered to the *Philosophical Magazine.*

8. Professor Davis shows that, taking two vectors as representative of two imaginary points, it is possible to pass through them an infinite number of imaginary lines. This is effected by proper projective changes in the system of coordinates. It is shown also that lines in different systems may have an infinite set of elements in common. The paper will be embodied in one to appear in the *Nebraska University Studies.*

9. In Professor Westfall's paper the continuity and boundedness of integral rational functions of infinitely many variables in a domain D_m: $\sum_{i=1}^{i=\infty} x_i^2 \leqq m$, is discussed. If $\lim f(x) = f(a)$ whenever $\lim \sum (x_i - a_i)^2 = 0$, for one point a of D_m, this holds for every point a, and the integral rational function is bounded in any given domain D_m. If $\lim f(x) = f(a)$ when $\lim x_i = a_i (i = 1, 2, 3, \cdots, +\infty)$ when $\sum_{i=1}^{i=\infty} x_i^2 \leqq m$ and $\sum_{i=1}^{i=\infty} a_i^2 < m$, this again holds true for any point x in any domain D_m. That is, semicontinuity or continuity at a single interior point is sufficient to ensure the same property at every point of any domain D_m in the case of integral rational functions.

10. In a paper presented at the summer meeting of the Society at Princeton, September 14, 1909, Professor Newson announced a fundamental theorem in the theory of linear groups. His present paper is devoted to the application of this theorem to the theory of linear groups in two variables. He is able to find all the continuous groups as previously determined by Lie, and all of the discontinuous groups as determined by Klein and also by Gordan.

11. By means of the known relation $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ Professor Brenke obtains a linear differential equation in S_k, where S_k denotes the sum of the first k terms of the series. On solving for S_k and passing to the limit there results the equation

$$\sum_1^\infty c^n J_n(x) = \tfrac{1}{2} e^{ax} \int_0^x [cJ_0(x) + J_1(x)] e^{-ax} dx ; \left[a = \frac{c^2 - 1}{c} \right].$$

The result holds for all values of x and c.

12. Fréchet proved, in his thesis, that the Heine-Borel theorem holds for countable families in any compact assemblage in which "limit" is definable by means of "distance" in the sense in which those terms are there used. Professor Hedrick points out in the present paper that the same theorem is true for any compact, closed assemblage whose derivative is closed.

13. Mr. Frizell defines two rules of combination of symbols,

— a higher rule distributive according to the inductive formulas

(1) $(a \circ \mu)b = ab \circ \mu b, \cdots,$ (2) $a(b \circ \mu) = ab \circ a\mu, \cdots,$

over a lower rule associative according to the inductive formula

$$(3) \qquad a \circ (b \circ \mu) = a \circ b \circ \mu, \cdots;$$

postulates two classes of symbols both possessing the fundamental group property as regards the rule denoted by the sign $\circ$, of which (4) one shall contain the symbol μ, and (5) the other shall contain the symbol $\mu\mu$, where (6) $\mu_i \circ \mu_j \neq \mu_i \cdots$; and demands (7) that the two classes shall be identical.

It follows that the well ordered infinite class defined by the above seven postulates constitutes an abelian semigroup with regard to each rule of combination and has a modulus for the higher rule, which is distributive over the lower.

14. In Dr. Ingold's paper an effort is made to determine how far arbitrary relations may be assigned between a tangent vector T (supposed to be a function of a parameter s) and its derivatives T', T'', T''', $\cdots$. A set of normal vectors N_1, N_2, N_3, $\cdots$ are defined by the relations

$$N_i = A_{i0}N_0 + A_{i1}N_1 + \cdots + A_{i\,i-1}N_{i-1} + A_{ii}T^{(i)},$$

where the scalar coefficients A_{ij} are so defined that the N_i form an orthogonal set of unit vectors.

It is shown first that the A_{ij} are expressible in terms of i quantities $1/r_i$ and their derivatives; that the $1/r_i$ are the curvatures of the curve thus defined; finally that the A_{ij} certainly exist if T is explicitly given, since the r_i, and hence also the A_{ij}, can be expressed in terms of T and its derivatives.

O. D. KELLOGG,
Secretary of the Section.

NOTE ON A NEW NUMBER THEORY FUNCTION.

BY MR. R. D. CARMICHAEL.

(Read before the American Mathematical Society, September 13, 1909.)

THE present note deals with the properties of a number theory function defined by means of Euler's ϕ-function in the following way:

$\lambda(p^a) = \phi(p^a)$ when p is an odd prime;
$\lambda(2^a) = \phi(2^a)$ if $a = 0$, 1, or 2; $\lambda(2^a) = \frac{1}{2}\phi(2^a)$ if $a > 2$;*
$\lambda(2^a p_1^{a_1} \cdots p_i^{a_i})$ = the lowest common multiple of $\lambda(2^a)$, $\lambda(p_1^{a_1})$, $\cdots$, $\lambda(p_i^{a_i})$, p_1, $\cdots$, p_i being different odd primes.

Throughout, in a congruence such as

$$x^a \equiv 1 \pmod{n}$$

it will be assumed that x is prime to n. Then we have the theorem

$$x^{\lambda(p^a)} \equiv 1 \pmod{p^a} \tag{1}$$

for every prime p and integer a. For, by Fermat's theorem, (1) is true when p is an odd prime and also when $p = 2$ and $a = 1$ or 2, in view of the definition of λ. Then we have to examine only the case where $p = 2$ and $a > 2$.

Now by Fermat's theorem we have

$$x^{\phi(2^a)} \equiv 1 \pmod{2^a}, \quad (a > 2).$$

But it is known that the foregoing congruence has no primitive root; that is, for any odd x the congruence is true when $\phi(2^a)$ is replaced by some factor of $\phi(2^a)$ less than the number itself. But $\frac{1}{2}\phi(2^a) = \lambda(2^a)$ is the largest factor of $\phi(2^a)$ less than itself and contains all other such factors. Then

$$x^{\frac{1}{2}\phi(2^a)} \equiv 1 \pmod{2^a}, \quad (a > 2).$$

Hence the theorem of congruence (1) is proved.

This result may be employed to obtain a simple demonstration of the following analog of Fermat's general theorem:

* It is in respect to this part of the definition alone that $\lambda(n)$ differs from $\psi(n)$ defined by Bachmann, Niedere Zahlentheorie, I, p. 157.

I. *For any given n the congruence*

$$x^{\lambda(n)} \equiv 1 \pmod{n}$$

is satisfied by every x prime to n.

For suppose

$$n = 2^a p_1^{a_1} p_2^{a_2} \cdots p_i^{a_i}$$

and let β be any number prime to n. Then since $\lambda(n)$ is a multiple of every $\lambda(p^a)$, $p = 2, p_1, \cdots, p_i$, we have

$$\beta^{\lambda(n)} \equiv 1 \pmod{2^a}, \qquad \beta^{\lambda(n)} \equiv 1 \pmod{p_1^{a_1}},$$
$$\cdots, \beta^{\lambda(n)} \equiv 1 \pmod{p_i^{a_i}}.$$

From these congruences the theorem follows.

If $a^{\lambda(n)}$ is the first power of a congruent to 1 modulo n, we may say that a is a primitive λ-root (mod n). To distinguish, we may speak of the usual primitive root as a primitive ϕ-root (mod n). It follows immediately from the theory of primitive ϕ-roots that primitive λ-roots always exist when n is the power of any prime; for this is but another statement of well-known results for the modulus p^a. The λ-function introduces a simplification and allows the principal theory of the existence of primitive roots to be summarized into the following theorem:

II. *In every congruence*

$$x^{\lambda(n)} \equiv 1 \pmod{n} \tag{2}$$

a solution g exists which is a primitive λ-root, and for any such solution g there are $\phi\{\lambda(n)\}$ primitive roots congruent to powers of g.

If any primitive root g exists, g^a is or is not a primitive root according as α is or is not prime to $\lambda(n)$; and therefore the number of primitive λ-roots which are congruent to powers of any such root g is $\phi\{\lambda(n)\}$.

The existence of a primitive λ-root in every case is easily shown by induction. If n is a power of a prime the theorem has already been established. We will suppose that it has been established when n is the product of powers of r different primes and show that the theorem still remains true when n is the product of powers of $r + 1$ different primes; and from this follows the theorem in general.

Put

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}},$$

and let h be a primitive λ-root of

$$x^{\lambda(p_1^{\alpha_1} \cdots p_r^{\alpha_r})} \equiv 1 \pmod{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}; \tag{3}$$

whence $h + p_1^{\alpha_1} \cdots p_r^{\alpha_r} x$ is another form of the same root if x is any integer. Likewise, if c is any primitive root of

$$x^{\lambda(p_{r+1}^{\alpha_{r+1}})} \equiv 1 \pmod{p_{r+1}^{\alpha_{r+1}}}, \tag{4}$$

another form of the root is $c + p_{r+1}^{\alpha_{r+1}} y$, where y is any integer. If x and y can be so chosen that

$$h + p_1^{\alpha_1} \cdots p_r^{\alpha_r} x = c + p_{r+1}^{\alpha_{r+1}} y,$$

either member of this equation will be a common primitive root of congruences (3) and (4); that is, a common primitive root of the two congruences may always be obtained provided that the equation

$$p_1^{\alpha_1} \cdots p_r^{\alpha_r} x - p_{r+1}^{\alpha_{r+1}} y = c - h$$

has always a solution in which x and y are integers. But since the coefficients of x and y are relatively prime, the equation has always a solution in integers.

Now let g be the common primitive λ-root of congruences (3) and (4) and write

$$g^a \equiv 1 \pmod{n},$$

where α is to be the smallest integer for which the congruence is true. Since g is a primitive λ-root of (3), α is a multiple of $\lambda(p_1^{\alpha_1} \cdots p_r^{\alpha_r})$. In the same way it is a multiple of $\lambda(p_{r+1}^{\alpha_{r+1}})$. But $\lambda(n)$ is the lowest common multiple of $\lambda(p_1^{\alpha_1} \cdots p_r^{\alpha_r})$ and $\lambda(p_{r+1}^{\alpha_{r+1}})$; therefore α is a multiple of $\lambda(n)$, and hence $\alpha = \lambda(n)$ in view of the analogue of Fermat's theorem already demonstrated; for $g = h + p_1^{\alpha_1} \cdots p_r^{\alpha_r} x = c + p_{r+1}^{\alpha_{r+1}} y$ is evidently prime to n. Therefore g is a primitive λ-root of

$$x^{\lambda(n)} \equiv 1 \pmod{n}.$$

The theorem announced follows by simple induction.

There is nothing in the preceding argument to indicate that the primitive λ-roots of (2) are all in a single set obtained by taking the powers of some root g; in fact this is not even usually

so when n contains more than one prime factor. By taking powers of a primitive root g a set of primitive roots is obtained which evidently is identical with the set obtained by taking powers of any other root belonging to the set. We may say then that the set thus obtained is the set belonging to g. Then

III. *If* $\lambda(n) > 2$, *the product of the primitive roots in the set belonging to any* g *is congruent to* 1 (mod n).

These primitive roots are

$$g,\ g^{c_1},\ g^{c_2},\ \cdots,\ g^{c_\nu},$$

where $1, c_1, c_2, \cdots, c_\nu$ are the integers less than $\lambda(n)$ and prime to it. If any one of these is c, another is $\lambda(n) - c$ when $\lambda(n) > 2$. Hence

$$1 + c_1 + c_2 + \cdots + c_\nu \equiv 0 \quad [\text{mod } \lambda(n)].$$

Therefore

$$g^{1 + c_1 + c_2 + \cdots + c_\nu} \equiv 1 \quad (\text{mod } n).$$

Hence the theorem.

Corollary. *The product of all the primitive* λ-*roots of* $x^{\lambda(n)} \equiv 1$ (mod n) *is congruent to* 1 (mod n) *when* $\lambda(n) > 2$.

When n is given it is of course a very easy matter to find $\lambda(n)$. But the inverse problem, to find every x such that

$$(5) \qquad \lambda(x) = a \quad \text{or} \quad \lambda^{-1}(a) = x,$$

is more difficult. We construct a method for solving this problem.

IV. *If* x_1 *is the largest value of* x *satisfying* (5), *any other solution* x_2 *is a factor of* x_1.

Suppose that p^α is the highest power of any prime p such that $\lambda(p^\alpha)$ is a factor of a. Then evidently p^α is a factor of x_1; but no higher power of p is a factor of x_2, and therefore the theorem follows. Hence the following method for solving the problem in consideration:

Obtain the largest solution x_1 *of* (2); *examine every divisor* d *of* x_1 *and retain those* d*'s for which* $\lambda(d) = a$. *These are all the solutions of* (5).

To make the rule effective we require a means of computing x_1. It is evident that the following method leads to the desired result: Separate a into its prime factors and find the highest power p^α of each prime p contained in a such that $\lambda(p^\alpha)$ is equal to or is a factor of a. Suppose that the following prime powers are found: $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_i^{\alpha_i}$. Then write out all the divisors of

a and take every prime q such that $q-1$ is equal to any one of these divisors, but q is not eqnal to any p; and say we have q_1, $q_2, \cdots, q_k$. Then

$$x_1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i} q_1 q_2 \cdots q_k. \tag{6}$$

V. Corollary. *If* y_1 *and* x_1 *respectively are the largest solutions of the equations*

$$\lambda^{-1}(ma) = y, \quad \lambda^{-1}(a) = x, \tag{5a}$$

where m *is any integer* > 1, *then* $y_1 > x_1$.

(It is to be observed that in equations (5) and ($5a$), a and ma are assumed to be numbers such that each equation has at least one solution.)

By aid of theorem IV and the rule based on it I have constructed the following table containing every n for each $\lambda(n) > 1$ and $\leqq 24$. It is interesting to notice that $\lambda(x) = 12$ has 84 solutions.

$\lambda(n)$	n
2	3, 4, 6, 8, 12, 24.
4	5, 10, 15, 16, 20, 30, 40, 48, 60, 80, 120, 240.
6	7, 9, 14, 18, 21, 28, 36, 42, 56, 63, 72, 84, 136, 168, 252, 504.
8	32, 96, 160, 480.
10	11, 22, 33, 44, 66, 88, 132, 264.
12	13, 26, 35, 39, 45, 52, 65, 70, 78, 90, 91, 104, 105, 112, 117, 130, 140, 144, 156, 180, 182, 195, 208, 210, 234, 260, 273, 280, 312, 315, 336, 360, 364, 390, 420, 455, 468, 520, 546, 560, 585, 624, 630, 720, 728, 780, 819, 840, 910, 936, 1008, 1040, 1092, 1170, 1260, 1365, 1456, 1560, 1638, 1680, 1820, 1872, 2184, 2340, 2520, 2730, 3120, 3276, 3640, 4095, 4368, 4680, 5040, 5460, 6552, 7280, 8190, 9360, 10920, 13104, 16380, 21840, 32760, 65520.
16	17, 34, 51, 64, 68, 85, 102, 136, 170, 192, 204, 255, 272, 320, 340, 408, 510, 544, 680, 816, 960, 1020, 1088, 1360, 1632, 2040, 2720, 3264, 4080, 5440, 8160, 16320.
18	19, 27, 38, 54, 57, 76, 108, 114, 133, 152, 171, 189, 216, 228, 266, 342, 378, 399, 456, 513, 532, 684, 756, 798, 1026, 1064, 1197, 1368, 1512, 1596, 2052, 2394, 3192, 3591, 4104, 4788, 7182, 9576, 14364, 28728.
20	25, 50, 55, 75, 100, 110, 150, 165, 176, 200, 220, 275, 300, 330, 400, 440, 550, 600, 660, 825, 880, 1100, 1200, 1320, 1650, 2200, 2640, 3300, 4400, 6600, 13200.
22	23, 46, 69, 92, 138, 184, 276, 552.
24	224, 288, 416, 672, 1120, 1248, 1440, 2016, 2080, 2912, 3360, 3744, 6240, 8736, 10080, 14560, 18720, 26208, 43680, 131040.

VI. *Let a be that divisor of α for which $\lambda^{-1}(a) = x$ has a greatest solution x_1 greater than such a solution when for a any other divisor of α is taken. Then x_1 is the largest divisor of $z^\alpha - 1$ for every z prime to the divisor.*

That x_1 divides $z^\alpha - 1$ follows from Theorem I. Let y_1 be any number greater than x_1. Then in view of the conditions in the proposition $\lambda(y_1)$ is not a divisor of α. Hence, from the foregoing theory of primitive roots, it follows that there is some number z such that $z^\alpha - 1$ is not divisible by y_1. Hence the theorem. (From V. it is seen that $a = \alpha$ when $\lambda^{-1}(a) = x$ has a solution.)

In a previous paper* I tabulated a function $M(a)$ for possible values of a up to $a = 150$. A reference to the definition of $M(a)$ there given will show that $2M(a)$ is identical with our present x_1, the largest solution of $\lambda^{-1}(a) = x$, provided this equation has a solution. That table will therefore serve for determining x_1 for $a \leqq 150$. Thus it is seen that the largest divisor of $z^{144} - 1$ for every z which is prime to the divisor is 685,933,859,520. Further, the table for $M(a)$ may also be used in continuing the table of the present paper.

Professor J. H. Jeans† and more recently Mr. E. B. Escott‡ have discussed the converse of Fermat's theorem, showing that the relation

$$e^{n-1} \equiv 1 \pmod{n}, \tag{7}$$

which is always true, when n is prime, for any value of e prime to n, is for any particular value of e true for values of n which are not prime. This result will be extended by proving the theorem that *there are values of composite n for which relation (7) is true when e is any number prime to n.* In view of the foregoing theory of the congruence

$$e^{\lambda(n)} \equiv 1 \pmod{n},$$

it is evidently necessary and sufficient for this result that n has the property

$$n - 1 \equiv 0 \quad [\mathrm{mod}\ \lambda(n)]. \tag{8}$$

When $n > 2$, $\lambda(n)$ is even; and therefore (8) can be true for composite n only when n is odd. Further, since $\lambda(n)$ is prime

* BULLETIN, ser. 2, vol. 15, no. 5 (February, 1909), p. 222.

† *Messenger of Mathematics*, vol. 27, p. 174.

‡ *Messenger of Mathematics*, vol. 36, p. 175.

to n it follows that n contains no repeated prime factor; and hence *n is a product of odd primes no one of which is repeated.*

That n is not the product of two odd primes is easily shown. Suppose $n = p_1 p_2$, $p_2 > p_1$. Then

$$\frac{p_1 p_2 - 1}{p_2 - 1} = p_1 + \frac{p_1 - 1}{p_2 - 1} + \text{integer}.$$

But $\lambda(n)$ contains the factor $p_2 - 1$ and is therefore not a divisor of $n - 1 = p_1 p_2 - 1$.

On the other hand it is easy to find values of $n = p_1 p_2 p_3$, satisfying relation (8). It is necessary and sufficient that

$$\frac{p_i p_j p_k - 1}{p_i - 1} \equiv \frac{p_j p_k (p_i - 1) + p_j p_k - 1}{p_i - 1} = \text{integer},$$
$$(i, j, k = 1, 2, 3 \text{ in some order});$$

that is, that $(p_j p_k - 1)/(p_i - 1) = \text{integer}$. The following values of n have been found by inspection using this relation:

$$3 \cdot 11 \cdot 17, \quad 5 \cdot 13 \cdot 17, \quad 7 \cdot 13 \cdot 31, \quad 7 \cdot 31 \cdot 73.$$

By a similar method one may seek values of n for which n is the product of four or more primes; but the work will not be carried out here.

An example given by Lucas,* illustrating the failure of the converse of Fermat's theorem, belongs to a different class of exceptions. He shows that

$$2^{n-1} \equiv 1 \pmod{n}, \text{ when } n = 73 \cdot 37.$$

Here $\lambda(n) = 72$ while $n - 1 = 36 \cdot 75$; or $n - 1$ is a multiple of $\frac{1}{2}\lambda(n)$. Then we can easily find other values than 2 and its powers for which the preceding congruence is true. In fact every number prime to n belongs to some index which is a divisor of $\lambda(n)$. But every divisor of $\lambda(n) = 72$ except 8, 24, 72 is a divisor of $n - 1$. Hence the congruence $a^{n-1} \equiv 1 \pmod{n}$ is true for any integer a prime to n and not belonging to the index 8, 24, or 72 (mod n). Most of the examples given by Escott in the paper already referred to are similar to this one. These, however, are not so interesting as those for which congruence (7) is true for any e prime to n.

* Théorie des nombres, p. 422.

BAIRE'S LEÇONS D'ANALYSE.

Leçons sur les Théories générales de l'Analyse. Par RENÉ BAIRE. Tome I: *Principes fondamentaux, Variables réelles.* 1907. 8vo. 17 figures, x + 232 pp. 8 fr. Tome II: *Fonctions analytiques, Équations différentielles, Applications géométriques, Fonctions elliptiques.* 1908. 8vo. 35 figures, x + 347 pp. 12 fr. Paris, Gauthier-Villars.

THE prefaces of these books express in vigorous and convincing language the author's beliefs and plans. Baire has abundantly demonstrated his right to strong opinions ; what he thinks regarding comparatively elementary instruction is not to be despised. As a specimen, I quote the following to avoid loss of force in translation : Rigeur et simplicité ne sont nullement inconciliables, si l'on prend nettement le parti de faire pénétrer, dans l'enseignement des principes fondamentaux, certaines idées qui ont été acquises à la Science dans l'étude de questions d'ordre plus élevé. Pour en prendre un exemple frappant, la notion de bornes supérieures et inférieures d'un ensemble, qui commence seulement à être vulgarisée."

This bugle call to the standard of rigor may affright some to whom rigor and difficulty seem synonymous : precisely to such persons Baire's treatise will be a revelation — it *is* simple. That it is also reasonably accurate, Baire's name and the preceding quotation guarantee ; indeed one's expectation outruns the author's intention, and one notes the careful avoidance of difficult questions far more than any tendency to finesse.

A very special interest attaches to the introductory work and to the treatment of the foundations of the subject, both because the presentation is somewhat novel, and because the main body of the work is strictly limited to rather usual topics which give rise to little comment.

The first chapter is a treatment of the fundamental concepts of irrational numbers, sets of points, limits, and continuity. As noted in the preface, this chapter is substantially a reproduction of Baire's Théorie des nombres irrationels, des limites et de la continuité, published in 1905 by Vuibert et Nony. As a whole it is clear, exact, and elegant, and may well serve any student as an introduction to this subject.

A consistent treatment of irrationals by the Dedekind cut process is followed by brief treatments of the bounds of an assem-

blage and of the limits of sequences. In the latter topics $\pm\infty$ are recognized as bounds, or as limits "in the extended sense"; whether such infinite limits are excluded or included either is stated or is determinable from the context, but the student might go astray occasionally, for example, in section 20, page 13.

In defining continuity, the sequence notion is used on page 21, but it is shown later, on page 46, that this definition coincides with the usual ε definition. The definition of uniform continuity is phrased in ε form immediately. These definitions are given for several variables, and essentially for any point set; they are then used to establish the four fundamental operations of algebra by means of the fundamental principle of extension of definition (page 28) from the rational to the real numbers.

There follow interesting treatments of concrete quantities, of relations of geometry to algebra, of sets of points, and of the properties of continuous functions. Among other fundamental topics, I shall mention the proof that the two definitions of continuity are equivalent, and the theorem on inverse functions (page 51). The latter, together with the fundamental theorem of extension, is used to establish such functions as x^y, $\log x$, and so on. The chapter ends with a very brief, but satisfactory, treatment of elementary theorems on series of constant terms.

The fundamental principles of the differential and integral calculus are welded in Chapter II into a single link in the mathematical chain; in fact, the traditional separation of those topics is characterized in the preface of the second volume as "superannuated." The sums which lead to the integral are introduced on page 74 via the law of the mean, which, together with other usual properties of derivatives, has been carefully proved. Though this does "lead us to study such sums a priori," the device lacks the ring of naturalness — the weld is obviously artificial. Still, the process leads directly to the fundamental properties of the integral without resort to geometry; the most important of these properties are derived here with characteristic brevity and clearness.

The remainder of the chapter contains rather usual theorems on elementary, improper, and parametric integrals, implicit functions and functional dependence, derivatives and differentials of higher order, differential equations and total differentials. Indeed, I need only mention one or two points to show that the author was looking rather toward broad presentation under

somewhat easy restrictions than toward the extreme niceties. Thus the original definition of integral (page 74) is restricted to continuous functions (extended later in the case of improper integrals); the theorem on implicit functions assumes the existence and continuity of all the first derivatives; and so on.

There are even one or two cases of what might seem carelessness in another, but which are doubtless rather evidences of Baire's desire to avoid complicating detail. For example, the hypothesis that dy/dx or else dx/dy should exist and be different from zero is suppressed on page 65. On page 72, section 84, the condition that f'_x, etc., be continuous, which is necessary, *for this proof*, is omitted. On page 117, n is used in two senses, for it is necessary to assume, *for this proof*, that the derivatives of order $n+1$ are continuous. The hypothesis that f'_x, f''_{xx}, etc., be continuous is omitted on page 121 in the theorem on d^2u. Finally, in section 94, page 81, a proof of the formula for change of variable in an integral is given in new form; it is quite satisfactory, but to the author's hypothesis that $x = \phi(t)$ possess a continuous derivative may well be added the provision that t_0 and t_1 be so chosen that x remains in the interval $x_0 \leqq x \leqq x_1$.

The author's opinions are on the whole against the use of higher differentials. Even the one advantage recognized (preface, page vi) — that of writing relations valid for any later choice of independent variables — is by no means impossible with the derivative notations, and the fact mentioned in lines 18–19 of page 125 seems to rob this advantage of its force.

Among the most interesting and novel features of the whole work are the very elegant presentations of the ideas of length, area, and volume in Chapter III. A precise treatment of length is followed by an axiomatic development of area — which is not used in the original presentation of integration. While the ideas are not new, the presentations are so; they will be found interesting even to the initiated. A still simpler presentation seems possible, however, in the case of the area under a continuous curve $y = f(x) \geqq 0$. For let us assume the following axioms:

(1) The area of any rectangle is the product of its base and its height.

(2) The area of the figure formed by placing a finite number of rectangles in juxtaposition is the sum of the areas of those rectangles.

(3) The area of any region, if it exists, is a unique number.

(4) The area of any part of a region is less than or at most equal to the area of the whole region.

With these axioms, it follows immediately that the area between $x = a$, $x = b$, $y = 0$, $y = f(x)$ is necessarily $\int f(x)\,dx$, whenever that integral exists in Riemann's sense. Such a presentation, while admittedly not quite so far reaching as that of the text, would be wholly in keeping with the spirit of the work as a whole, in avoiding extreme cases.

Before passing to the question of volume, the double integral is defined, and its properties are discussed. The proof of the formula for change of variables is mentioned in the preface, and deserves notice here. The proof is given first for linear transformations, and is extended to the general case by use of the law of the mean. It is evident that Baire considered this presentation very carefully; the only objection appears to be its length, which seems out of proportion to some other parts in view of the fact that other proofs — artificial, to be sure — are available.

The treatment of improper double integrals (pages 177–179) is so brief that it may be misleading; the impression that circles may be used exclusively is not strictly justified by the text, but that impression will doubtless be received by a superficial reader.

The discussion of volume is exactly similar to that of area. It is followed by a treatment of triple integrals and a very extended treatment of the area of a surface. Especially the latter will be found interesting, but it is clearly out of proportion to the rest of the work. An excellent presentation of line and surface integrals, together with the theorems of Green and of Stokes, concludes this chapter and the first volume.

Volume II opens with a rather brief treatment of functions of complex variables. The presentation is general rather than an exposition of any one school; the demonstrations are conducted by methods of real variables whenever desired; emphasis is laid on the developability of any function which has a derivative; and only theorems of unquestionable prominence are given. Functions of several complex variables receive considerable attention; theorems are proved for several variables when convenient; and the results are used in proving such theorems as that for differentiating under the integral sign.

The most interesting portion of this chapter is the section on infinite series, which includes the theory of real series, since only series of constant terms were treated in Volume I. Baire makes extensive use of what he calls "normally convergent"

series, i. e., series which satisfy the Weierstrass test for uniform convergence. It is shown that any uniformly convergent series may be transformed into a normally convergent series by properly grouping consecutive terms, and the usual forms of statement are deduced from the results for normally convergent series. The new phrase is certainly convenient; its use in this book abundantly justifies its introduction.

Chapter V contains a treatment of differential equations, confined for the most part to elementary methods of actual solution. It includes, however, an existence proof for a set of linear equations (pages 86–89), and a very excellent treatment of characteristics of differential equations of the first order (pages 148–164). Partial differential equations of higher order are barely mentioned.

Chapter VI deals with applications to geometry. Again the work is quite restricted to very usual theorems, the topics considered being the usual elementary theorems regarding such ideas as lines of curvature, asymptotic lines, applicability of surfaces, geodetic lines, and so on. The presentation is far more traditional than in the previous chapters.

The final chapter is entitled elliptic functions, but a considerable portion of it is devoted to infinite products. The treatments are quite elementary and traditional, as in several chapters of this volume. Here, as in the chapter on complex variables, little is said of geometric representation of the Riemann type — another example of the policy of rigid exclusion of all dispensable materials.

On the whole, Baire's work fulfils the promise of his prefaces; the work is simple, restricted to rather fundamental topics, and yet accurate. The manner of presentation is often quite new, and is always clear and effective. There is some lack of uniformity, some loss of proportion, of one topic as compared with another; but this must remain a question of opinion, possibly of tradition, even of prejudice. Thus the foundations are laid wide and strong and as if for eternity. That the superstructure is even less imposing than in many of the older Cours may be disappointing; it would seem reasonable and reassuring, however, to assume that Baire wished to build so as to leave for others not the task of rebuilding from the very ground what he has done here, but rather that of starting where he stops, assured that the foundations he has laid will stand the strain of an enormous superstructure.

E. R. HEDRICK.

INFINITE SERIES.

Lehrbuch der unendlichen Reihen. Von Dr. Niels Nielsen. Vorlesungen gehalten an der Universität Kopenhagen. Leipzig, Teubner, 1909. viii + 287 pp. Price 12 M.

This excellent elementary textbook shows the careful working over which is necessary to make a course of lectures fit the requirements of the classroom. From the beginning to the end the author constantly has in mind the students before him. The foundations do not presume a partial structure already in the students' minds, but start from the bare fundamentals all must possess to understand the course at all. For this reason the work is divided into three parts: theory of sequences, series with constant terms, series with variable terms. We may say that the first part defines what a series must mean, and what it can give us; the second part discusses the management of the particular value of a series for a given value of the variable; the third part discusses the sweep of values due to different values of the variable. The development proceeds leisurely and is well illustrated with examples. The references are sufficiently numerous to incite the student to follow up the subject in original papers, but not so exhaustive as to overwhelm him.

The first part contains six chapters, in order: rational numbers, irrational numbers, real sequences, complex sequences, applications to elementary transcendental functions, doubly infinite sequences. The conception of rational numbers and their combinations under the four processes of arithmetic is the beginning. From this is developed immediately the idea of rational sequence, and limit. It is then shown that any periodic decimal fraction represents a rational number, and that non-periodic decimals represent limits but not rational numbers. An irrational number a is defined to be the non-rational limit of two approximation series h_n, l_n, such that

$$h_n > a > l_n, \quad \lim (h_n - a) = 0, \quad \lim (a - l_n) = 0.$$

The irrational ω, $h_n > \omega > l_n$, is greater than the irrational ω', $h'_n > \omega' > l'_n$, if for a certain n, $l_n > h'_n$; and ω is less than ω', if for a certain n, $h_n < l'_n$. In any other case, for each n, $h_n \geqq l'_n$ and $h'_n \geqq l_n$ and $\omega = \omega'$. The four fundamental operations are defined for irrationals by defining the limits of $h_n \pm h'_n$, $l_n \pm l'_n$,

or $h_n\, h'_n$, $l_n\, l'_n$, $h_n : h'_n$, $l_n : l'_n$, respectively. With these definitions it is shown that every irrational has an incommensurable decimal expression, and that irrational sequences lead to no new varieties of number. Thus the rational sequence really closes the number system. The development followed here is a special case of that of G. Cantor.*

Sequences of real numbers are classified as convergent or fundamental series; divergent, which are either proper or improper according as the sequence from a certain term on retains the same sign or not; and oscillating. The limits of the sum, difference, product, and quotient of two fundamental series are shown to be the sum, difference, product, and quotient, respectively, of the two limits. The general test of convergence is proved in these terms: †

The necessary and sufficient condition for the convergence of the sequence $a_0, a_1, \cdots, a_n, \cdots$ *is that for any preassigned arbitrary small positive number* ϵ *there exists a positive integer* N, *such that for* $n \geqq N$

$$| a_{n+p} - a_n | < \epsilon$$

for any positive integer p.

The author now introduces the conceptions of monotone sequence, of limit points, of upper and lower boundaries, of superior and inferior limits. He closes the chapter with a theorem of Abel: ‡

If there is given a monotonic non-increasing sequence ϵ_n, *and if from a given sequence* $\cdots, a_n, \cdots$ *we form a new sequence* $\cdots, s_n, \cdots$, *where* $a_p = s_p - s_{p-1}$, *then for every* n

$$\epsilon_1 \cdot g_n \leqq a_1 \epsilon_1 + \cdots + a_n \epsilon_n \leqq \epsilon_1 \cdot G_n,$$

where G_n and g_n are respectively the upper and lower boundaries of the sequence s_n.

In the next chapter, on complex sequences, the complex number is treated as a couple (a, b). The usual formulas of combination are developed, and the convergent complex sequence is defined to be one such that the sequence of the moduli of its terms is convergent. A theorem of Jensen,§ which is a generalization of one of Cauchy, is given:

* *Math. Annalen*, vol. 5 (1875), p. 128.

† Du Bois-Reymond: Allgemeine Funktionentheorie, vol. 1, p. 6, 260.

‡ *Crelle*, vol. 1 (1826), p. 314.

§ *Tidsskrift for Mathematik*, ser 5, vol. 2. (1884), pp. 81–84. *Comptes Rendus*, vol. 106 (1888), pp. 833–836.

If the sequence $\cdots, a_n, \cdots$ *satisfies the two conditions*

$$(1) \qquad \lim_{n \doteq \infty} |a_n| = \infty,$$

and for every n

$$(2) \quad |a_1| + |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}| < B. |a_n|,$$

where B *is a fixed determinate number independent of* n; *if moreover the fundamental series* $\phi_1, \phi_2, \cdots \phi_n, \cdots$ *has the finite limit* $\Phi = \lim_{n \doteq \infty} \phi_n$, *then we also have*

$$\Phi = \lim_{n \doteq \infty} \left(\frac{a_1 \phi_1 + (a_2 - a_1)\phi_2 + \cdots + (a_n - a_{n-1})\phi_n}{a_n} \right).$$

From this we may deduce the formula

$$\lim_{n \doteq \infty} \left(\frac{b_n}{a_n} \right) = \lim_{n \doteq \infty} \left(\frac{b_n - b_{n-1}}{a_n - a_{n-1}} \right),$$

where $b_n = a_1 \phi_1 + (a_2 - a_1)\phi_2 + \cdots + (a_n - a_{n-1})\phi_n$.

For $a_n = n$, we have a theorem of Cauchy :*

If in the sequence $\cdots, b_n, \cdots \lim_{n \doteq \infty} (b_n - b_{n-1})$ *is finite and determinate, then*

$$\lim_{n \doteq \infty} \left(\frac{1}{n} b_n \right) = \lim_{n \doteq \infty} (b_n - b_{n-1}).$$

In the second part, which treats of real series, a series is defined to be a sequence $\cdots, s_n, \cdots$ of sums of terms $u_1, \cdots, u_n, \cdots$. Convergence of a series is distinguished from summability. If the new sequence $S_n = (s_1 + s_2 + \cdots + s_n)/n$ is formed and if it is a convergent sequence, then the series $u_1, u_2, \cdots, u_n$, which gives $\cdots, s_n, \cdots$ is said to be summable. A generalization of this is also suggested, through the Jensen theorem mentioned above. Convergence which is independent of the order of the terms is unconditional; otherwise it is conditional. Absolute convergence exists when the series of absolute values of the terms converges. It is shown that unconditional convergence and absolute convergence imply each other.

Certain cases of convergence are now considered. These are: the well-known theorem of Leibniz on alternating series;

* Analyse algébrique, 1821, p. 54.

comparison series; a theorem of Pringsheim to the effect that *if* $\cdots, a_n, \cdots$ *is monotone increasing and divergent, then the series whose terms are*

$$u_n = \frac{a_{n+1} - a_n}{a_{n+1} \cdot a_n^\rho}$$

is convergent for every positive ρ; logarithmic tests; a theorem of Jensen: *

If $\lim_{n=\infty} |a_n| = \infty$, *and* $|a_1| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}| < B \cdot |a_n|$, *but otherwise the values of* a_n *are quite arbitrary, then*

$$\sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{a_n}$$

is divergent; several conclusions are drawn from this, embodying theorems of Dini; two theorems ascribed to both Du Bois-Reymond † and Dedekind ‡:

1. *If* $\cdots, a_n, \cdots$ *and* $\cdots, b_n, \cdots$ *are two infinite series of arbitrary complex numbers which satisfy only the conditions that both*

$$\sum_{n=0} a_n \text{ and } \sum_{n=0}^{\infty} |b_n - b_{n+1}|$$

converge, then $\sum a_n b_n$ *converges.*

2. *If* $\cdots, a_n, \cdots$ *and* $\cdots, b_n, \cdots$ *satisfy only the conditions* $\lim_{n=\infty} b_n = 0$, $\sum |b_n - b_{n+1}|$ *converges, and* $\sum a_n$ *oscillates between finite boundaries, then* $\sum a_n b_n$ *converges.*

The chapter following is devoted to elementary tests of convergence. This seems to be a purely artificial separation of the subject treated in this chapter and the preceding. The tests given are (1) Kummer's: §

The series of positive terms $\sum u_n$ *is convergent if it is possible to determine a series of positive terms* $\cdots, \phi_n, \cdots$ *such that for* $n \geqq N$

$$\phi_n \cdot \frac{u_n}{u_{n+1}} - \phi_{n+1} \geqq \alpha > 0.$$

Likewise divergent if

$$\phi_n \cdot \frac{u_n}{u_{n+1}} - \phi_{n+1} < 0.$$

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* *Tidsskrift for Mathematik* (1884), ser. 5, vol. 2, p. 85.

† Neue Lehrsätze über die Summen unendlicher Reihen, p. 10.

‡ Dirichlet, Vorlesungen über Zahlentheorie, § 101, 3 Aufl., 1879.

§ *Crelle*, vol. 13 (1835), p. 172.

(2) Cauchy's: *

The series of positive terms $\sum u_n$ is convergent or divergent according as

$$\liminf_{n=\infty} \frac{u_n}{u_{n+1}} > 1 \quad \text{or} \quad \limsup_{n=\infty} \frac{u_n}{u_{n+1}} < 1.$$

(3) Duhamel's: †

If for the series of positive terms $\sum u_n$ we have a development of the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1+\alpha}{n} + \cdots,$$

the series converges or diverges according as $\alpha > 0$ or $\alpha < 0$.

(4) Raabe's: ‡

If for the series of positive terms $\sum u_n$ we have

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{a}{n^{1+\delta}} + \cdots,$$

where a is finite and δ is an arbitrary small but assignable positive number, then $\sum u_n$ diverges.

(5) The logarithmic test.

Following these is a chapter on the multiplication of series, which is given a somewhat original treatment.

After an excellent chapter on transformation of series for more rapid numerical computation, the question of convergence is again taken up. The preceding theorems are widened and generalized and the discussion is led to the general criteria of Pringsheim. From his criteria of the first and second kinds are shown to follow all the more elementary criteria. It seems unnecessary to trace the development further. It is evidently along teachable lines, and will give the student a clear understanding of the question of convergence. This part is closed with a treatment of infinite products, infinite continued factors, and double series.

The last part treats of series of variable terms. In the first chapter the idea of point set (Menge), is developed, with its usual accompanying terms. In the second chapter the idea of

* Analyse algébrique, p. 134.

† *Journ. de Math.*, vol. 4 (1839), pp. 214–221.

‡ *Journ. von Ettinghausen und Baumgartner*, vol. 10 (1832); *Crelle*, vol. 11, pp. 309–311. *Journ. de Math.*, vol. 6 (1841), pp. 85–88.

function of one or two real variables, and of a complex variable is developed. The function is handled as a point set of functional values. The general principle of convergence is stated in these terms:

The necessary and sufficient condition for the existence of $f(a-0)$, *and thus for the existence of a limit* A *for the function* $f(x)$ *as* x *approaches* a, *is that it be possible to find for any preassigned arbitrary small number* $\epsilon > 0$ *a number* $\delta > 0$ *such that if* $a - \delta \leqq x \leqq a$, $a - \delta \leqq y \leqq a$, *then* $|f(x) - f(y)| < \epsilon$.

A similar condition is stated for a function of a complex variable. The definition of, and conditions for, continuity follow. It is shown that any rational function of continuous functions of x is continuous, save for those values which make the resulting denominator vanish. Also if $y = f(x)$ is continuous for $x = a$, and $z = \phi(y)$ is continuous for $y = b = f(a)$ then $z = F(x) = \phi[f(x)]$ is continuous for $x = a$. The existence of maxima and minima is then proved as usual.

The next chapter treats of uniform convergence. Seven theorems are given to the effect that

1. *A convergent sequence of continuous functions of* x *which is uniformly convergent in a given interval defines a continuous function of* x *over this interval.*

2. *An infinite convergent series is uniformly convergent over an interval if the remainder after the nth term is convergent independently of the values of* x *in the interval.*

3. *An infinite convergent series of continuous functions of* x *which is uniformly convergent over an interval defines a continuous function of* x.

4, 5, 6, 7. These relate to the convergence of infinite products.

To illustrate these theorems the gamma function is briefly treated. The following chapters discuss trigonometric series, power series, Dirichlet series, and faculty series. In the latter chapter Professor Nielsen has incorporated some original work. The 210 problems are very carefully selected to throw light on the developments which they follow. The text as a whole is rather carefully graded as to difficulty and yet is full enough for a first course. Some few errors remain uncorrected but these are easily apparent to the reader.

JAMES BYRNIE SHAW.

THE COLLINEATIONS OF SPACE.

Die Lehre von den geometrischen Verwandtschaften. Dritter Band: die eindeutigen linearen Verwandschaften zwischen Gebilden dritter Stufe. By RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1909, viii + 574 pp.

THE third volume of Professor Sturm's treatise has about the same range of discussion for three dimensions as the preceding ones had for one and two respectively,* the size of the volume being explained by the very full treatment given to systems of correlations. This volume is not provided with a separate index nor a preface, but does contain a glossary of 165 technical words not found in the preceding ones, and the table of contents of all four volumes.

The first chapter (pages 1–352) is concerned with the ordinary problems of collineation and correlation in space, being somewhat similar to the treatment found in the corresponding parts of Reye's Geometrie der Lage, but much more extensive. The properties of central and axial perspective, affinity, similarity, and congruence are first brought out to discuss the collineations and polarity defined by linear complexes. All of this leads up to the theorem that a general collineation can be resolved into three axial involutions.

The two kinds of collineation which leave a given quadric invariant, and in particular a space cubic curve are given thirty pages. This could have been materially shortened by a more liberal use of analytic methods,† the development being almost exclusively synthetic. In fact the need of still another treatise, having the same general content as this first chapter, but proceeding algebraically, is keenly felt. It could be regarded as an appendix to the work of Professor Sturm.

Metrical, particularly focal, properties follow. Here a number of results are necessarily stated in algebraic form, but mostly without proofs. A knowledge of projective analytic geometry of quadrics is presupposed. Besides lines of curva-

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* See the reviews in volume 15 of the BULLETIN; that of Volume 1 on page 135, and of Volume 2 on page 252.

† As for example, that employed by Wiman; "Ueber die algebraischen Curven von den Geschlechtern $p=4$, 5, and 6, welche eindeutige Transformationen in sich besitzen," *Bihang till Kongl. Svenska Vetenskaps-Akad. Handlingar*, 1895, and particularly, Miss Van Benschoten; "Birational transformations of algebraic curves of genus four," *Amer. Jour. Math.*, vol. 31 (1909), pp. 213–252.

ture and curves of contact with an enveloping developable, geodesics on a quadric are defined and shown to be covariant in the collineations of a system of confocal quadrics. A few general theorems concerning the commutativity of two collineations are followed by a more thorough discussion of groups generated by central harmonic homologies. Every axial involution can be generated in this way. Cyclic collineations and cyclic correlations of degree four are treated at length.

Over sixty pages (287–352) are devoted to the discussion of groups of collineations. The first topic is that of collineations in inscribed tetrahedral position, i. e., such that a tetrahedron exists having the property that the image of each vertex lies in the opposite face. It is shown that the existence of one such tetrahedron insures the existence of ∞^9 of them, three vertices being arbitrary. The article on cyclic collineations with non-planar cycles distinguishes between cycles of odd and of even period, and discusses those of period 4, 5, and 6 at length. These themes have been exhaustively treated in a number of recent dissertations from Breslau and Strassburg.*

In case of period six the principal distinction is between four imaginary invariant points on the one hand, and two real and two imaginary ones on the other. In the article on the general group concept a number of properties are mentioned, but the fundamental question, under what conditions are two collineations commutative, is not touched.†

Besides the regular body groups, the G_{32} defined by six linear complexes mutually in involution, and its linear subgroup of order 16 are treated in detail. The chapter closes with a comprehensive synthetic treatment of the configuration $(12_6, 16_3)$ composed of 12 planes, 12 points and 16 lines, such that each of the 16 lines lies in three of the planes and passes through six points, and every point lies in six planes, defining a desmic

* Particularly H. Küppers, "Kollineationen, durch welche fünf gegebene Punkte des Raumes in dieselben fünf Punkte transformiert werden" (Münster, 1890, 79 pp.); H. Reim, "Wie müssen zwei projektivische Punktfelder aufeinander gelegt werden, damit entsprechend kongruente Polygone cyclisch zusammenfallen?" (Breslau, 1879, 28 pp.); R. Krause, "Ueber senäre cyklische Kollineationen" (Strassburg, 1903, 58 pp.); E. Gässler, "Ueber senär cyklische Korrelationen in der Ebene und im Raume" (Strassburg, 1903, 39 pp.); J. Cordier, "Gruppe von 96 Kollineationen (Strassburg, 1905, 58 pp.).

† The theorems given by Reye regarding two commutative linear substitutions are only partly correct. The theorem of Stéphanos for binary fields can be readily extended to any number of dimensions, but other forms can also appear in space of three dimensions.

system. This system belongs to a group of 2304 collineations and correlations, which contains subgroups of order 1152 and of order 576. The simple group of collineations of order 168 first discovered by Klein, and that of order 360 found by Valentiner are mentioned only in a footnote. Notwithstanding the excellent discussion in the present treatise and the many valuable memoirs mentioned in it, there is still much important work to be done before the theory of linear and birational transformations can be fully translated into the language of substitutions.

The second chapter (pages 353–574) is concerned with linear systems of curves and surfaces, singular collineations, problems in enumerative geometry and linear systems of linear transformations. It begins with a system of plane curves having one degree of freedom, then two, etc., each successive case being generated by the fan formation from the preceding one. While this treatment is an excellent one, it could be much improved or at least its results made more useful for others by the addition of an algebraic formulation of both processes and results. The section on the generation of curves and surfaces by means of projective pencils follows Cremona somewhat, but is more readable and much more extensive. The treatment of the important subject of restricted systems is more systematic but less extensive than that found in Salmon's Algebra, the only other book containing it. The theory of poles and polars is essentially that of Cremona, but here again, is much longer. Incidentally a generous synthetic treatment of covariant curves and surfaces including the Hessian, the Steinerian, the Cayleyan, and their Jacobian is introduced.

The long sections on singular correlations and collineations begin with the adjustment of two correlative spaces into a polar space. As first singular form appears the polarity with regard to a cone, or its dual, a conic, and a pair of planes or a pair of points. If now the superposed position of the two spaces be dissolved, and with it the involutorial correspondence, we have the three singular correlations, called by Hirst the central, the planar, and the axial. A well-known special case of the last named is the reciprocity in a special linear complex. By dualizing one of the spaces we obtain the singular collineations of the first kind, but only two forms, as the first and second are really equivalent. A central homology with invariant 0 furnishes an example of the first kind (ordinary artist's perspective)

and an axial collineation with parameter 0 is an illustration of the second kind.

From this foundation it is now possible to reduce the successive cases to depend on those of singular plane collineations which were treated at length in Volume 2. But complete enumeration is a hopeless task; after allowing for all possible duplications, over 4000 signatures exist, each one deserving some attention.

The results of this investigation are employed to determine the number of conditions imposed by homology, axial involution, affinity, etc., and then in turn the possible singular transformations in these various types. Linear systems of space correlations, of polar spaces and duality in a pencil of linear complexes are then treated, and the singular and degraded cases included among them are determined.

The idea of apolar linear systems of space correlations leads to a number of new configurations, and serves to systematize a number of theorems found by Rosanes, Reye, Voss, Kohn, and others.

Given two correlations C, Γ having the property that one tetrahedron $\alpha'\beta'\gamma'\delta'$ in the space Σ' exists such that its image $A_1B_1C_1D_1$ in Γ is inscribed in its image $ABCD$ in C; in the two spaces Σ, Σ' there are then ∞^9 tetrahedra having this property; by performing the operations in inverse order (the C polar of the Γ polar) we have ∞^9 tetrahedra circumscribed about the given ones. The correlations Γ and C are called apolar, C is said to support Γ and Γ to lie in C.

If C_1, C_2 are two correlations, each of which supports Γ, then every correlation of the pencil determined by them will also support it. Similarly for nets, webs, and in general k-fold systems. Simultaneously Γ may be any correlation of an i-fold system, such that $i + k = 14$. This idea is now applied to a singular correlation, i. e., Γ is now polarity with regard to a cone having its vertex at S, and S' the center in Σ'. The points S, S' are conjugate in C. It now follows that every pencil of correlations contains one which supports a given one.

The same procedure is then carried out for collineations, by dualizing one of the spaces and applying (S, s'), (S', s) connexes as was done in the second volume. Finally a long discussion (57 pages) is given to linear systems of projective configurations which support each other.

Notwithstanding the size of the volume many topics are

treated very briefly, and some knowledge of various special operations is presupposed. A familiarity with the preceding volumes would not be sufficient preparation for the intelligent reading of the present one.

VIRGIL SNYDER.

A SYNOPTIC COURSE FOR TEACHERS.

Elementarmathematik vom höheren Standpunkte aus. Von F. KLEIN. Teil I: *Arithmetik, Algebra, Analysis.* Vorlesung gehalten im Wintersemester 1907–08. 5 + 590 pp. Teil II: *Geometrie.* Vorlesung gehalten im Sommersemester 1908. 6 + 515 pp. Ausgearbeitet von E. HELLINGER. Autogr. Leipzig, in Kommission bei B. G. Teubner, 1908–09.

THE volumes under review contain a course of lectures intended for prospective teachers of mathematics in the secondary schools of Germany. The objects of the course and the reasons for giving it are so well stated in the introduction to the first volume and are of such vital interest in their application to conditions in our own country, that it seems desirable to quote at length.

"In recent years" — thus does Professor Klein begin his first lecture — "a widespread interest has developed among university teachers of mathematics and the natural sciences regarding the proper training of teachers for our secondary schools. This movement is of quite recent date; for a long period previously our universities were concerned exclusively with the higher science without any reference to the needs of the secondary schools and, in fact, without attempting to bring about a connection with secondary mathematics. But what is the result of such a practice? The young student at the outset of his university work is brought face to face with problems that do not serve to remind him of what he has previously studied and naturally he proceeds to forget all of it quickly and thoroughly. On the other hand, if after leaving the university he enters upon his work as a teacher, he is required to give instruction in the established courses in elementary mathematics and, as he is unable without assistance to bring his new work into relation with his advanced mathematics, he soon adopts the old traditional methods and his university studies become merely a more or less pleasant memory which has no influence on his teaching.

"At the present time the attempt is being made to destroy this *double discontinuity* which has certainly been in the interest neither of the secondary schools nor of the university. This is to be done, on the one hand by infusing into our secondary school courses new ideas in keeping with the modern development of our science and of culture in general . . ., on the other hand by giving due consideration in our universities to the needs of the prospective teacher. One of the most important means to this end, it seems to me, is a synoptic course of lectures of the kind I begin today. . . . My purpose will be throughout to exhibit *the mutual connection between the problems of the various branches of mathematics* which is not always sufficiently emphasized in the special courses devoted to them, as well as to emphasize their relations to the problems of elementary mathematics. Thereby I hope to make easier for you what I should like to designate as the real purpose of your university study of mathematics; viz., that *you may be able in a large measure to draw inspiration for your teaching from the great body of knowledge that has here been presented to you.*"

It is true also in this country that university courses in mathematics are generally given without direct reference to the needs of those who expect to teach the subject in our secondary schools. The result is that described by Professor Klein: the mathematics studied in the university has little or no influence on the teaching of elementary mathematics. An increasing number of prospective high school teachers are taking advanced courses and advanced degrees at our universities in preparation for their future work. As a result it seems probable that at no distant date a master's degree at least will be required of candidates for positions on the faculties of our best high schools. This is of course as it should be. But it imposes on our universities the duty of offering these men and women carefully planned courses of study that will give them a thoroughly practical and helpful preparation for their later work. The problems connected herewith have received but little attention hitherto in this country. In Germany, on the other hand, they have been widely discussed during the past decade. These discussions culminated in the report of the German commission of 1907 which contains a comprehensive plan of study for the prospective teacher of mathematics or the natural sciences.*

*Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte, Leipzig, 1908, pp. 264–306.

This report is of great interest to us in this country and might well serve as a basis for the discussions of an American commission instructed to consider these questions.

The desirability of such a synoptic course as part of the training of a teacher is so obvious that it seems highly probable that courses of this character will be given at our own universities as soon as they begin seriously to consider the needs of those who are preparing to teach. In considering the volumes under review, which give us in detail what their distinguished author believes should form the content of such a course, it seems desirable, therefore, to keep in mind their possible use as a basis for courses of this kind in America.

From what has been said of the nature of the course, it will already be clear that the prospective teacher is to take this course in his last year at the university. It is assumed that he has previously taken courses in analytic geometry, calculus, differential equations, descriptive and projective geometry, the theory of numbers, the theory of curves and surfaces, and the theory of functions, in addition to courses in mechanics, experimental and theoretical physics, chemistry, philosophy, pedagogy, logic and psychology.* Can we expect one of our students to cover this ground, assuming him to spend one year in graduate work, without making his undergraduate course too one-sided? I think we can. Many of our students now cover the work in analytic geometry and calculus, in chemistry and in experimental physics by the end of their sophomore year. The junior year might then bring one semester each (three hours weekly) of differential equations, solid analytic geometry, the theory of numbers, and descriptive geometry, and a full year course in analytic mechanics. The senior year should then contain a full year course in the theory of functions of a complex variable and a full year course in theoretical physics. The work in philosophy, logic, psychology, and pedagogy may be distributed through the junior and senior years. The year of graduate work might then be devoted to curves and surfaces, projective geometry and a synoptic course of the kind now before us. This outline would seem to leave an adequate amount of time during the four undergraduate years for general cultural courses. Many desirable modifications of this outline will suggest themselves. My present purpose is accomplished if I have shown that without im-

* This list is taken from the report of the German commission referred to above.

pairing his undergraduate course along general cultural lines a student can prepare himself along the lines of the plan laid down by the German commission in the time usually required for the obtaining of a master's degree.

We may now turn to the consideration of the contents of the volumes under review. The limitations of space (and of time) prevent a detailed description. We must content ourselves with a general survey, to which may be added the discussion of one or two features which seem to merit special attention. The first volume is divided into three main parts devoted respectively to "Arithmetik" (i. e., the theory of numbers in a broad sense), algebra, and analysis. In some 80 pages the author considers in order the operations with the natural numbers, the fundamental laws to which these operations are subject, the logical foundations of arithmetic (as we use the term in this country, i. e., the German Rechnen), the technique of numerical calculation with the natural numbers (the thoroughly practical point of view is here emphasized by the detailed description of the mechanism of a calculating machine), the successive extensions of the number concept by the introduction of the negative, fractional, and irrational numbers. There follows a section of some 40 pages on the theory of numbers proper, treating in particular the properties of prime numbers, decimal fractions, continued fractions, Fermat's last theorem, cyclotomic problems, and closing with an elementary proof that the regular inscribed polygon of seven sides cannot be constructed with a ruler and compass. Then follows a section of about 40 pages on ordinary and higher complex numbers, with a more detailed consideration of quaternions. This completes the first main part.

It seems desirable at this point to say a word regarding the author's method throughout this volume. In accordance with his avowed purpose of exhibiting everywhere the mutual relations of the various branches of mathematics we have throughout the first volume a wealth of geometric illustrations. We have geometric interpretations of the fundamental laws of algebra (with a vigorous protest against the practice of using such interpetations as proofs in cases where they do not apply), Klein's beautiful geometric representation of a continued fraction, a geometric formulation of the problem of the pythagorean numbers and its extension to that of Fermat's theorem already mentioned. We have already spoken of the application of

analysis to prove the impossibility of constructing with ruler and compass an inscribed regular heptagon. We are given also the usual geometric representation of complex numbers. Finally the multiplication of quaternions is interpreted as rotation and stretching in space. True to his other avowed object, the author also gives at the end of the discussion of each topic what he conceives to be its relation to elementary mathematics and its bearing on the problems confronting the teacher. In the latter particular the method of the first volume differs essentially from that of the second, where all matter relating to the teaching of the elementary branches is reserved for the end and is discussed at length in an appendix. This arrangement is made desirable, to some extent, by the sequence of topics adopted. But the reviewer is inclined to believe that the plan of the first volume is the more effective. Finally, in this connection we must not fail to call attention to the prominent place given throughout every discussion to the history of the subject under consideration. The historical setting is made an essential feature of the presentation and contributes greatly to a clear understanding and the stimulating effect of the whole.

In order to prepare his hearers further in this direction, the author here interposes a general survey of the modern development of mathematics. It is a mere sketch, but it is a masterpiece. In some twenty pages he calls attention first to two fundamental tendencies in the growth of our science. The first tendency has for its object the development of a given branch of mathematics for its own sake and with its own methods. If this tendency alone obtained, mathematics would appear as a group of distinct theories which may show here and there incidental points of contact, but which have no organic unity. The second tendency on the other hand has for its object precisely to wipe out the boundaries between the various so-called branches, and conceives of mathematics as a unified whole, in which the results and methods of every branch are common property of the whole. From this point of view the two fundamental divisions of analysis and geometry become fused into a single whole in which every theorem of either may be regarded as a theorem of the other. It seems to us that the author might have been even more emphatic on this point than he is, a subject to which we will return presently in the discussion of his treatment of the modern developments of the foundations of the science. After calling attention to these two tendencies, the

author sketches briefly the history of mathematics from ancient to modern times with reference to them. This discussion, if printed, would hardly fill more than half a dozen ordinary octavo pages; nevertheless, the author is able to give a remarkably vivid picture of the successive stages in the development of mathematics.

After this interlude, he takes up the second main part of his lectures. In his discussion of algebra he confines himself exclusively to the theory of equations, in particular to the use of graphical and in general of geometric methods in this theory. The first part of the discussion is devoted to the description of certain graphical methods for the solution of real equations with real roots. By the use of the principle of duality two methods of representation are obtained, according as the coordinates are interpreted as point or as line coordinates. This discussion contains much of interest also to the student of analytic geometry and forms an excellent example of the interplay of two branches of mathematics when applied to a special problem. The second part of the treatment of the theory of equations is devoted to the consideration of the equation from the point of view of the theory of functions of a complex variable, the representation by means of conformal mapping on two spheres occupying the central position.

The third main part of the volume, that on analysis, begins with a discussion of the logarithm. After sketching the origin and development of the theory in historical sequence which brings out clearly why it is that the irrational number e should appear as the base of the system of "natural" logarithms, professor Klein concerns himself in detail with the difficulties attendant on the introduction of logarithms in secondary instruction. This is followed by an exposition of how he would like to see this subject treated in elementary courses. His plan contemplates the definition of the logarithm of a as the area between the hyperbola $xy = 1$, the x-axis, the ordinate $x = 1$ and the ordinate $x = a$; *i. e.*, by the relation

$$\log x = \int_1^x \frac{dx}{x}.$$

Whatever may be said of the practicability of this procedure in Germany, there can be little doubt that it is utterly impracticable in this country, at least with the present organization of

our secondary school curricula. With us, logarithms are needed first as an aid in numerical computation in our courses in trigonometry. At the time when it thus becomes necessary to introduce logarithms, the pupil is not even familiar with the hyperbola, to say nothing of his total ignorance of the notion of an integral. However much we may be in sympathy with the movement looking toward an early introduction of the notion of function, graphical representation, and of the elements of the infinitesimal calculus, it appears to the reviewer quite impossible to make these concepts all familiar to a pupil by the time it first becomes desirable to consider logarithms. Judging by a recent German review, conditions in that country do not seem to be very different from our own in this regard.* The reviewer, moreover, is of the opinion that nothing is gained by an attempt to introduce the natural logarithms before the regular course in the calculus. As to the best method of introducing logarithms, when they become desirable as an aid to computation, opinions may differ. The chief pedagogical difficulty at this time is to make clear to the pupil that corresponding to every positive real number there is a logarithm to a given positive base, say 10. The reviewer has found that the following plan works well in practice: Supposing the base to be a, the successive powers

$$(1) \qquad \cdots, a^{-3}, \ a^{-2}, \ a^{-1}, \ 1, \ a, \ a^2, \ a^3, \ \cdots$$

and the corresponding exponents

$$(2) \qquad \cdots, -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3, \ \cdots$$

form a geometrical and an arithmetical progression, with the property that to the product of any two numbers of the former corresponds the sum of the corresponding numbers of the latter. Moreover, this property is readily seen to hold for any geometrical progression and any arithmetical progression, provided the number 1 is in the former and corresponds to the number 0 in the latter. If then any number of geometric means be inserted between two numbers of (1) and the same number of arithmetic means be inserted between the corresponding terms of (2), the two sequences of numbers will still have the fundamental property mentioned. It remains only to show that by carrying

*Cf. C. Färber, *Archiv der Mathematik und Physik*, vol. 15 (July, 1909), p. 75.

this process far enough, any given positive number may be approximated to as closely as we please in the sequence (1).

This is the only place, it should be added, in the two volumes in which it appears that one of Professor Klein's suggested pedagogical reforms is impracticable in this country. The section on the logarithm closes with a consideration of this function from the point of view of the theory of functions of a complex variable. Naturally the exponential function is disposed of in the same connection.

There follows an extended section on the trigonometric functions (and also of the hyperbolic functions) mainly from the point of view of the theory of functions of a complex variable. The applications of the trigonometric functions also receive considerable attention, especially as regards the theory of small oscillations and (more extensively) the representation of an arbitrary function by Fourier series. This third and last main part on analysis closes with a treatment of the infinitesimal calculus as such. The sequence of topics is again historical, the comment critical. Some twenty pages (a little less than half of this section) is devoted to Taylor's theorem. Throughout there is much emphasis on geometric representation; the connection of the calculus with the theory of finite differences and the problems of interpolation is clearly exhibited, and many interesting comments on pedagogical questions are inserted.

The first volume closes with an appendix in which the author proves the transcendental character of e and π, and then gives a beautifully clear account of the fundamental ideas in the theory of sets (Mengenlehre).

In the second volume the author sets himself the problem of giving a survey of the whole field of geometry. As has already been said all pedagogical questions are here reserved for separate discussion in an appendix. The method is almost exclusively analytic, the applications are very largely in the field of mechanics. As in the first volume, there are also here three main parts: Part I: The simplest geometric forms; Part II: Geometric transformations; Part III: Systematic development and the foundations of geometry.

Part I begins with a discussion of the well known expressions of length, area, and volume as determinants, in which the significance of the algebraic sign of these expressions receives particular attention. This is followed by a detailed discussion of Grassmann's systematic use of determinants for the definition

of geometric quantities. We have here a very readable exposition of the comparatively little known ideas that Grassmann developed in his Ausdehnungslehre. The notions here developed are at once coordinated with fundamental concepts in mechanics. Throughout also the history of the subject is emphasized. Vector analysis, Plücker's line coordinates, n dimensional geometry are among the topics introduced. As a detail we may mention that Poncelet is given the whole credit as the discoverer of the principle of duality, while Gergonne, who would seem to deserve as much credit in this discovery, is not even mentioned.

Part II discusses in order and with considerable detail the affine and the projective transformations, with many interesting applications to geometry and in particular to the problems of perspective drawing. Here the author makes a strong plea that every teacher of mathematics should be familiar with the principles of descriptive geometry, a plea with which the reviewer is in full sympathy. This is followed by a briefer discussion of higher transformations, the transformations by reciprocal radii, and the several methods in use in the construction of geographical maps receiving special attention. The most general continuous point transformations are then briefly considered in connection with problems of analysis situs. After a section of less than twenty pages devoted to the dualistic and contact transformations, the author devotes some thirty pages to the introduction and use of imaginary elements into geometry, von Staudt's classical work in this respect receiving detailed consideration.

Part III, finally, itself falls into two parts. The first of these is devoted to the systematic organization of the field of geometry. Klein's well-known use of the group concept as a means of geometric classification is of course fundamental. On the analytic side the geometry associated with a given group then becomes the theory of invariants for this group. We are, therefore, given an extended survey of this theory and its applications to geometry. This section closes with the special developments regarding the characterization of the affine and metric geometries within the general projective geometry. The second part is devoted to the logical foundations of geometry. In many respects this is the most interesting part of the volume; in one fundamental particular, however, it is the least satisfying. The discussion of the two methods of building up

the ordinary metric geometry on the one hand by making the group of rigid displacements fundamental, on the other by choosing as undefined the notions of distance, angle, and congruence is beautifully clear. The discussion of the foundations as they appear in Euclid's Elements that follows, with its critical comment and the clear description in historical sequence of the developments to which the problems involved gave rise, forms a remarkably stimulating bit of reading.

What appears to us as unsatisfactory in Professor Klein's treatment is due to his point of view. This applies as well to his treatment of the foundations of analysis. This point of view is summed up in the statement that Professor Klein can see no virtue in the purely formal and abstract treatment of these questions. He expresses himself very clearly on this point on several occasions. To regard the undefined elements as mere symbols devoid of meaning (except such as is implied in the explicit assumptions concerning them) and to regard the unproved propositions as mere arbitrary assumptions appears to Professor Klein as "the death knell of all science" (volume 2, page 384). The reviewer is obliged to take emphatically a diametrically opposite view. We should say this formal point of view has given to science a new and powerfully vital principle. It is without doubt, so it seems to us, the most powerful unifying principle in our science today. We will try briefly to justify these assertions. The formal logical consequences of a set of assumptions concerning certain undefined symbols constitute what we will here call an *abstract theory*. If any concrete meaning is assigned these symbols for which all the assumptions appear to be satisfied, we obtain what we will call a *concrete representation* of the abstract theory. From the point of view of the foundations such a concrete representation is regarded as a consistency proof of the assumptions. We are in full accord with what Professor Klein has to say as to the logical limitations of such a "proof" and the important metaphysical problems to which the formal point of view gives rise (volume 1, pages 33–36); we are glad that he has insisted on these matters so emphatically. However, our present interest is in *the application of the formal treatment outside of the field of the logical foundations*. A given abstract theory may have many different concrete representations. These different concrete representations are then unified by the underlying abstract theory. From this point of view geometry and analysis are coex-

tensive, a fact which in a course of lectures of the kind here considered we might have expected to see strongly emphasized.

Professor Klein would probably take issue with us right here. To him geometry is essentially intuitional (anschaulich); if it is not intuitional it is not geometry, he would probably say. That he holds this point of view is abundantly shown. He insists for example that geometric methods are less precise and rigorous than analytic methods, a statement which is certainly true only under the crude intuitional view of geometry. But this is only a quibble of words. Professor Klein is of course justified in using the word geometry to apply only to that which is "anschaulich" (he does speak of n dimensional geometry, however), if he so desires. Whatever we call it, however, there is such a thing as an abstract theory underlying geometry, which to avoid circumlocution we call geometry. The methods of this abstract geometry are just as precise and rigorous as those of (abstract) analysis, no more and no less. The unifying power of this point of view, to mention only one example, is seen in the fact that projective geometry and certain tactical configurations (triple systems, etc.), with only a finite number of elements appear as concrete representations of the same abstract theory. The formal point of view is of the greatest advantage also in the consideration of imaginary elements in geometry. After describing von Staudt's method of introducing complex elements into geometry, whereby certain involutions are taken to represent complex points and calling attention to the fact that the whole theory of complex projective geometry could be built up on this basis, Professor Klein remarks (volume 2, page 270): "In most cases, however, the application of this geometric interpretation (of complex elements) would, in spite of its theoretical advantages, be attended with so many complications that one may well be satisfied with its theoretical possibility and that one will return to the naiver point of view to the effect that a complex point is a set of imaginary coordinates with which one may in a certain way operate as with the real points." From the abstract point of view a complex point is just the same sort of a point in complex geometry as a real point; in fact this distinction is entirely meaningless, until a chain has been selected as the real chain. It is unnecessary to multiply examples. The advantages of a formal treatment are briefly expressed as follows: If an abstract theory is developed concerning a set of mere symbols, then this

theory applies to *every* concrete representation of these symbols for which the fundamental assumptions are satisfied. These concrete representations of a given abstract theory may be many and varied in character; the abstract theory serves to unify them all.

It would seem that a clear understanding of this point of view is of particular importance to the prospective teacher. It is now well recognized that a knowledge of the foundations of mathematics is essential in the best preparation of the teacher, and the abstract point of view, if not absolutely necessary, greatly facilitates a clear understanding of the problems here presented. That the non-euclidean geometries serve, though not as conveniently, to describe the properties of our intuitional space, is merely due to the fact that the points, lines, etc., of the latter may be regarded as satisfying all the assumptions lying at the basis of each of these geometries.

The volume closes with a very interesting account of the movements toward reform in the teaching of elementary geometry as they have developed during the past years in England, France, Italy, and Germany. The work as a whole is a remarkable example of the distinguished author's mastery of the art of clear and stimulating exposition. We sincerely hope that it will have a wide influence, also in America, in arousing an active interest in a more serviceable preparation of our teachers.

J. W. Young.

CORRECTION.

The following misprints occur in page 122 of Dr. Onnen's paper in the December number of the Bulletin:

Lines 4–5. *For* . . . dividing n times by any integer a . . . *read* . . . dividing n times by a any integer. . . .

Line 6 from the bottom. *For* . . . an integral value for n . . . *read* . . . an integer. . . .

NOTES.

The opening (January) number of volume 11 of the *Transactions of the American Mathematical Society* contains the following papers: "Theorems on simple groups," by H. F. Blichfeldt; "Infinite discontinuous groups of birational

transformations which leave certain surfaces invariant," by V. SNYDER; "Proper multiple integrals over iterable fields," by E. B. LYTLE; "On a class of hyperfuchsian functions," by C. F. CRAIG; "Periodic orbits about an oblate spheroid," by W. D. MACMILLAN.

AT the sixty-first meeting of the American association for the advancement of science, held at Boston, December 27, 1909, to January 1, 1910, Professor A. A. MICHELSON was elected president, Professor E. H. MOORE was elected vice president and chairman of Section A, and Professor G. A. MILLER was re-elected secretary of the section. The next meeting of the association will be held at Minneapolis.

AT the meeting of the London mathematical society held on December 9 the following papers were read: By T. H. BLAKESLEY, "Exhibition of an instrument for solving cubic equations;" by A. B. BASSET, "The connection between the theories of the singularities of surfaces and double refraction;" by W. BURNSIDE, "On the representation of a group of finite order as a group of linear substitutions with rational coefficients;" by A. L. DIXON, "The eliminant of the equations of four quadric surfaces."

THE course of lectures on "Graphical methods in mathematics and physics," delivered, 1909–1910, by the Kaiser Wilhelm professor in Columbia University, Professor CARL RUNGE of the University of Göttingen, is to be published in book form by Columbia University. The subject is one which has not received due attention in this country or abroad. A considerable amount of the material contained in the lectures is original with Professor Runge. The methods developed have many important applications in astronomy, physics, engineering, and various departments of technology.

THE firm of Martin Schilling in Halle announces a number of new models: Series XXX, numbers 6 and 7, plaster models of two surfaces of order 12, class 10, applicable to the surface of the paraboloid of revolution, constructed under the direction of Professor Darboux by Dr. E. ESTANAVE; Series XXXIII, numbers 2 and 3, two thread models of the discriminant surface of a quartic equation, constructed at the suggestion of Pro-

fessor Klein and under the direction of Professor F. Schilling by R. HARTENSTEIN; Series XXXV, numbers 1–27, twenty-seven card models of spherical quadrilaterals, constructed with the cooperation of Professor Schilling by Dr. W. IHLENBURG; Series XXVI, numbers 19–23, new models in descriptive, projective, and analytic geometry, constructed by Professor F. SCHILLING. The last series consists of spherical blackboards of various sizes, together with mountings and movable rings, and two one-sheeted hyperboloids with their asymptotic cones.

THE academy of sciences of Paris has contributed 500 francs towards the expense of erecting a monument to Laplace at Beaumont en Auge, where he was born in 1746.

AT the meeting of the academy of sciences of Paris on December 6, the Leconte prize (2,000 francs) was awarded to M. RITZ for his contributions to mathematical physics and to mechanics.

THE prize of 800 francs announced by the Belgian academy for an important contribution to the differential geometry of ruled space (see BULLETIN, volume 15, page 44) has been awarded to Professor E. J. WILCZYNSKI for his memoir on "The general theory of congruences."

THE December number (volume 18, numbers 11–12) of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* contains a reproduction of the Abel monument which was unveiled at Christiania, October 17, 1908. The total contributions to the cost of the monument amounted to about $10,000, all but $1,000 of which were subscribed in Norway.

THE December number (volume 2, number 2) of *The Mathematics Teacher* contains a report of the organization of the American work of the international commission on the teaching of mathematics, including the names of the 237 members of sub-committees. In Germany this work is assigned to individuals instead of to committees, each person preparing an exhaustive monograph. Besides a number of minor reports of organization and progress, one complete report has already appeared, that by Dr. W. LIETZMANN on "Material and method in mathematical instruction in northern Germany, based upon existing text-books."

DURING the year 1906–1907 the following persons received the degree of doctor of philosophy from the German universities, with mathematics as the major subject. The title or the dissertation is added.

Berlin.

KERL, O. "Voranschläge der Genauigkeit beim trigonometrischen Punkteinschalten."

KNÖPP, K. "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze."

Breslau.

STEROSTZIK, H. "Ueber eine von Steiner gefundene, noch wenig beachtete Eigenschaft der Leitstrahlen der Kegelschnitte und über Kurven, die mit den einen Kegelschnitt doppelt berührenden Kreisen zusammenhängen."

VOGT, W. "Korrelative Räume bei gegebener Punktkernfläche."

WEISS, E. "Anzahlbestimmungen für das Strahlennetz (lineare Kongruenz)."

WIESING, O. "Ueber eine zwei-zweideutige Verwandtschaft zwischen zwei Ebenen und ihr Analogon im Raume."

Erlangen.

EGERER, H. "Ueber die Curve der Ecken der Vierseite, die von den gemeinsamen Tangenten eines festen Kegelschnitts und der Kegelschnitte eines Büschels gebildet werden."

HAUSER, W. "Ueber Resultanten- und Diskriminantenbildung in der Theorie der elliptischen Thetafunktionen."

Giessen.

HENSEL, G. "Ueber permutable Gruppenbasen aus zwei Elementen."

LANGE, M. "Die Verteilung der Elektrizität auf zwei leitenden Kugeln in einem zu ihrer Zentrallinie symmetrischen elektrostatischen Felde."

THAER, C. "Ueber Invarianten, die symmetrischen Eigenschaften eines Punktsystems entsprechen."

Göttingen.

BORN, M. "Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen."

BROGGI, U. "Die Axiome der Wahrscheinlichkeitsrechnung."

CRATHORNE, A. R. "Das räumliche isoperimetrische Problem."

GILLESPIE, D. C. "Anwendungen des Unabhängigkeitsgesetzes auf die Lösung der Differentialgleichungen der Variationsrechnung."

HASEMAN, C. "Anwendung der Theorie der Integralgleichungen auf einige Randwertaufgaben in der Funktionentheorie."

LEBEDEFF (Miss), W. "Die Theorie der Integralgleichungen in Anwendung auf einige Reihenentwickelungen."

WILLERS, F. A. "Die Torsion eines Rotationskörpers um seine Achse."

Greifswald.

APFELSTEDT, M. "Ueber eine Gattung von projektiven Transformationsgruppen in fünf Veränderlichen."

BROSZAT, W. "Ueber Scharen von ∞^4 Flächen im R_3, die durch Berührungstranformation in Scharen von ∞^4 Kurven, überführbar sind."

REICHEL, W. "Ueber trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde."

ROELCKE, O. "Ueber die Bäcklundsche Transformation der Flächen konstanter Krümmung."

Heidelberg.

SPEYERER, K. "Ueber Wärmeströmung in dünnen, frei anstrahlenden Platten."

Jena.

DURHOLD, P. "Ueber ein Kreisbündel sechster Ordnung."

Kiel.

HANSEN, O. "Ueber die äquiforme Geometrie im Bündel."

Königsberg.

FOETHKE, E. Anwendung des erweiterten euklidischen Algorithmus auf Resultantenbildungen."

Marburg.

HÜTTIG, F. "Arithmetische Theorie eines Galoischen Körpers."

OETTINGER, E. "Ueber stationäre Gasbewegungen mit Berücksichtigung der inneren Wärmeleitung."

Munich.

DINGLER, H. "Beiträge zur Kenntnis der infinitesimalen Deformation einer Fläche."

FUCHS, F. "Beiträge zur Theorie der elektrischen Schwingungen einer leitenden Rotationsellipsoides."

MÜNICH, K. "Ueber nicht-euklidische Cykliden."

SCHÜBEL, H. "Aufstellung von nicht-euklidischon Minimalflächen."

Münster.

GESSNER, E. "Ueber die Asymptotenkurven einer Schar Konoidflächen im allgemeinen und die des Cylindroids im besonderen."

LANGENKAMP, O. "Ueber Saccheri's Untersuchungen des Parallelenaxioms."

Rostock.

DOLGE, P. "Ueber Bernouillische Zahlen und Funktionen, welche zu einer Fundamentaldeterminante gehören und deren Anwendung auf die Summation unendlicher Reihen."

HELLWIG, M. "Untersuchung über die Lage der Inzidenzpunkte bei Reflexion und Refraktion an Ebene, Kugel und Kreiszylinder für zwei feste Punkte im Raume (Licht- und Augenpunkt)."

Strassburg.

ALTERAUGE, L. "Ueber lineare Relationen zwischen hypergeometrischen Integralen."

ENDERS, M. A. "Ueber die Darstellung der Raumkurve vierter Ordnung vom Geschlecht 1 durch Thetafunktionen."

HUPPERZ, R. W. "Analytische Untersuchung der allgemeinen Schraubenregelfläche. Eine monographische Studie."

MAGENER, F. W. A. "Anallagmatische Punktkoördinaten im Kegelgebüsch und ihre Anwendung auf die nicht-euklidische Geometrie."

RUTHINGER, M. "Die Irreducibilitätsbeweise der Kreisteilungsgleichung."

SAUER, K. "Zur Funktionentheorie auf dem algebraischen Gebilde $s = \sqrt[3]{f_{3n}(z)}$."

SCHUMACHER, H. A. "Ueber eine Riemann'sche Funktionenklasse mit zerfallender Thetafunktion."

ZAHN, K. A. K. "Constructive Bestimmung der Hauptaxen und der Umrisse einer Fläche zweiten Grades, die durch einen Kreis und vier Punkte des Raumes bestimmt ist."

Tübingen.

HOFFMANN, G. "Das Abel'sche Theorem für die elliptischen Integrale."

LÜFFLER, E. "Beiträge zur Theorie der Schnittpunkte algebraischer Kurven."

For the year 1907–1908 the list is as follows.

Berlin.

SCHEE, W. "Ueber irreguläre Potenzreihen und Dirichletsche Reihen."

Bonn.

KUMMER, A. "Ueber eine Gattung von projektiven Transformationsgruppen in sechs Veränderlichen."

Breslau.

SCHMIDT, R. "Ueber zweite Polarflächen einer allgemeinen Fläche 4. Ordnung."

Erlangen.

NOETHER, (Miss) E. "Ueber die Bildung des Formensystems der ternären biquadratischen Form."

Giessen.

KEINEMANN, E. "Die gemischte Kegelschnittschar (das Kegelschnittsystem $s(1p, 3l)$)."

KÜBEL, K. G. J. "Anwendungen einer anschaulichen Darstelluug des Imaginären."

JENZ, O. "Ueber singuläre Asymptotenkurven."

Göttingen.

CAIRNS, W. D. W. "Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen."

HELLINGER, E. "Die Orthogonalinvarianten quadratischer Formen von unendlich vielen Variabeln."

KÖNIG, R. "Die Oscillationseigenschaften der Eigenfunktionen der Integralgleichung mit definitivem Kern und das Jacobische Kriterium der Variationsrechnung."

LAUMANN, T. "Ueber den Isomorphismus von Gruppen linearer Substitutionen mit reellen und mit komplexen Koeffizienten."

SWIFT, E. "Ueber die Form und Stabilität gewisser Flüssigkeitstropfen."

WEYL, H. "Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems."

Halle.

BRANDES, H. "Ueber die axiomatische Einfachheit mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsbeweise des Pythagoräischen Lehrsatzes."

ERFURTH, P. "Die Komplementärflächen der pseudosphärischen Schraubenflächen."

MAHLO, P. "Topologische Untersuchungen über Zerlegung in ebene und sphärische Polygone."

MORGENSTERN, A. "Beiträge zur numerischen Lösung der Gleichung fünften Grades."

ZÖLLICH, H. "Beiträge zur Theorie der ganzen transzendenten Funktionen der Ordnung Null."

Jena.

HEILAND, F. "Hüllflächen einer Schar von Regelflächen zweiter Ordnung."

Kiel.

DIECK, W. "Zur Klassifikation der Punktepaare und Kegelschnittbüschel."

GURSKI, V. "Ueber den Zusammenhang zwischen den partikulären Lösungen der einzelnen Gebiete bei der hypergeometrischen Differentialgleichung dritter Ordnung mit zwei endlichen singulären Punkten."

HASS, P. "Zur Definition des Begriffs der eindeutigen analytischen Funktion."

Königsberg.

ARNDT, B. "Ueber die Verallgemeinerung des Krümmungsbegriffs für Raumkurven."

DORNER, O. "Ueber Teiler von Formen."

JANZEN, O. "Ueber einige stetige Kurven, über Bogenlänge, linearen Inhalt und Flächeninhalt."

KALUZA, T. "Die Tschirnhaustransformation algebraischer Gleichungen mit einer Unbekannten."

KISCHKE, R. "Ueber Fehlerabschätzung bei unendlichen Produkten und deren Anwendungen."

Leipzig.

SCHILLER, G. "Die Bewegung einer homogenen Kugel auf einer materiellen Parabel unter dem Einfluss der Schwerkraft."

Munich.

CRAMER, F. H. "Ueber die Erniedrigung des Geschlechts abelscher Integrale, insbesondere elliptischer und hyperelliptischer, durch Transformation."

HORN, C. "Konforme Abbildung eines von gewissen Kurven begrenzten Flächenstücks auf den Einheitskreis."

MOHRMANN, H. "Beiträge zur Theorie der Singularitäten des algebraischen Linien-Complexes beliebigen Grades."

WALEK, K. "Binäre kubische Transformation und Complexe."

Münster.

KREFT, W. "Beiträge zur Goursatschen Transformation der Minimalflächen."

Rostock.

PEECK, H. "Ein Beitrag zur Theorie der gebrochenen Fokaldistanzen."

WEISSE, E. "Anwendungen der elliptischen Funktionen auf ein Problem aus der Theorie der Gelenkmechanismen."

Strassburg.

BRAND, E. L. "Ueber Tetraëder, deren Kanten eine Fläche zweiter Ordnung berühren."

BRESSLAU, H. S. S. "Dirichlet's Satz von der arithmetischen Reihe für den Körper der dritten Einheitswurzeln."

KEMPF, A. "Tetraëder, deren Kanten eine Fläche F_2 zweiter Ordnung berühren."

Tübingen.

KÖSTLIN, E. "Ueber eine Deutung der Gleichung, die wischen dem Bogen und dem Neigungswinkel der Tangente zm Endpunkt des Bogens einer ebenen Kurve besteht."

REIFF, —. "Rollen einer Kugel in einem Zylinder ohne Einwirkung der Schwerkraft."

SCHWARTZ, R. "Der Eisensteinsche Satz über die Koefficienten der Reihenentwicklungen algebraischer Funktionen."

THE mathematical society of Berlin announced a special session to be held January 8, 1910, to commemorate the centenary of the birth of E. E. KUMMER. The principal address was given by Professor K. HENSEL, of the University of Marburg.

THE Göttingen academy of sciences announces that it has made an award of 100 Marks from the interest of the Wolfskehl foundation to Dr. A. WIEFERICH, of Münster, who has succeeded in proving that the equation $x^p + y^p = z^p$ cannot be solved in terms of positive integers, not multiples of p, if $2^p - 2$ is not divisible by p^2. (*Crelle's Journal*, volume 137.) "This surprisingly simple result represents the first advance since the time of Kummer in the proof of the last Fermat theorem."

PROFESSOR M. J. M. HILL, of the University of London, has been elected honorary fellow of St. Peter's College, Cambridge.

DRS. A. N. WHITEHEAD and H. F. BAKER have been appointed chairmen of examiners for parts I and II, respectively, of the mathematical tripos for 1910.

AT the annual meeting of the Royal society of London on November 30, the Copley medal was conferred upon Dr. G. W. HILL, of West Nyack, New York, and a royal medal upon Professor A. E. H. LOVE, of Oxford University.

ON the occasion of the dedication of the new buildings at the University of Stockholm on December 16 honorary doctorates were conferred upon Professors H. POINCARÉ and P. PAINLEVÉ, of the University of Paris.

AT the seventy-fifth anniversary of the University of Brussels on November 27, the degree of doctor of laws was conferred upon Professor H. POINCARÉ, of the University of Paris.

DR. H. C. MCWHEENEY has been appointed professor of mathematics at University College, Dublin.

PROFESSOR R. SALIGER, of the German technical school at Prague, has been appointed professor of mechanics at the technical school at Vienna.

PROFESSOR L. SCHLESINGER, of the University of Klausenburg, has been elected to membership in the academy of sciences of Halle.

PROFESSOR G. KOWALEWSKI, of the University of Bonn, has been appointed professor of mathematics at the German technical school at Prague.

DR. A. TIMPE has been appointed docent in mechanics at the technical school at Aachen.

DR. TH. KALUZA has been appointed docent in mathematics at the University of Königsberg.

PROFESSOR G. BOCCARDI, of the University of Turin, has been promoted to a full professorship of mathematical astronomy.

PROFESSOR C. SEVERINI, of the University of Catania, has been promoted to a full professorship of analytic geometry.

PROFESSOR T. BOGGIO, formerly of the University of Messina, then at the Institute at Florence, has been appointed associate professor of rational mechanics at the University of Turin.

PROFESSOR E. SOLER, formerly of the University of Messina, has been appointed to a full professorship of theoretical geodesy at the University of Padua.

PROFESSOR M. ABRAHAM, of the University of Göttingen, has been appointed professor of rational mechanics at the technical school of Milan.

DR. U. SCARPIS has been appointed docent in algebraic analysis at the University of Bologna.

DR. E. LAURA has been appointed docent in rational mechanics at the University of Turin.

DR. F. SIBIRANI has been appointed docent in the calculus at the University of Bologna.

THE degree of LL.D. has been conferred by Columbia University on the Kaiser Wilhelm Professor, CARL RUNGE.

PROFESSOR C. J. KEYSER has been appointed head of the department of mathematics at Columbia University to succeed Professor J. H. VAN AMRINGE on the latter's retirement at the close of the present academic year.

PROFESSOR A. S. CHESSIN will deliver at the University of Pennsylvania in February a course of lectures on "The gyrostat and its modern applications."

PROFESSOR E. MILLER, of the University of Kansas, will retire from active service at the close of the present academic year under the conditions of the Carnegie foundation.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

AUTONNE (L.). Sur les groupes de matrices linéaires non invertibles. Paris, 1909. 8vo. 80 pp. Fr. 5.00

BECK (H.). See BÔCHER (M.).

BLUMENTHAL (O.). Principes de la théorie des fonctions entières d'ordre infini. Paris, Gauthier-Villars, 1909. 8vo. 7 + 150 pp. Fr. 5.50

BÔCHER (M.). Einführung in die höhere Algebra. Deutsch von H. Beck. Mit einem Geleitwort von E. Study. Leipzig, Teubner, 1910. 8vo. 12 + 348 pp. Cloth. M. 7.00

BOLZA (O.). Vorlesungen über Variationsrechnung. Umgearbeitete und stark vermehrte deutsche Ausgabe der "Lectures on the calculus of variations" desselben Verfassers. 3te Lieferung. Leipzig, Teubner, 1909. 8vo. 9 pp. + pp. 541–705. M. 5.00
In 1 volume, cloth. M. 20.00

BOREL (E.). Leçons sur la théorie de la croissance, professées à la Faculté des Sciences de Paris, recueillies et rédigées par A. Denjoy. Paris, Gauthier-Villars, 1909. 8vo. 6 + 172 pp. Fr. 5.50

BRIOSCHI (F.). Opere matematiche, pubblicate per cura del comitato per le onoranze a F. Brioschi. Vol. V (ultimo). Milano, Hoepli, 1909. 4to. 12 + 556 pp. L. 30.00

BÜCHER, neue, über Naturwissenchaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet.) Mitgeteilt Herbst 1909. Leipzig, Hinrich. 8vo. Pp. 49–67. M. 0.30

DENJOY (A.). See BOREL (E.).

DETTE (W.). Analytische Geometrie der Kegelschnitte. Leipzig, Teubner, 1909. 8vo. 6 + 232 pp. Cloth. M. 4.40

EMDE (F.). See JAHNKE (E.).

GALILEO (G.) Le opere. Edizione nazionale sotto gli auspicî di Sua Maestà il Re d'Italia. Vol. XX (ultimo). Firenze, Barbèra. 1909. 4to. 589 pp.

GANS (R.). Einführung in die Vektoranalysis mit Anwendungen auf die mathematische Physik. 2te Auflage. Leipzig, Teubner, 1909. 8vo. 10 + 126 pp. Cloth. M. 3.60

GRAF (J. H.). Einleitung in die Theorie und Auflösung der gewöhnlichen Differentialgleichungen nebst vielen Uebungsbeispielen. Bern, Wyss, 1910. 8vo. 6 + 115 pp. M. 2.00

GRUHN (P.). Mathematische Formelsammlung. Hannover, Jänecke, 1909. 8vo. 62 pp. M. 1.20

GUYGOU (E.). Notes sur les approximations numériques. 3e édition. Paris, 1909. 8vo. 32 pp. Fr. 1.00

IGNATOWSKY (W. VON). Die Vektoranalysis und ihre Anwendung in der theoretischen Physik. (2 Teil.) Teil I: Die Vektoranalysis. Leipzig, 1909. 8vo. 128 pp. M. 2.60

JAHNKE (E.) und EMDE (F.). Funktionentafeln mit Formeln und Kurven. Leipzig, 1909. 8vo. 12 + 176 pp. Cloth. M. 6.00

JOPKE (A.). Synthetische Untersuchungen über lineare Kegelschnittsysteme 1ster, 2ter und 3ter Stufe. (Diss.) Breslau, 1909. 8vo. 56 pp. M. 1.80

KARST (L.). Lineare Funktionen und Gleichungen. Lichtenberg, 1909. 4to. 44 pp. M. 1.50

KIEPERT (L.). Grundriss der Differential- und Integralrechnung. 1ter Teil: Differentialrechnung. 11te unveränderte Auflage des gleichnamigen Leitfadens von M. Stegemann. Hannover, Helwing, 1910. 8vo. 20 + 818 pp. M. 12.50

KLIEM (F.). Ueber Oerter von Treffgeraden entsprechender Strahlen in eindeutig und linear verwandten Strahlengebilden 1ster bis 4ter Stufe. (Diss.) Breslau, 1909. 8vo. 52 pp. M. 1.50

KOMMERELL (V. und K.). Allgemeine Theorie der Raumkurven und Flächen. 1ter Band, 2te Auflage. (Sammlung Schubert, XXIX.) Leipzig, Göschen, 1909. 8vo. 8 + 172 pp. Cloth. M. 4.80

LANDAU (E.). Handbuch der Lehre von der Verteilung der Primzahlen. Leipzig, Teubner, 1909. 8vo. Vol. I, 18 + 564 pp. M. 20.00
Vol. II, 9 pp. + pp. 565–961. M. 14.00

LINDEMANN (F.). Ueber den sogenannten letzten Fermatschen Satz. Leipzig, Veit, 1909. 8vo. 5 + 82 pp. M. 3.50

PICKFORD (A. G.). Elementary projective geometry. Cambridge, 1909. 8vo. Cloth. 4s.

RIQUIER (C.). Les systèmes d'équations aux dérivées partielles. Paris, Gauthier-Villars, 1910. 8vo. 23 + 590 pp. Fr. 20.00

SCHIEDE (J.). Die Begriffe der Funktion und des Differentialquotienten in der Gymnasialprima. Köslin, 1909. 4to. 31 pp. M. 1.50

SERRET (J. A.). Cours d'algèbre supérieure. 6e édition. Vols. I–II. Paris, Gauthier-Villars, 1910. 8vo. Vol. I, 13 + 648 pp. Vol. II, 12 + 694 pp. Les deux volumes. Fr. 25.00

STARKE (P.). Planimetrische Konstruktionsaufgaben mit Verwendung der harmonischen Teilung und des Apollonischen Kreises. Leipzig, 1909. 4to. 71 pp.

STEGEMANN (M.). See KIEPERT (L.).

STUDY (E.). See BÔCHER (M.).

STURM (R.). Die Lehre von den geometrischen Verwandtschaften. 4ter Band: Die nichtlinearen und die mehrdeutigen Verwandtschaften zweiter und dritter Stufe. (B. G. Teubner's Sammlung, XXVII, 4.) Leipzig, Teubner, 1909. 8vo. 10 + 486 pp. Cloth. M. 20.00

THIELE (T. N.). Interpolationsrechnung. Leipzig, Teubner, 1909. 8vo. 12 + 175 pp. M. 10.00

WELLISCH (S.). Theorie und Praxis der Ausgleichungsrechnung. 1ster Band: Elemente der Ausgleichungsrechnung. Mit einem Bildnisse von K. F. Gauss. Wien, Fromme, 1909. 8vo. 11 + 276 pp. M. 10.00

WILLOT (J.). Exposé des fondements de la géométrie. Paris, Gauthier-Villars, 1908. 8vo. 53 pp. Fr. 1.75

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). A new algebra. Parts 1-4 with answers. London, Macmillan, 1909. 8vo. 544 pp. Cloth. 4s.

CHILD (J. M.). See BARNARD (S.).

EDGETT (G. C.). Exercises in geometry. Boston, Heath, 1909. 12mo. 6 + 81 pp. Cloth. $0.40

FABRIS (V.) Nozioni elementari di algebra, ad uso della terza classe delle scuole tecniche e della prima classe delle scuole normali. 7a edizione riveduta e corretta dall'autore. Torino, Botta, 1909. 8vo. 39 pp. L. 0.80

FENKNER (H.). Lehrbuch der Geometrie für den Unterricht an höheren Lehranstalten. Ausgabe B. Für Realschulen. (In 2 Teilen.) 2ter Teil: Raumgeometrie und Trigonometrie. Nebst einer Aufgabensammlung. Berlin, Salle, 1910. 8vo. 4 + 88 pp. M. 1.40

FUCINI (C.). Algebra elementare, per le scuole secondarie inferiori. 3a edizione. Genova, Gioventù, 1910. 8vo. 79 pp.

GAUSS (F. G.). Die Teilung der Grundstücke insbesondere unter Zugrundelegung rechtwinkliger Koordinaten. Nebst vierstelligen logarithmischen und trigonometrischen Tafeln, einer Quadrattafel, sowie einer Multiplikations- und Divisionstafel. 5te Auflage. 2 Teile. Berlin, Decker, 1909. 8vo. 195 + 80 pp. Cloth. M. 7.60

HOVESTADT (H.). See KILLING (W.).

HOWARD (H. E.). Easy practical mathematics. London, Longmans, 1909. 8vo. 1s.

JENSON (O.). See WALTHER (F.).

KAMBLY und ROEDER. Planimetrie, neu bearbeitet von A. Thaer. Ausgabe B. Für Realanstalten und Gymnasien mit mathematischem Reformunterricht. 148–151ste Auflage der Kamblyschen Planimetrie. Breslau, Hirt, 1909. 8vo. 240 pp. M. 2.50

KILLING (W.) und HOVESTADT (H.). Handbuch des mathematischen Unterrichts. 1ter Band. Leipzig, Teubner, 1910. 8vo. 8 + 456 pp. Cloth. M. 10.00

KUTNEWSKY (M.). See MÜLLER (H.).

LEMBCKE (K.). Allgemeine Arithmetik und Algebra in ihrer Beziehung zu einander und zu den höheren bürgerlichen Rechnungsarten. 2te, verbesserte Auflage. Wismar, Hinstorff, 1910. 8vo. 12 + 189 pp. M. 3.00

MAHLERT (A.). See MÜLLER (H.).

MINET (A.) et PATIN (L.). Cours pratique d'arithmétique, de système métrique et de géométrie. Cours élémentaire, première et deuxième années. 14e édition, revue et corrigée. Paris, Nathan, 1909. 16mo. 192 pp.

MONNET (G.). See NAUD (L.).

MÜLLER (H.). und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. 2ter Teil. Ausgabe A, für Gymnasien. 3te Auflage. Leipzig, Teubner, 1909. 8vo. 10 + 287 pp. M. 2.80

MÜLLER (H.) und MAHLERT (A.). Lehr- und Uebungsbuch der Arithmetik und Algebra für Studienanstalten. Ausgabe A: Für gymnasiale Kurse. 2ter Teil. Für die oberen drei Klassen. Leipzig, Teubner, 1909. 8vo. 6 + 139 pp. Cloth. M. 2.00

NAMPON (G.). Cent problèmes de géométrie et d'algèbre des examens du brevet supérieur avec solutions. Paris, Hachette, 1909. 16mo. 195 pp. Fr. 1.25

NAUD (L.) et MONNET (G.). Douze cents problèmes (arithmétique, système métrique, géométrie et algèbre) à l'usage des candidats aux examens de l'administration des postes et des télégraphes. Paris, 1909. 8vo. 96 pp. Fr. 1.00

NICHOLSON (J. W.). School algebra, with answers. New York, American Book Co., 1909. 12mo. 316 pp. Cloth. $1.00

NOODT (G.). Leitfaden der ebenen Geometrie, nach modernen Grundsätzen auf Grund der Bestimmungen über die Neuordnung des höheren Mädchenschulwesens vom 12. XII. 1908 bearbeitet. 2ter Teil. (Klasse 2 und 1.) Bielefeld, Velhagen, 1909. 8vo. 6 + 136 pp. M. 1.60

PATIN (L.). See MINET (A.).

ROEDER. See KAMBLY.

SCHÜLKE (A.). Vierstellige Logarithmentafeln. 7te Auflage. Leipzig, 1909. 8vo. 28 pp. M. 0.60

THAER (A.). See KAMBLY.

VINTÉJOUX (F.). Eléments d'arithmétique, de géométrie et d'algèbre. 8e édition, revue. Paris, Hachette, 1909. 16mo. 7 + 576 pp. Fr. 2.50

WALTHER (F.) und JENSON (O.). Mathematischer Lehr- und Uebungsgang für höhere Mädchenschulen, Lyzeen und Studienanstalten. Abteilung B: Für die Oberstufe der höheren Mädchenschule. 2 Teile. Leipzig, Brandstetter, 1909. 8vo.
1. Geometrie. 6 + 179 pp. M. 3.00
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III. APPLIED MATHEMATICS.

BARKOW (R.). Grundzüge der mechanischen Wärmetheorie. Für den Selbstunterricht bearbeitet. Berlin, Hachfeld, 1909. 8vo. 3 + 39 pp. M. 1.50

BRILL (A.). Vorlesungen zur Einführung in die Mechanik raumerfüllender Massen. Leipzig, Teubner, 1909. 8vo. 10 + 236 pp. Cloth. M. 8.00

CRANDALL (C. L.). The transition curve, by offsets and by deflection angles. (With this is bound Crandall's field book for railroad surveying.) 2d edition, revised and enlarged. New York, Wiley, 1909. 12mo. Cloth. $2.00

FILLOUX (L.). Mémoire sur l'intégration mécanique de l'hodographe. Paris, Berger-Levrault, 1909. 8vo. 27 pp.

FÖPPL (A.). Vorlesungen über technische Mechanik. 3ter Band: Festigkeitslehre. 4te Auflage. Leipzig, Teubner, 1909. 8vo. 16 + 426 pp. Cloth. M. 10.00

FRANKLIN (W. S.). Electric waves. London, Macmillan, 1909. 8vo. Cloth. 10s.

HALE (W. J.). The calculations of general chemistry; with definitions, explanations and problems. New York, Van Nostrand, 1909. 12mo. 7 + 9 + 174 pp. Cloth. $1.00

HEDRICK (E. R.) and KELLOGG (O. D.). Applications of the calculus to mechanics. Boston, Ginn, 1909. 8vo. 4 + 116 pp. Cloth. $1.00.

JAMESON (J. M.). Elementary practical mechanics. London, Longmans, 1909. 8vo. Cloth. 6s.

LANCHESTER (F. W.). Aerodynamik. Ein Gesamtwerk über das Fliegen. Aus dem Englischen von C. und A. Runge. 1ter Band. Leipzig, Teubner, 1909. 8vo. 14 + 360 pp. Cloth. M. 12.00

LUDWIG (P.). Elemente der technologischen Mechanik. Berlin, Springer, 1909. 8vo. 57 pp. M. 3.00

MALCOLM (C. W.). A textbook on graphic statics. London, Spon, 1909. 8vo. 4s. 6d.

RUIZ CASTIZO (J.). Los principios fundamentales de la mecánica racional. Un primer capítulo de la dinámica. (Publicado en la Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales.) Madrid, 1909. 51 pp. P. 2.50

RUNGE (C. und A.). See LANCHESTER (F. W.).

RUSSELL (A.). Théorie des courants alternatifs. Traduit de l'anglais par G. Seligmann-Lui. Vol. II. Paris, Gauthier-Villars, 1910. 8vo. 556 pp. Fr. 18.00

SALVERT (VICOMTE DE). Mémoire sur l'attraction du parallélepipède ellipsoïdal. Vol. I. Paris, Gauthier-Villars, 1909. 8vo. 12 + 340 pp. Fr. 7.00

SELIGMANN-LUI (G.). See RUSSELL (A.).

THE SIXTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE sixteenth annual meeting of the Society was held at Boston on Tuesday, Wednesday, and Thursday, December 28–30, 1909, in affiliation with the American association for the advancement of science. Tuesday afternoon was devoted to a joint session with Sections A and B of the Association. A joint session was held with section A on Wednesday morning, the programme consisting of Professor Keyser's vice-presidential address "On the thesis of modern logistic," a report by Professor D. E. Smith on "The work of the International Commission on the teaching of mathematics," and the papers numbered (1) and (2) in the list below. Separate sessions of the Society were held on Wednesday afternoon and on Thursday morning and afternoon. On Tuesday evening several members took advantage of an invitation to attend the dinner and smoker of the Association of mathematical teachers in New England. The annual dinner of the Society took place on Wednesday evening, forty-seven members gathering for this agreeable occasion. Much credit for the success of the meeting must be given to the local committee on arrangements, Professors Tyler, Bartlett, and Bouton.

The total attendance at the annual meeting included the following sixty-one members:

Dr. C. S. Atchison, Professor D. P. Bartlett, Dr. E. G. Bill, Professor G. D. Birkhoff, Professor C. L. Bouton, Professor E. W. Brown, Dr. J. E. Clarke, Mr. G. R. Clements, Professor F. N. Cole, Professor L. L. Conant, Professor J. L. Coolidge, Mr. C. H. Currier, Mr. F. F. Decker, Dr. F. J. Dohmen, President E. A. Engler, Professor T. C. Esty, Professor F. C. Ferry, Professor W. B. Fite, Professor W. A. Garrison, Miss A. B. Gould, Professor F. L. Griffin, Professor J. N. Hart, Professor C. N. Haskins, Professor L. A. Howland, Professor E. V. Huntington, Dr. L. C. Karpinski, Professor O. D. Kellogg, Professor C. J. Keyser, Professor Gaetano Lanza, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Dr. J. V. McKelvey, Professor H. P. Manning, Professor W. H. Metzler,

Professor G. A. Miller, Professor J. A. Miller, Mr. H. H. Mitchell, Professor W. A. Moody, Professor C. L. E. Moore, Professor G. D. Olds, Dr. F. W. Owens, Dr. H. B. Phillips, Dr. Arthur Ranum, Professor R. G. D. Richardson, Mr. W. J. Risley, Professor E. D. Roe, Professor Mary E. Sinclair, Professor D. E. Smith, Miss M. E. Trueblood, Professor H. W. Tyler, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. K. Whittemore, Professor D. T. Wilson, Professor E. B. Wilson, Dr. Ruth G. Wood, Professor T. W. D. Worthen, Mr. W. C. Wright, Professor Alexander Ziwet.

Ex-President H. S. White and Professor E. W. Brown occupied the chair alternately during the several sessions. The Council announced the election of the following persons to membership in the Society: Professor R. M. Barton, Dartmouth College; Dr. J. R. Conner, Johns Hopkins University; Miss Eva M. Smith, London, England. Nine applications for membership were received.

The reports of the Treasurer, Auditing Committee, and Librarian have recently appeared in the Annual Register. The membership of the Society has increased during the year from 601 to 618, including at present 58 life members. The number of papers presented at all meetings during the year 1909 was 149. The total attendance of members at the meetings was 311. The Treasurer's report shows a balance of $8003.78, of which $3581.70 is credited to the life-membership fund.* Sales of the Society's publications during the year amounted to $1748.90. The Library has increased to nearly 3300 volumes. A catalogue of the Library, corrected to January 1, 1910, has been issued as a separate publication.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor L. E. Dickson,
	Professor J. I. Hutchinson.
Secretary,	Professor F. N. Cole.

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*The total income of the Society since January 1, 1895, has been $56,299.22, of which $30,310.94 is credited to members' dues. Disbursements in the same period have been $49,412.11, of which $39,008.47 has been expended for printing the *Bulletin*, *Transactions*, and other publications of the Society. The total returns from publications have been $13,508.99. Editorial expenses have been $2750.15, administrative expenses $7653.14.

Treasurer, Professor J. H. Tanner.
Librarian, Professor D. E. Smith.

Committee of Publication,

Professor F. N. Cole,
Professor E. W. Brown,
Professor Virgil Snyder.

Members of the Council to serve until December, 1912,

Professor D. R. Curtiss, Professor J. C. Fields,
Professor L. P. Eisenhart, Professor P. F. Smith.

The following papers were read at this meeting:

(1) Professor F. L. Griffin: "Certain tests comparing areas and other geometrical magnitudes."

(2) Professor G. A. Miller: "Groups generated by two operators s_1, s_2 satisfying the equation $s_1s_2^2 = s_2s_1^2$."

(3) Dr. H. M. Sheffer: "Total determinations of deductive systems with special reference to the algebra of logic."

(4) Professor R. G. D. Richardson: "The Jacobi criterion in the calculus of variations and the oscillation of solutions of m linear differential equations of the second order with m parameters."

(5) Dr. J. V. McKelvey: "The groups of birational transformations of algebraic curves of genus five."

(6) Professor J. L. Coolidge: "The representation by means of circles of the imaginary elements of a three-dimensional domain."

(7) Dr. L. C. Karpinski: "Jordanus Nemorarius and John of Halifax."

(8) Mr. H. H. Mitchell: "The subgroups of the collineation group of the finite plane, $PG(2, p)$."

(9) Professor W. H. Jackson: "Differential and integral equations arising out of the theory of radiation."

(10) Professor G. D. Birkhoff: "The stable solutions of the problem of three bodies."

(11) Professor W. D. Cairns: "The solution of the Lagrange equation in the calculus of variations by means of integral equations."

(12) Dr. Arthur Ranum: "On the line geometry of r emannian space."

(13) Dr. H. F. MacNeish: "Linear polars of quantics which are completely reducible to the product of linear forms."

(14) Professor E. V. Huntington: "An elementary explanation of the precession of a gyroscope."

(15) Professor C. J. Keyser: "Relational groups."

(16) Professor Edward Kasner: "Thomson and Tait's theorem on conservative forces."

(17) Professor Edward Kasner: "Note on Lamé's families connected with dynamics."

(18) Dr. Arthur Ranum: "On Clifford parallels and Clifford surfaces in riemannian space."

Dr. Sheffer was introduced by Professor Osgood. In the absence of the authors, Dr. MacNeish's paper was read by Professor Veblen, and the papers of Professors Jackson, Cairns, Keyser, and Kasner were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this note Professor Griffin observes that the process used in an earlier paper * for comparing areas, lengths, etc., of central orbits, can be applied to the purely geometrical problem of making similar tests in many cases where two curves are given by either polar or rectangular equations or by differential equations of the first or second order.

2. In 1878 Cayley published a note on the groups which may be generated by two operators s_1, s_2 satisfying the equation $s_1s_2 = s_2^2s_1^2$ and observed that it is not possible to represent all the operators of such a group in the form $s_1^\alpha s_2^\beta$ except when the group is cyclic. In 1905 Netto considered the same relation in *Crelle* and observed that either the orders of s_1, s_2 are equal to each other or the order of one of these operators is twice the order of the other. In a recent number of the *Quarterly Journal* Professor Miller extended these results, but did not give many general theorems. In the present paper he gives several fundamental theorems which are implied in the above relation, and by means of these he obtains the known results much more easily and also arrives at a number of new results. In particular he shows that either the first or second of the three generational relations $s_1^5 = s_2^5 = 1$, $s_1s_2^2 = s_2s_1^2$ given by

* Read December 30, 1908.

Netto is redundant. That is, the two conditions $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ imply that $s_2^5 = 1$.

The following are some of the theorems established by Professor Miller in the present article: If two operators satisfy the relation $s_1 s_2^2 = s_2 s_1^2$, their squares are of the same order as their product, and they generate a group whose commutator subgroup is generated by two conjugate commutators and whose commutator quotient group is cyclic. If both of these operators are of odd order, they generate a solvable group whose commutator subgroup is either cyclic or the direct product of two cyclic groups. If α belongs to exponent 7 with respect to a prime number p and if $\alpha + 1 \equiv \alpha^2 \bmod p$, then $p = 29$ and $\alpha = 24$. Similar characteristic properties are established for the primes 5, 11, 19 as incidental results. It is also proved that $(s_1 s_2^{-1})^{4n} = s_2^{-1}(s_1^6 s_2^{-6}) s_2$ whenever s_1, s_2 are both of odd order.

3. Basal determinations ("postulate definitions") of various deductive systems — for example, of "ordinary" algebra, of the "algebra of logic," and of geometry — in terms of conveniently chosen (or basal) operations or relations have been developed recently. The present paper discusses the problem of the *total* determinations of deductive systems, that is, the total sets of element classes, relations, and propositions that are possible for a given system. The paper is thus intended as an introduction to a general theory of systems.

In particular, Dr. Sheffer solves the problem for the deductive system called the algebra of logic. He finds the total set of relations (and operations) which can serve as basal relations (and operations) for that algebra. Of this total set, the relations and operations employed hitherto — "logical addition," "logical multiplication," and "inclusion" (Huntington); the "between"-relation (Kempe); and the O-relation (Royce) — are shown to be isolated cases.

4. The relation between the Jacobi criterion of the calculus of variations and the oscillation of the solution of the self-adjoint differential equation of the second order $(pu') + qu + \lambda ku = 0$ was the subject of a paper read by Professor Richardson at the summer meeting of the Society. In a second paper of a series to appear in the *Mathematische Annalen* the results have been

extended to the case of m linear differential equations

$$(p_i u_i')' + q_i u_i + (\lambda r_{i1}(x) + \cdots + \pi r_{in}(x))u_i = 0 \quad (i = 1, 2, \cdots, m)$$

with the m parameters $\lambda, \cdots, \pi$. These differential equations are considered as the Lagrange equations of a calculus of variations problem, the functions $u_i(x)$ satisfying the boundary conditions $u_i(0) = u_i(1) = 0$ and a certain quadratic condition or the quadratic and nm linear conditions. The Jacobi criterion in each case determines exactly the number n of oscillations of the function $u_i(x)$ in the interval 0, 1.

5. The purpose of Dr. McKelvey's paper was to find the normal curves of hyperspace of genus 5 and the forms of the plane curves into which they can be projected; also to find the groups of birational transformations under which they are invariant. The plane curves of this genus are in general sextics with 5 double points. They become nodal quintics when a g_3^1 exists. The equations of the sextics were obtained by means of the quadratic relations among their adjoint curves of order 3. The curve of genus 5 is the only one which is completely defined by the quadratic relations among its adjoints of order $n - 3$. It was shown that the transformations of the sextic are related to the linear transformations of a certain plane quintic, except when the quadratic relations among the adjoints are also invariant. The largest group obtained was of order 192.

6. In 1872 Laguerre showed how the imaginary points of the finite three-dimensional domain could be represented by means of real circles of positive or negative radius. Professor Coolidge's paper exhibited two other methods of circular representation which afford a better separation between conjugate imaginaries, and a simpler representation of the simplest point systems.

7. The introduction into Europe of the Hindu methods in arithmetic is closely associated by Moritz Cantor in his Geschichte der Mathematik with the names of Leonard of Pisa and Jordanus Nemorarius. The prominent place given by Cantor to Jordanus rightfully belongs to John of Halifax (Sacrobosco). Dr. Karpinski's paper presents some new material in regard to the Algorismus by Jordanus and some notes on copies

found in American libraries of the Algorismus by Sacrobosco, and emphasizes the important rôle played by John of Halifax in spreading the system of the Hindus, taught by the Arabs.

8. In Mr. Mitchell's paper the determination is made of the subgroups of the collineation group in three variables, when the coefficients of the transformations $\rho x'_i = \Sigma a_{ij} x_j$ $(i, j = 1, 2, 3)$ are integers reduced modulo p, p being a prime. The order of the group is $(p^2 + p + 1)(p^2 + p)p^2(p - 1)^2$.

The problem is treated geometrically, the $p^2 + p + 1$ sets of marks $(\alpha\beta\gamma)$, upon which the group may be represented as a permutation group, being regarded as the points of a finite plane. A self-conjugate subgroup consisting of all transformations with determinant unity exists if p has the form $3n + 1$. There are subgroups leaving fixed a point, a line, a triangle, and a conic. The only additional subgroups are the G_{360}, the Hessian G_{216}, the G_{168}, and two subgroups of the Hessian, G_{72} and G_{36}. The methods employed in the determination of the groups of the latter class apply equally well to the case of finite groups in the ordinary geometry.

9. Schuster has discussed the transmission of radiant heat when the isothermal surfaces are parallel planes and conduction is neglected. The problem was reduced by him to the solution of the following equation :

$$\left(1 + \frac{\partial}{\partial t}\right)\frac{\partial^2 E}{\partial \mu^2} = \frac{\partial E}{\partial t}, \tag{1}$$

where E is the total density of radiation at any point. The aim of the discussion was to describe only the broad features of the phenomena and the following assumptions were made : (i) E is independent of the time, (ii) the radiation flows only normally to the isothermal planes, (iii) the absorption is independent of the wave length.

In the present paper, Professor Jackson finds that the removal of restriction (i) gives the length of time necessary to establish the steady state to a given degree of approximation, while the removal of restrictions (ii) and (iii) leads respectively to the following equations, which are reducible to ordinary integral equations of the second kind with symmetric kernel :

$$\left(1 + \frac{\partial}{\partial t}\right)\left(1 - x^2\frac{\partial^2}{\partial \mu^2}\right)E = \int_0^1 E dx, \tag{2}$$

where t, x, μ are independent variables,

$$(3) \qquad \left(1+\frac{\partial}{\partial t}\right)\left(1-f(\kappa)\frac{\partial^2}{\partial m^2}\right)E=\int_0^1 E d\kappa,$$

where t, κ, m are independent variables.

10. Professor Birkhoff discusses the stable solutions of the problem of three bodies in the light of an earlier paper on stability presented by him at the Princeton meeting, September 14, 1909.

11. The method of integral equations is applied by Professor Cairns to the Lagrange problem: To determine $y_1, \cdots, y_n$ as functions of x so that they make

$$I=\int_a^b F(y_1', \cdots, y_n';\, y_1, \cdots, y_n;\, x)dx$$

a minimum and at the same time satisfy given differential equations

$$\phi_i(y_1', \cdots, y_n';\, y_1, \cdots, y_n;\, x)=0 \quad (i=1, 2, \cdots, m<n).$$

(This includes isoperimetric problems as a special case.)

The requirement that the second variation

$$\int_a^b Q dx$$

shall be positive, calls for the existence of functions satisfying the equations (given for simplicity's sake for $n=2$)

$$L_1(u)+\lambda k_1 u_1+\lambda'\left(\frac{\partial\phi}{\partial y_1}-\frac{d}{dx}\frac{\partial\phi}{\partial y_1'}\right)=0,$$

$$L_2(u)+\lambda k_2 u_2+\lambda'\left(\frac{\partial\phi}{\partial y_2}-\frac{d}{dx}\frac{\partial\phi}{\partial y_2'}\right)=0,$$

$$u_1'\frac{\partial\phi}{\partial y_1'}+u_2'\frac{\partial\phi}{\partial y_2'}+u_1\frac{\partial\phi}{\partial y_1}+u_2\frac{\partial\phi}{\partial y_2}=0,$$

where

$$L_i(u) \equiv \tfrac{1}{2}\left(\frac{d}{dx}\frac{\partial Q}{\partial u_i'} - \frac{\partial Q}{\partial u_i}\right).$$

A system H_{ij} of *generalized* Green's functions shows this set of equations to be equivalent to the system of integral equations

$$\phi_1(x) = \lambda \int_a^b \{H_{11}(x, \xi)\phi_1(\xi) + H_{12}(x, \xi)\phi_2(\xi)\} d\xi + x_1' g_1(x),$$

$$\phi_2(x) = \lambda \int_a^b \{H_{21}(x, \xi)\phi_1(\xi) + H_{22}(x, \xi)\phi_2(\xi)\} d\xi + x_2' g_2(x),$$

$$\lambda' \int_a^b \Phi(x)dx + \lambda \int_a^b \Psi(\phi_1(x), \phi_2(x))dx = 0,$$

g_i, Φ, and Ψ being known functions.

Finally, the solution of these is reduced by a system of orthogonal functions

$$\psi_{11}(x), \quad \psi_{12}(x), \quad \cdots; \quad \psi_{21}(x), \quad \psi_{22}(x), \quad \cdots;$$

$$\int_a^b (\psi_{1i}^2 + \psi_{2i}^2)dx = 1, \quad \int_a^b (\psi_{1i}\psi_{1j} + \psi_{2i}\psi_{2j})dx = 0 \qquad (i \neq j)$$

to a generalized system of linear equations in an infinite number of variables already developed by the writer.

The theory is completely developed, with a full solution of the original minimum problem.

12. In this paper Dr. Ranum gives a classification of linear complexes, of congruences of order one and class one, and of quadric surfaces having real generators, in riemannian space. He also studies in detail three special classes of quadric surfaces, namely Clifford surfaces, surfaces of revolution, and normal surfaces (those having one generator of each set perpendicular to every generator of the other set).

13. By projective methods Dr. MacNeish obtains a recursion sequence of geometric constructions for the linear polar of a point as to a linear k-ad of points, as to a k-line in the plane, and in general as to a k-hedron in n-space. Then, using symbolic

notation, he shows analytically that the linear polars, obtained synthetically above, harmonize with the analytic polar theory for the n-ary k-ic which is the product of linear factors. A simple application of these results gives a construction for the linear polar of algebraic curves, surfaces, and spreads. A second application is the consideration of certain configurations from the standpoint of linear polarity. The quadrangle-quadrilateral configuration in the plane is generalized and a self dual configuration in n-space is obtained consisting of an $(n + 2)$-point and an $(n + 2)$-hedron. A second generalization gives an associated k-point and k-hedron in n-space which is reciprocal only for $k = n + 2$. The invariantive conditions for reciprocity in the case of 4 collinear points is then considered and some interesting new geometric interpretations of the concomitants of the binary quartic form are obtained.

14. Professor Huntington's paper on the gyroscope obtains by elementary methods the actual accelerations of the several particles of the rotating disc when the axle of the disc is "precessing" with angular velocity ψ', and hence provides a direct elementary proof of the well-known fact that the applied forces required to maintain this precession constitute a couple lying in a plane perpendicular to the plane of precession and having a moment equal to $C\omega\psi'$, where C is the moment of inertia of the disc about its axle and ω the angular velocity of spinning. The following rule for the direction of precession is believed to be new: If the applied force be thought of as due to the pressure of a shelf against the axle, the precession will take place in the direction in which the axle would tend to roll along the shelf. The paper also calls attention to an error in a recent text-book, in regard to the motion induced by the applied couple in case precession is prevented. As a matter of fact, if the gyroscope is not allowed to precess, the rotation caused by the applied couple will be exactly the same as if the disc were not spinning.

15. From the postulates (as those of Russell, "Théorie générale des relations," *Revue de Mathématiques*, volume 7, page 115) of the calculus of relations, it readily follows that relations constitute an infinite class of potence equal at least to that of the continuum. Of the infinitude of classes of relations there is a large finite number of classes that by virtue of their

comprehensiveness and obviousness may be characterized as fundamental. Such are, for example, the classes known as symmetric, asymmetric, non-symmetric, transitive, intransitive, non-transitive, multiform, uniform, biuniform, and so on, together with the classes resulting from logical addition and multiplication of given fundamental relational classes. Professor Keyser's paper is a preliminary report upon the problem of determining which of the fundamental classes of relations are groups under such rules of combination (logical addition and logical multiplication, for example) as are applicable to relations without exception. Of several scores of relational classes thus examined, it is found that a goodly percentage possess the group property under one or both of the rules mentioned. The paper with detailed results will be published at a later date.

16. If particles are projected from an arbitrary point in all directions with a given speed and acted upon by any field of force, a doubly infinite system of trajectories will be obtained. Professor Kasner shows that only in the conservative case will these curves admit orthogonal surfaces. A second peculiarity of conservative fields is the mutual orthogonality of the three circles of curvature having four-point contact. The main part of the paper relates to a question suggested by the general theorem of Thomson and Tait: With what speeds must particles be projected normally from an arbitrary surface, so that the trajectories described shall form a normal congruence? In general the only solution is that for which the sum of the potential and kinetic energies is constant. For certain exceptional surfaces however other laws are possible.

17. In his second paper Professor Kasner determines those conservative forces for which every system of surfaces of equal action (in the sense of Thomson and Tait) constitutes a family of Lamé, that is, part of a triply orthogonal system. The trajectories are found to be circles. The result may be stated quite simply in optical terms and may be connected with non-euclidean geometry.

18. It is well known that the geometry of a congruence of Clifford parallels corresponds to the point geometry of a sphere. Some of the consequences that Dr. Ranum draws from this

correspondence are the following. If the distances between three Clifford parallels are a, b, c, and if the flux-angles between the three rectangular Clifford surfaces joining them are α, β, γ, then these six quantities are related to one another exactly as if their doubles were the sides and angles of a spherical triangle; e. g., $\sin 2a : \sin 2b : \sin 2c = \sin 2\alpha : \sin 2\beta : \sin 2\gamma$. The locus of a line that is right parallel to one of the generators of a cone of order m and left parallel to the others in order is a ruled non-developable surface S of order $2m$. Any curve drawn on the surface S and everywhere orthogonal to its generators will meet any generator in points whose distance apart is constant for the given surface and proportional to its volume.

F. N. Cole,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

The twenty-sixth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago, on Friday and Saturday, December 31, 1909–January 1, 1910. Professor G. A. Miller, Chairman of the Section, presided at all of the sessions except at the opening on Friday morning, when Professor E. B. Van Vleck, Vice-President of the Society, occupied the Chair. The attendance at the various sessions included sixty-one persons among whom were the following forty-seven members of the Society:

Professor C. H. Ashton, Mr. W. H. Bates, Professor G. A. Bliss, Professor Oskar Bolza, Professor J. W. Bradshaw, Professor W. H. Bussey, Dr. Thomas Buck, Professor D. F. Campbell, Professor D. R. Curtiss, Professor J. F. Downey, Dr. Arnold Dresden, Mr. E. B. Escott, Mr. Meyer Gaba, Professor E. D. Grant, Mr. T. H. Hildebrandt, Professor F. H. Hodge, Professor T. F. Holgate, Professor Kurt Laves, Dr. A. C. Lunn, Mr. E. J. Miles, Dr. W. D. MacMillan, Dr. H. F. MacNeish, Professor G. A. Miller, Professor E. H. Moore, Professor C. N. Moore, Dr. R. L. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Professor Alexander Pell, Mrs. Anna J. Pell, Miss Ida M. Schottenfels, Professor G. A. Scott, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Professor A. W. Smith, Professor A. L. Underhill, Professor E. B. Van

Vleck, Dr. G. E. Wahlin, Professor E. J. Wilczynski, Professor R. E. Wilson, Professor B. F. Yanney, Professor A. E. Young, Professor J. W. Young, Professor J. W. A. Young.

On Friday evening forty members of the Society dined together at the Quadrangle Club, and enjoyed one of the most interesting occasions in the history of the Section, in the way of social intercourse and the promotion of acquaintance and good fellowship.

At the business meeting on Saturday morning the following officers of the Section were elected for the ensuing year: Professor L. E. Dickson, Chairman, Professor H. E. Slaught, Secretary, and Professor W. B. Ford, third member of the program committee. At this time also the following resolution was introduced by Professor Van Vleck and carried by unanimous vote: Resolved that the Chicago Section respectfully requests the Council of the Society to arrange, in accord with the powers granted to it in By-Laws III and VII, that the next annual meeting of the Society be held with the Chicago Section, and that the President's address be also there given.

The following papers were read at this meeting:

(1) Professor C. H. Ashton: "A new elliptic function."

(2) Professor C. N. Moore: "On the uniform summability of the developments in Bessel functions of order zero."

(3) Miss Hazel H. MacGregor: "Three-dimensional chains and a classification of the collineations in space."

(4) Mr. E. B. Escott: "Logarithmic series."

(5) Mr. E. B. Escott: "Calculation of logarithms."

(6) Professor E. J. Wilczynski: "On the problem of three bodies."

(7) Dr. A. C. Lunn: "An abstract definition of limit."

(8) Dr. A. C. Lunn: "Note on the existence of the instantaneous axis in a rigid body.

(9) Mrs. Anna J. Pell: "On an integral equation with an adjoined condition."

(10) Professor G. R. Dean: "Generalized plane stress."

(11) Mr. W. H. Bates: "The medium curvatures of R_n in S_{n+1}."

(12) Dr. G. E. Wahlin: "On the base of a relative field, with an application to the composition of fields."

(13) Professor D. R. Curtiss: "Note on a method of determining the number of real branches of implicit functions in the neighborhood of a multiple point."

(14) Professor E. B. VAN VLECK: "A functional equation for the sine."

(15) Professor E. B. VAN VLECK: "On certain extensions of Abel's functional equations and their relation to Weierstrass's algebraic addition theorem."

(16) Professor G. A. MILLER: "Groups generated by two operators each of which is transformed into a power of itself by the other."

(17) Professor W. A. MANNING: "The limit of the degree of primitive groups."

(18) Professor C. H. SISAM: "On three-spreads satisfying four or more linear partial differential equations of the second order."

(19) Dr. L. I. NEIKIRK: "Groups of rational fractional transformations in a general field."

(20) Professor J. W. YOUNG: "On the discontinuous zeta groups defined by the rational normal curves in a space of n dimensions."

(21) Dr. R. L. BÖRGER: "On the Galois group of the reciprocal sextic equation."

(22) Professor F. R. MOULTON: "The singularities of the solution of the two-body problem for real initial conditions."

(23) Professor J. B. SHAW: "On hamiltonian products."

(24) Mr. A. R. SCHWEITZER: "On the geometry of the projective line."

(25) Mr. A. R. SCHWEITZER: "On the dimensional extension of Grassmann's extensive algebra" (preliminary report).

The papers of Dr. Neikirk and Miss MacGregor were presented by Professor J. W. Young. In the absence of the authors the papers of Professors Dean, Manning, and Shaw, Dr. Börger, and Mr. Schweitzer were read by title.

Besides the above papers, informal reports from two committees of the International Commission on the teaching of mathematics were presented, one by Professor D. R. Curtiss on "Courses of instruction in universities," and one by Professor E. B. Van Vleck on "Preparation of instructors for colleges and universities." Considerable interesting discussion followed these reports.

Abstracts of the formal papers follow below. The numbering corresponds to that of the titles in the list above.

1. In Professor Ashton's paper, a set of four functions $O(z)$

is defined by the functional equation

$$O(z) = [1 - ae^{\frac{2\pi i}{w_1}(z+w_2)}]\, O(z + w_2),$$

under the condition that

$$\lim_{n=\infty} O(z + nw_2) = 1.$$

From this definition the developments of the function $O(z)$ as an infinite product and as an infinite series are determined and some of its properties are studied. Its development in another infinite product by means of Weierstrass's theorem is also obtained and from this it is simply expressed as an infinite product of gamma functions. Its multiplication theorems are determined. Its relation to the σ or θ function is shown to be similar to the relation of the reciprocal of the gamma function to the sine, and finally it is shown that all of the elliptic functions and the constants which occur in a discussion of these functions can be expressed very simply in terms of O functions.

2. In this paper Professor Moore establishes the following theorem : If the function $f(x)$ is finite and integrable in the interval $c \leqq x \leqq 1$, and has a derivative that is finite and integrable in the interval $0 \leqq x \leqq c$, where c is any positive constant < 1, then the development of $f(x)$ in Bessel functions of order zero will be uniformly summable to the value of $f(x)$ throughout the interval $0 \leqq x < x_0$, where x_0 is any positive constant $< c$.

It has already been shown by Professor Moore in a previous paper * that if $f(x)$ is finite and integrable in the interval $0 \leqq x \leqq 1$, the development will be uniformly summable to $f(x)$ in any closed interval lying in the interval $0 \leqq x < 1$, which does not include a point of discontinuity of the function and does not include the origin. The behavior of the series in the neighborhood of the origin was not discussed in the previous paper.

3. The classification of the collineations in space, on the assumption that two collineations are equivalent if one can be transformed into the other by a linear homogeneous transformation with complex coefficients, is well-known. If, however,

* *Transactions Amer. Math. Society*, vol. 10 (1909), pp. 391–435.

the coefficients are restricted to real numbers, the number of types of collineations will be increased. This classification brings up the important additional problem as to the conditions under which a collineation with complex coefficients can be transformed into one with real coefficients.

In a recent paper * Professor J. W. Young has considered these two problems for the complex line from the point of view of projective geometry, the notion of a linear chain being fundamental. In a subsequent paper,† by making use of the idea of a two-dimensional chain, Professor Young considered these problems for the complex plane.

Miss MacGregor applies the same principle of classification to the non-singular collineations in a complex space of three dimensions and considers the corresponding problems. She gives the classification of the collineations in space into nineteen distinct types, each of which leaves a three-dimensional chain invariant. Any such collineation may be represented with real coefficients. The necessary and sufficient conditions that a collineation be of this type are derived and the corresponding systems of invariant chains are determined.

4. The first paper by Mr. Escott completes the one read by him in April, 1904, before the Chicago Section. In the ordinary logarithmic series

$$\log\frac{X+d}{X-d}=2\left[\frac{d}{X}+\frac{1}{3}\left(\frac{d}{X}\right)^3+\cdots\right]$$

the problem is to express X in the form of polynomials in x so that both $X+d$ and $X-d$ shall have rational linear factors. The solution of this problem gives interesting applications of the elementary theory of numbers. A number of series of this kind have been developed by others, but there has been no systematic attempt to solve the problem. Mr. Escott shows how an indefinite number of series may be obtained where X is of degree 1 to 7, and gives four examples where X is of degree 10, some of the factors in the latter case being quadratic.

5. In his second paper Mr. Escott shows the application of the series in the preceding paper to the computation of loga-

* "The geometry of chains on a complex line," *Annals of Mathematics*, vol. 11 (1909), pp. 33–48.

† Read before the Society at its April meeting, 1908.

rithms. Huyghens has given a list of numbers differing by unity, having small factors, which may be used to calculate the logarithms of numbers as far as 100. This table is improved and extended to 200.

6. The problem of three bodies has been almost exclusively studied from the analytical point of view. In the present paper, Professor Wilczynski formulates a number of questions suggested by geometry, some of which are capable of direct and final answers, while others of a more difficult character are merely indicated. It is intended that all of these problems be subjected to a more thorough investigation in the future.

The center of mass of the three bodies is supposed to be at rest. Each of the masses will describe a certain curve, the straight lines joining them in pairs will generate three ruled surfaces, the plane of the three masses will envelop a cone. In this first paper the linear differential equations are set up which characterize the projective differential properties of some of these loci. Particular attention is paid to the question: can the ruled surface generated by the straight line joining two of the bodies be developable? It is found that in order that this may be so, the mutual distances must satisfy a certain differential equation of the second order, but the question of the compatibility of this equation with those of the problem of three bodies is provisionally left open. Can this developable be a cone? Leaving aside the trivial cases in which the cone degenerates into a plane or a straight line, whose existence is obvious, it turns out that the vertex of the cone must be at infinity, i. e., the cone is necessarily a cylinder. Moreover, the triangle formed by the three bodies must in that case constantly remain isosceles, the two equal sides being those which join the two bodies which are situated upon the same element of the cylinder to the third. It is then shown that there actually exist solutions of the problem of three bodies for which the triangle always remains isosceles, and that some of these give rise to the "cylindrical solutions" just indicated. The third body, in the case of such a cylindrical solution, always describes a plane curve similar to the corresponding plane section of the cylinder. It does not appear that the plane section of the cylinder can be determined in general as a simple curve, but the differential equations which characterize it projectively are obtained.

Another question which receives a simple answer is that as

to the existence of solutions in which one of the bodies describes an asymptotic curve upon one of the ruled surfaces generated by the line which joins it to one of the other bodies. The criteria for plane orbits are also indicated.

It is proposed to found a new general theory of perturbations upon a combination of the theory of osculating conics, cubics, and quartics of plane and space curves, with the method of the variation of constants.

7. In his first paper Dr. Lunn gives a general abstract setting of the concept of limit, designed so as to exhibit as particular cases the ordinary special limiting processes of analysis. The dependent variable is assumed to be numerical, real or n-dimensional complex, but the independent variable is taken as an abstract set of elements, undefined except as it is subject to an order relation satisfying certain postulates. The abstract theory, based on the abstract analogue of the proposition of Du Bois-Reymond, is devoted primarily to several theorems relating to necessary and sufficient conditions for interchange of two limiting processes. These yield as special cases such theorems as those on the integration and differentiation of infinite series, the differentiation of a definite integral, and the behavior of the solutions of differential equations as functions of a parameter.

8. In his second paper Dr. Lunn defines a rigid body with fixed origin as a non-coplanar set of points each at a constant distance from the origin and such that the scalar product of the vectors from the origin to any two points is constant. A purely analytic vector proof is given that there exists a single vector w such that the velocity of any point is given by $v = w \times r$.

9. Mrs. Pell shows that the solution of an orthogonal integral equation with an adjoined condition depends on the solution of an orthogonal equation without any condition.

10. In the case of an elastic plate under the action of a system of forces parallel to the faces and uniformly distributed across the edges, Professor Dean finds that the cubical expansion is a linear function of the two coordinates parallel to the faces; the coefficients being arbitrary constants which are determined by means of given surface tractions or displacements. This function when introduced into the equations of equilibrium

in terms of the displacements reduces the system to three independent equations, two of which are of Poisson's form in two dimensions, the third having a single term and being satisfied by the linear function which expresses the cubical expansion.

The solutions of the Poisson equations are expressed as surface and line integrals. The results substituted in the generalized form of Hooke's law give the normal components of stress and the components of shear.

Using Hertz's method of finding the pressure between two bodies in contact, thus determining the surface tractions and displacements, we have a complete and workable solution of the problem of determining the stresses in riveted and pin joints and in other problems of technical mechanics.

Errors frequently made in applying the theory of elasticity to technical mechanics, chiefly in assigning boundary conditions, are pointed out and means of avoiding them are suggested.

11. At any point of a surface in ordinary space, the curvatures of the two lines of curvature are the roots ρ_1 and ρ_2 of the equation

$$\begin{vmatrix} D + E\rho & D' + F\rho \\ D' + F\rho & D'' + G\rho \end{vmatrix} = 0 .$$

After division by the coefficient of ρ^2, the absolute term of this equation is the gaussian curvature of the surface, and the coefficient of ρ its medium curvature.

Similarly, at any point of a hypersurface R_n in the euclidean space S_{n+1}, the curvatures of the n lines of curvature are the roots $\rho_1, \cdots, \rho_n$ of the equation

$$(1) \qquad \begin{vmatrix} \alpha_{11} + a_{11}\rho & \alpha_{12} + a_{12}\rho \cdots \alpha_{1n} + a_{1n}\rho \\ \alpha_{21} + a_{21}\rho & \alpha_{22} + a_{22}\rho \cdots \alpha_{2n} + a_{2n}\rho \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_{n1} + a_{n1}\rho & \alpha_{n2} + a_{n2}\rho \cdots \alpha_{nn} + a_{nn}\rho \end{vmatrix} = 0 ,$$

in which the a's are the first fundamental quantities and the α's the second.

Equation (1) is of the nth degree in ρ and may be written

$$(2) \qquad \rho^n + K_1\rho^{n-1} + \cdots + K_{n-1}\rho + K_n = 0 ,$$

in which K_n is the Kronecker-Gaussian curvature of the hyperspace, and has been fully treated. In this paper, Mr. Bates studies the other coefficients of (2), which are called the medium curvatures of R_n in S_{n+1}.

12. If K is an algebraic field of degree N and k a subfield of degree n, Dr. Wahlin's paper shows that there exist in K $N/n = r$ numbers $\Omega_i (i = 1, 2, \cdots, r)$, and in k r ideals O_i, such that every integer in K can be expressed in the form

$$\sum_{i=1}^{N} \mu_i \Omega_i,$$

where the μ_i are integers of O_i, and moreover all numbers so represented are integers in K. It is then shown that the relative discriminant of K is equal to

$$O_1^2 \cdot O_2^2 \cdots O_r^2 \cdot |\Omega_i^{(k)}|^2,$$

where the index k is used to designate the relative conjugates to Ω_i.

These results are then used to deduce an expression for the discriminant of a field compounded from two fields in the case where the degree of the compounded field is equal to the product of the degrees of the two fields divided by the degree of the greatest subfield common to these two fields.

13. In a paper read before the Society in April, 1908, Professor Curtiss developed a method for determining the number of real branches of an implicit function $f(x, y) = 0$ in the neighborhood of a multiple point. It was there shown that if the point (x_0, y_0) is of multiplicity n, and if $\partial^n f(x_0, y_0)/\partial y^n \neq 0$, then the number of real branches passing through that point depends on the number of changes of sign in the values of $f(x, y)$ computed on each real branch of the implicit function $\partial f(x, y)/\partial y = 0$ in the neighborhood of (x_0, y_0). In the present note it is shown that if a_2 and b_2 have any but a certain finite number of values the implicit function $a_2 f(x) + b_2 f(y) = 0$ can be used in place of $f(y) = 0$, the restriction $\partial^n f(x_0, y_0)/\partial y^n \neq 0$ being thereby removed and other simplifications effected.

14. Cauchy has shown that the only real continuous solutions of the functional equation

$$\phi(x + y) + \phi(x - y) = 2\phi(x)\phi(y)$$

(other than the trivial solutions $\phi(x) \equiv 0$ and $\phi(x) \equiv 1$) are $\sin cx$ and $\sinh cx$. Professor Van Vleck gives in his paper a functional equation defining *uniquely* the sine function, and from this equation the properties of the function follow with great ease and rapidity.

15. In one of his earliest published papers Abel showed that if a differentiable function $f(x, y)$ possesses the property that $f[x, f(y, z)]$ is symmetric in x, y, z, then there exists a function $\phi(x)$ such that

$$\phi(f[x, y]) = \phi(x) + \phi(y). \tag{1}$$

Professor Van Vleck extends the theorem to a class of multiform functions $f(x, y)$ and derives a necessary and sufficient condition that an algebraic equation $G[\phi(x + y), \phi(x), \phi(y)] = 0$ shall present an actual (and not an impossible) addition theorem. Two generalizations of (1) were also discussed, the first of which has the form

$$\phi(f[x, y]) = f_1[\phi(x), \phi(y)].$$

16. Two special cases of the groups generated by two operators each of which is transformed into a power of itself by the square of the other have been considered in earlier papers by Professor Miller; namely, when the square of each of the two generators transforms the other generator either into itself or into its inverse. The object of the present paper is to obtain some fundamental theorems relating to the general case where the two operators s_1, s_2 satisfy the conditions

$$s_1^{-2}s_2s_1^2 = s_2^{\alpha}, \quad s_2^{-2}s_1s_2^2 = s_1^{\beta}.$$

If at least one of the two numbers α, β is even, the corresponding operator is of odd order and hence it must be generated by its square. In this case the group G generated by s_1, s_2 may be generated by a cyclic group and an operator transforming this cyclic group into itself. As many properties of these groups are well known, Professor Miller confines his attention to the cases in which both α and β are odd.

After observing that the subgroup H generated by s_1^2, s_2^2 is invariant under G and that $s_1^{2(\beta-1)}$, $s_2^{2(\alpha-1)}$ are invariant operators under G, it is proved that the orders of s_1, s_2 must divide $2(\alpha - 1)(\beta - 1)$. Hence each of these orders has an upper limit whenever α, β are both different from unity and only

then. It is proved that the fourth derived of G is unity and hence G is always solvable. It is also proved that the orders of s_1, s_2 are divisors of $2(\beta - 1)^2$, $2(\alpha - 1)^2$ respectively. The general theorems are illustrated by means of the special categories of groups which result when $\alpha = \beta = 1$ and when $\alpha = \beta = 5$. In the latter case the first derived group is abelian.

17. In *Liouville's Journal* for 1871 Jordan demonstrated the fundamental theorem :

"If a primitive group G (not containing the alternating group) contains a substitution A which displaces only m letters, the degree of G cannot exceed a certain limit."

It is a matter of great interest to reduce this limit as much as possible. In two subsequent papers * Jordan has given us expressions for the limit which are much lower than that first published. A further reduction † was recently made by Professor Manning, and the paper presented at this meeting gives a still lower limit. A special case of his general theorem may be stated thus :

The degree of a simply transitive primitive group which contains a substitution of prime order p on q cycles ($q < 2p+3$) cannot exceed the greater of the two numbers $pq+q^2-q$, $2q^2-p^2$.

18. In this paper, Professor Sisam first determines under what conditions a three-spread in space of n dimensions can satisfy more than four linear partial differential equations of the second order. He then shows that if it satisfies four such equations, it has at each point four tangents having contact of the second order with the three-spread. The rest of the paper is devoted to the determination of the conditions under which two or more of these three-point tangents at a generic point may be consecutive.

19. This paper is an extension of that presented to the Chicago Section at the April, 1909, meeting by Dr. Neikirk. Several cases arise and the results of the former paper are extended to include these. The last part of the paper is devoted to finding a transformation $S^{(i)}$ which represents the product of the powers of several substitutions $S^{(i)} = R_1^{y_1^{(i)}} R_2^{y_2^{(i)}} \cdots R_k^{y_k^{(i)}}$.

* *Bulletin Soc. Math. de France*, vol. 1 ; *Crelle's Journal*, vol. 79.

† BULLETIN, vol. 13 (1907), p. 373.

$S^{(i)}$ is given by a polynomial in x and the exponents $y_r^{(i)}$ $(r = 1, 2, \cdots, k)$.

20. A rational normal curve C_n in a linear space S_n of n dimensions is left invariant by a three parameter group G of collineations $x_i' = \Sigma a_{ij} x_j$ $(i, j = 0, 1, \cdots, n)$ in S_n which is isomorphic with the group of all linear fractional substitutions on a single variable ζ. This isomorphism is most readily effected by interpreting ζ as the parameter of the points on C_n, so that to every collineation of G corresponds a unique ζ-substitution, and conversely. This fact has suggested a method of defining arithmetically certain discontinuous ζ-groups, a problem of the highest importance in the theory of automorphic functions. The method consists in determining G for a given C_n and then considering the discontinuous subgroup of G obtained by restricting the coefficients a_{ij} to be integers with determinant $|a_{ij}| = 1$. If this subgroup contains collineations other than identity, the C_n may be called an *integral* C_n, and the corresponding ζ-group will be discontinuous. That such integral C_n's exist follows at once from the fact that the *canonical* C_n, whose equations are $x_i = \zeta^{n-i}$ $(i = 0, 1, \cdots, n)$ gives rise in this way to the elliptic modular group.*

The case $n = 2$ of this method is the only one which has received detailed treatment.† By the consideration of a certain invariant J it follows that the discontinuous ζ-groups defined by the C_2's cannot contain elliptic substitutions of periods other than 2, 3, 4, and 6. By calculating the corresponding invariant for the general case C_n, Professor Young in a recent paper ‡ showed that the period e of any elliptic substitution occurring in a discontinuous ζ-group defined by a C_n must satisfy the relation

$$\sin \frac{(n+1)\pi}{e} = J \sin \frac{\pi}{e},$$

where J is any integer, positive, negative, or zero. In particular, the values $e = n,\ n + 1,\ n + 2$ satisfy this equation for $J = -1, 0, +1$. The present paper is devoted to the proof that integral C_n's really exist whose discontinuous ζ-groups con-

* J. W. Young, "On a class of discontinuous ζ-groups," etc., *Rendiconti del Circolo mat. Palermo*, vol. 23 (1907).

† Fricke-Klein, Automorphe Funktionen, v. 1, p. 533.

‡ BULLETIN, vol. 14 (1908), p. 367.

tain elliptic substitutions of periods n and $n+1$. The method of proof is direct in that it shows how to obtain the equations of a C_n whose group contains a given substitution of period n or $n+1$.

21. In this paper Dr. Börger finds the Galois group of the reciprocal sextic equation

$$x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 = 0$$

for the domain $R(1)$ of the coefficients. The group is a subgroup of the primitive substitution group G^6_{48} having three systems of imprimitivity. Criteria for the irreducibility of the equation are also found.

22. The differential equations which are satisfied by the relative motion of two bodies subject to the newtonian law of gravitation define, when the initial values of the dependent variables are given, certain analytic functions of the independent variable t. The position and character of the singularities of these functions depend upon the properties of the differential equations and the numerical values of the initial conditions. In Professor Moulton's paper the positions of the singular points are found for all possible real initial conditions, their changes of position are determined as the initial values of the dependent variables are varied, the character of the singularities is found in all cases, the Riemann surfaces are constructed, and (except in a certain degenerate case) a variable is defined as a function of t in terms of which the coordinates can be expanded as power series convergent for all real values of t.

23. Professor Shaw's paper is in abstract as follows: Let there be a system of symbols, α, β, γ, $\cdots$ either finite or infinite in number. The product of two of these, as α, β, is the symbol $\alpha\beta$, and is of the second rank. The product of $\alpha_1, \cdots, \alpha_n$ is $\alpha_1\alpha_2 \cdots \alpha_n$, and is of rank n. These products may be associative or not, but they are distributive; that is $(l\alpha + m\beta)\gamma = l\alpha\gamma + m\beta\gamma$, etc., where l, m are ordinary numbers. If we restrict the product further we give it a special name and prefix a symbol, as $S \cdot \alpha\beta\gamma$, $V \cdot \alpha\beta\gamma$, from quaternions. These are called partial products.

The first partial product is defined by $I \cdot \alpha\beta$, which is such that it is a scalar, and

$$I \cdot \alpha\beta = I \cdot \beta\alpha\,;\; I \cdot (m\alpha)\beta = mI \cdot \alpha\beta,\; I \cdot (\alpha + \beta)\gamma = I \cdot \alpha\gamma + I \cdot \beta\gamma.$$

Also

$$\begin{aligned} I \cdot \alpha_1\alpha_2 \cdots \alpha_{2m} = I\alpha_1\alpha_2 \cdot I \cdot \alpha_3 \cdots \alpha_{2m} - I \cdot \alpha_1\alpha_3 \cdot I \cdot \alpha_2\alpha_4 \cdots \alpha_{2m} \\ + I \cdot \alpha_1\alpha_4 I \cdot \alpha_2\alpha_3\alpha_5 \cdots \alpha_{2m} \cdots \\ + (-1)^i I \cdot \alpha_1\alpha_i \cdot I \cdot \alpha_2 \cdots \alpha_{i-1}\alpha_{i+1} \cdots {}_{2m} + \cdots \\ + \cdot I \cdot \alpha_1\alpha_{2m} I\alpha_2 \cdots \alpha_{2m-1}. \end{aligned}$$

These forms turn out to be Pfaffians, and are such that

$$I \cdot \alpha_1 \cdots \alpha_{2m} = I \cdot \alpha_2 \cdots \alpha_{2m}\alpha_1 = I \cdot \alpha_{2m} \cdots \alpha_1.$$

Next we define the alternate product $A_m \cdot \alpha_1 \cdots \alpha_m$, such that $A_m \cdot \alpha_1 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_m = - A_m \cdot \alpha_1 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_m$. It is of rank m. The Joly products are then defined by the statements

$$A_i \cdot \alpha_2 \cdots \alpha_{i+2h} = \Sigma \pm A_i \cdot \alpha_{j_1}\alpha_{j_2} \cdots \alpha_{j_i} \cdot I \cdot \alpha_{j+1} \cdots \alpha_{j_{i+2h}},$$

where $j_1 < j_2 \cdots < j_i$ and $j_{i+1} < j_{i+2} \cdots < j_{i+2h}$ and the $i + 2h$ subscripts are permuted in all possible ways, the sign being $+$ or $-$ according to the number of inversions.

Again

$$\begin{aligned} A_{h+i-2c}(A_h \cdot \alpha_1 \cdots \alpha_h A_i \cdot \beta_1 \cdots \beta_i) = \Sigma \pm A_{h+i-2c} \cdot \alpha_{j_1} \\ \cdots \alpha_{j_{h-c}}\beta_{k_1} \cdots \beta_{k_{i-c}} \cdot I(A_c\alpha_{j_{h-c+1}} \cdots \alpha_{j_h} A_c\beta_{k_{i-c+1}} \cdots \beta_{k_i}). \end{aligned}$$

(Example: $A_1 \cdot A_2\alpha_1\alpha_2 A_3\beta_1\beta_2\beta_3 = \beta_1 I \cdot A_2 \cdot \alpha_1\alpha_2 A_2 \cdot \beta_2\beta_3 - \beta_2 I \cdot A_2 \cdot \alpha_1\alpha_2 A_2 \cdot \beta_1\beta_3 + \beta_3 I \cdot A_2 \cdot \alpha_1\alpha_2 A_2 \cdot \beta_1\beta_2$.)

Finally, the Hamilton product of $\alpha_1, \cdots, \alpha_m$ is $H \cdot \alpha_1 \cdots \alpha_m = A_m \cdot \alpha_1 \cdots \alpha_m + A_{m-2} \cdot \alpha_1 \cdots \alpha_m + \cdots$, understanding that A_{m-m} or $A_0 \cdot$ is I. These Hamilton products are associative, that is, for example, $H(H \cdot \alpha_1\alpha_2)(H \cdot \alpha_3\alpha_4\alpha_5) = H \cdot \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = H \cdot \alpha_1 H\alpha_2\alpha_3\alpha_4\alpha_5 =$ etc. The properties of these various expressions are then studied.

24. Mr. Schweitzer pointed out features of his axioms for the projective line analogous to his descriptive axioms for the

line (the system ${}^1R_1$) and for the plane (the system ${}^2R_2$). In particular, he showed how his planar descriptive system could be logically combined with linear projective axioms in such a way that the resulting system might be descriptive or projective, but not necessarily either.

25. In his second paper, Mr. Schweitzer presented a preliminary report on a study of the memoirs of F. Riesz * and H. Hahn † in relation to the generalization of Grassmann's extensive algebra ‡ to a denumerably infinite number of dimensions.

H. E. SLAUGHT,
Secretary of the Section.

THE SIXTY-FIRST MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE sixty-first meeting of the American Association for the Advancement of Science was held in Boston during the convocation week, December 27, 1909, to January 1, 1910. The president of the meeting was President D. S. Jordan, of Stanford University. The address of the retiring president, Professor T. C. Chamberlin, entitled "A geologic forecast of the future opportunities of our race" was given in Sanders Theatre, Cambridge, on the evening of the opening day.

Comparatively few papers on pure mathematics appeared on the separate program of Section A (mathematics and astronomy) because of the fact that the American Mathematical Society held its annual meeting in affiliation with the Association. The address of the retiring vice-president, Professor C. J. Keyser, of Columbia University, entitled "The thesis of modern logistic," was given on Wednesday morning at a joint session of Section A and the American Mathematical Society. At the same session Professor D. E. Smith presented a report on the work of the International Commission on the teaching of mathematics.

Another joint session was held on Tuesday afternoon under the auspices of the mathematicians and the physicists. During

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* *Math. Naturw. Berichte aus Ungarn*, 1905, pp. 309, 341–343, etc.

† *Wiener Berichte*, Abt. II^A, vol. 116, pp. 601, 609–610, 642, etc.

‡ See also *American Journal*, October, 1909, pp. 365–410.

this session Professor C. Runge, Kaiser Wilhelm exchange professor of mathematics at Columbia University, read a paper "On the determination of latitude and longitude in a balloon," and Professor E. W. Brown read the first of his two papers in the list below. These joint sessions constituted the most noteworthy features of the program of Section A, and it is to be hoped that they may have tended to attract more investigators to the border land between mathematics and physics, where our country seems to be especially in need of workers.

The officers of Section A were: vice-president, E. W. Brown; secretary, G. A. Miller; councilor, G. B. Halsted; member of the general committee, H. W. Tyler; sectional committee, C. J. Keyser, E. W. Brown, G. A. Miller, E. O. Lovett, Harris Hancock, F. R. Moulton, Winslow Upton. On the recommendation of the sectional committee the following twenty-two members of the American Mathematical Society were elected fellows of the Association: R. P. Baker, W. E. Brooke, Thomas Buck, Arthur Crathorne, I. M. DeLong, C. E. Dimick, F. J. Dohmen, J. F. Downey, L. P. Eisenhart, J. C. Fields, B. F. Finkel, F. L. Griffin, A. G. Hall, C. N. Haskins, T. F. Holgate, J. I. Hutchinson, D. N. Lehmer, W. D. MacMillan, C. A. Noble, F. G. Reynolds, Charlotte A. Scott, A. W. Smith.

The following twenty papers were read before the Section:

(1) Professor C. J. Keyser: "The thesis of modern logistic."

(2) Professor C. Runge: "On the determination of latitude and longitude in a balloon."

(3) Professor E. W. Brown: "On certain physical hypotheses sufficient to explain an anomaly in the moon's motion."

(4) Professor D. E. Smith: "The work of the International Commission on the teaching of mathematics."

(5) Mr. C. G. Abbot: "The value of the solar constant of radiation."

(6) Professor F. W. Very: "A new mode of measuring the intensities of spectral lines."

(7) Professor F. W. Very: "The absorption of light by the ether of space."

(8) Professor F. W. Very: "The fireball of October 7, 1909."

(9) Professor E. W. Brown: "On a recent hypothesis and the motion of the perihelion of Mercury."

(10) Professor J. A. Miller and Mr. W. R. Marriott: "The heliocentric position of certain coronal streams."

(11) Professor D. P. TODD: "The mutual relation of magnifying power, illumination, aperture, and definition in telescopic work."

(12) Professor G. B. HALSTED: "La contribution non-euclidienne à la philosophie."

(13) Mr. H. E. WETHERILL: "Declination of the moon for Greenwich mean time."

(14) Mr. H. W. CLOUGH: "Meteorological waves of short period and allied solar phenomena."

(15) Professor MILTON UPDEGRAFF: "Recent work with the 6-inch transit circle of the United States Naval Observatory."

(16) Mr. V. M. SLIPHER: "Peculiar star spectra indicating selective absorption of light in space."

(17) Mr. R. M. STEWART: "Personality with the transit micrometer."

(18) Professor F. W. VERY: "Water vapor on Mars."

(19) Professor C. L. DOOLITTLE: "The existence of anomalous fluctuations in the latitude as shown by simultaneous observations with the zenith telescope and the reflex zenith tube of the Flower Observatory."

(20) Mr. LEON CAMPBELL: "Visual observations of variable stars at the Harvard College Observatory."

In the absence of their respective authors the papers by Professor Todd, Mr. Wetherill, and Mr. Slipher were read by title, and that of Mr. Stewart was read by Dr. O. J. Klotz. The remaining papers were read by their authors. Professor Keyser's vice-presidential address appeared in *Science*, December 31, 1909. Abstracts of the other papers of mathematical interest are given below. They bear numbers corresponding to those of the titles in the list given above.

2. Professor Runge remarked that the problem of finding the geographical position in a balloon from observations of the sun is very different from the same problem on a ship for the reason that in a balloon there is no dead reckoning. In a balloon the only way of getting the geographical position from the sun is by observing both altitude and azimuth at the same time. The accuracy with which the azimuth of the sun may be observed is rather rough; it would be difficult to obtain it within less than one tenth of a degree. Therefore the reduction of the observations need not be very accurate. At the

same time it is essential that the reduction should be made very quickly. For the time since the moment the observations were taken introduces an uncertainty which may be expressed by the area of a circle whose radius is equal to the distance through which the balloon may have traveled.

One naturally would therefore turn to graphical methods for the reduction of the observations. The reduction consists in finding the latitude ϕ and the hour angle t from the declination δ, the azimuth a, and the altitude h. Professor Runge proposes to find first the latitude ϕ from δ, a, h and then the hour angle t from δ, a, ϕ. In both cases we have to deal with the representation of an equation between four variables. Both of these equations may be written in the following form:

$$f(p) + h(r, s)g(q) = k(r, s),$$

where p, q, r, s denote the four variables. That is to say, two of the variables enter the equations in separate functions $f(p)$, $g(q)$, and the equation is linear in these functions, the coefficients being any functions of the other two variables. Equations of this kind may be represented graphically by the "méthode des points alignés" of Maurice d'Ocagne,* taking $f(p)$ and $g(q)$ as line coordinates, making $f(p)$ equal to the ordinate of the point of intersection of the straight line with the axis of ordinates and $g(q)$ equal to the gradient of the straight line, that is, the tangent of its angle with the axis of abscissas. In this way the rectangular coordinates of the point whose equation in line coordinates is the given equation become

$$x = h(r, s), \quad y = k(r, s).$$

For any given value of p the different values of q correspond to straight lines which form a pencil of rays whose center is on the axis of ordinates at the particular value defined by p, and any alteration of p would simply shift the center along the axis without altering the pencil of rays in any other way. The whole diagram may therefore be obtained by drawing two figures, one containing the curves $r =$ constant and $s =$ constant and the other containing the pencil of rays, and placing these two figures in the proper way one over the other. It so happens that the variable p is the declination of the sun which

* Traité de Nomographie.

may be regarded as constant during the ascent of the balloon. The aeronaut would therefore merely use a definite superposition of figures. These are photographed on transparent plates and a blue print is taken by copying the plates, one after another, on the same paper in proper position.

The aeronaut has one blue print to read off the latitude, a second to read off the hour angle after he has found the latitude. The equations are

$$\sin \delta + \cos \phi \cos h \cos a = \sin \phi \sin h,$$

$$\tan \delta + \sec \phi \sin t \cot a = \tan \phi \cos t.$$

In the first equation the curves $\phi =$ const. and $h =$ const. are the ellipses $x = \cos \phi \cos h$, $y = \sin \phi \sin h$. In the second equation the curves $\phi =$ const. and $t =$ const. are the confocal ellipses and hyperbolas

$$x = \sec \phi \sin t, \quad y = \tan \phi \cos t.$$

3. Newcomb has shown that there is a difference between the observed and the theoretical position of the moon which can be roughly represented by a term of period about 270 years and coefficient 13″. In the present paper Professor Brown examined numerous hypotheses sufficient to explain the term, in order to clear the ground of those which seemed to be of doubtful value and to bring forward those which appeared sufficiently reasonable to merit tests from observations of a different nature. He gave an account of three of these hypotheses, stating that a minute libration of the moon would be sufficient provided it took place in the moon's equator and had the proper period. The supposition of magnetic attraction practically demanded a periodic change in the magnetic moment of the earth or of the moon. If this were rejected, it was necessary to suppose that the mean place of the lunar magnetic axis was near the lunar equator and that the oscillations of its position took place in the plane of the equator. The observed secular change of the earth's magnetic axis could not produce the phenomenon without demanding a larger motion of the lunar perigee than observations warrant. On the border line between two sets of hypotheses is a curious fact, namely, that if the period of the solar rotation coincides very nearly with one of the principal lunar periods, a minute equatorial ellipticity of the sun's mass would be sufficient to explain the term. So

far as known these hypotheses do not conflict with any observed phenomena, but they cause some theoretical difficulties.

4. The International Commission on the teaching of mathematics was suggested some years ago but the first steps in its organization were not taken until April, 1908. At that time the Fourth International Congress of Mathematicians, then in session in Rome, empowered Professor Klein of Göttingen, Sir George Greenhill of London, and Professor Fehr of Geneva to appoint such a commission, and to arrange for it to report at the next congress to be held at Cambridge in 1912. As a result three commissioners have been selected from each of the leading countries and the work has actively begun. Those of the United States are Professors D. E. Smith, W. F. Osgood, and J. W. A. Young. It is expected that each of these countries will submit a very full report of the nature of the work in mathematics, from the kindergarten through the college, with some discussion of the range of advanced work in the universities. In the United States the investigation is carried on by means of fifteen committees, each divided into subcommittees. Professor Smith stated that about two hundred and seventy-five people are engaged in the work and the subcommittee reports will be submitted during the present winter. The committee reports will be submitted before the summer of 1910, and the national report by Easter, 1911. This paper is to appear in the *American Mathematical Monthly.*

9. Professor Brown's second communication consisted of a brief account of the hypotheses of Seeliger brought forward to account for the outstanding large residual in the motion of the perihelion of Mercury and the small residuals in the secular motions of the four minor planets. An analysis of the nature of the three hypotheses and a comparison of the number of arbitrary constants introduced, with the number of residuals to be accounted for, were also given.

10. Professor Halsted's memoir describes the meaning and growth of non-euclidean geometry, sketches its history and founders, and points out that philosophy has found a new criterion in this subject. The memoir is to appear in French in the *Mémoires de la Société des Sciences physiques et naturelles de Bordeaux.*

The next regular meeting of the Association will be held at the University of Minnesota under the presidency of Professor A. A. Michelson, of the University of Chicago. Professor E. H. Moore, of the University of Chicago, was elected vice-president and chairman of Section A; Professor G. A. Miller, of the University of Illinois, continues as secretary. Professor E. R. Smith, of the Brooklyn Polytechnic Preparatory School, was elected member of the sectional committee for five years.

G. A. MILLER,
Secretary of Section A.

SHORTER NOTICES.

Rara Arithmetica. By DAVID EUGENE SMITH, Teachers College, Columbia University. Second edition. Boston, Ginn and Company, 1909. xviii + 507 pp. Cloth. Price, $4.50.

THE title page of this splendid volume modestly states that "the work is a catalogue of the arithmetics written before the year 1601 with a description of those in the library of George Arthur Plimpton of New York." Another appropriate title might be, A brief history, on the bibliographical plan, of the genesis and content of sixteenth century arithmetic.

As a bibliography this work is more extensive than any of its predecessors, and is nearly complete for the formative period between 1472 and 1601. There are mentioned over five hundred and fifty different works, which number swells nearly to twelve hundred by the inclusion of the various editions. About four hundred and fifty of the different books are genuine arithmetics, while the others deal partially with algebra, astrology, or the calendar. In determining the significance of this number one thinks at once of De Morgan's Arithmetical Books, the best of the earlier authorities on the subject, and recalls that this work gives only seventy arithmetics printed before 1600. The list in Professor Smith's Rara Arithmetica is much more extensive than those of Graesse, Hain, and Coppinger, and contains more Italian titles than are given by Riccardi in his Bibliotheca Mathematica Italiana and more German ones than are included by Murhard in his Bibliotheca Mathematica.

But this work is more than a scholarly, well edited digest of all the earlier bibliographies; it is the result of the examination

of the original works in so far as they are extant and are to be found in the libraries of Europe or America. In particular, it is an extensively illustrated catalogue of the arithmetical collection of Mr. Plimpton. This collection is already well known to many scholars, and it will now become known to other students of mathematics wherever Professor Smith's work may circulate. It is the third great private collection of rare arithmetical books of the sixteenth century and surpasses its predecessors, the Libri and Boncompagni libraries, in respect to this period. It contains over three hundred arithmetics, while De Morgan with the aid of the British Museum was able to consult less than one hundred of those printed before 1601, including all editions. It is probable that Mr. Plimpton's library does not lack more than a dozen or so extant sixteenth century arithmetics that went through two editions.

The work is arranged chronologically by first editions, but this classification is supplemented by indices of reference, both alphabetical and geographical, making readily accessible every important fact in the volume. In describing a specimen, the first edition is always taken, if available, and discussed under these heads: title, colophon, description, editions. The description includes the size of the page and the text in centimeters; the number of blank, numbered, and unnumbered pages; the style of numbering; and the place of publication and date. The title pages of all the works of special value are reproduced in fac-simile, besides many pages and selections that show the evolution of arithmetic; and in addition to these there are twelve plates of particular interest to the student of early mathematical literature.

But Professor Smith's work is more than a catalogue, it is a condensed history of arithmetic during its formative period. The numerous notes not only point out the significant features of the specimens under discussion, but they form a comparative study of the subject from many points of view, for example, the comparison of the arithmetic of the Latin schools with that of the trade schools, the comparison of the arithmetics of different nations, and the relation of abacus reckoning to figure reckoning. The following excerpt illustrates this feature of the work:

"This work (Scheubel's Arithmetic) is the production of a scholar rather than a man conversant with the demands of business. While Scheubel tried to write a mercantile arithmetic, the result was far removed from the needs of the common people.

It carries the work in subjects like the roots so far that the ordinary Rechenmeister could not have used it. Moreover, it is written in Latin and is much more extended than the work of Gemma Frisius, so that it appealed neither to the business school nor to the ordinary classical school. A great deal of attention is given to exchange, the rule of three, and the extracting of roots of high order. Attention is also given to problems which would now form part of algebra, and there is a brief treatment of geometry from the standpoint of mensuration.

"While Scheubel is not much appreciated to-day, he was really ahead of his time. He tried to banish the expression 'rule of three' and to substitute 'rule of proportion.' His explanation of square root is in some respects the best of the century, and he dismisses with mere mention the 'duplatio' and 'mediatio' of his contemporaries. He extracts various roots as far as the 24th, finding the binomial coefficients by means of the Pascal triangle a century before Pascal made the device famous."

As to its usefulness, this is a work which no bibliographer of rare books will fail to consult. It will become an authoritative source for writers of mathematical history and the standard reference book on sixteenth century arithmetic for scholars in mathematics everywhere. It would be wasteful of the reviewer's space to speak of the author, because his special fitness is known to practically every student of the history of mathematics, and his scholarship stamps with authority all of his productions.

LAMBERT L. JACKSON.

Coordinate Geometry. By HENRY BURCHARD FINE and HENRY DALLAS THOMPSON. New York, The Macmillan Company, 1909. 8vo. 8 + 300 pp.

IT was generally considered by the writers of the earlier American text-books on analytical geometry and by those who then taught the subject that the material for a first course consisted of the chief metrical properties of the separate species of conic sections. There is a marked similarity between the text in these books and the easier portions of Chapters I, II, VI, X, XI, XII of Salmon's Treatise on Conic Sections (edition of 1869). Within recent years, however, there has been a marked tendency among some of the teachers to regard the acquisition of these isolated facts about parabolas, ellipses, and

hyperbolas as a means instead of an end. They are eager to infuse into even a first course some of the spirit of the so-called "modern" geometry and to replace a few of the less important facts of the old course by some of the simplest of the "new" ideas. Many of the recent texts have reflected more or less of this tendency. The advocates of the older method argue that it is necessary to master their facts before the newer ideas can be grasped. They also urge that their material, if thoroughly learned, will occupy all the time allotted; and that any attempt to introduce additional ideas will only result in lack of mastery and consequent confusion. The leaders of the innovation reply that the natural result of progress in mathematics, as in other subjects, is gradually to replace the less important old facts by the more important new ones. They also insist that some of the new ideas will tend to unify the mass of metrical facts.

In looking over a new book on analytical geometry, it is natural, therefore, to inquire whether there is any departure from the conventional body of material and whether there is any striking feature in the method of presentation. There was a strong tendency for some years to introduce into the elementary texts on analytical geometry, as well as those on calculus, many more or less complicated problems from various subjects, especially from physics and statistics. The discussion of loci and graphs gave an excuse for prolonged excursions into domains bordering on the mathematical territory. That a moderate use of such problems is stimulating to thought and is necessary to give some idea of the practical applications of mathematics is probably not often questioned. However, in some cases at least, this practice was undoubtedly overdone. In the present text almost the opposite extreme is found. There are practically no problems showing the applications of mathematics. In Chapter XI, on equations and graphs of certain curves, the path of a projectile is discussed. Perhaps there may be other applications, but they were not apparent in a first survey. There are places where such problems and illustrations are particularly useful. For example, in the first chapter, on coordinates, some illustrations of the use of coordinates and coordinate paper in the charts employed in many businesses to-day would add life to the chapter, without in any way lessening its dignity.

The authors state that it is in deference to usage that the chapter following that on the straight line is devoted to the

circle. They urge that it would be better to omit this until the chapters on the parabola and ellipse have been studied. Their reason is that thus "the student sooner realizes the power of the method of the coordinate geometry through seeing it employed in investigating *new* material." On the other hand, in justification of the retention of a short chapter on the circle in this place, it may be argued that, as the *method* is new, the student must gain some facility in its use by employing it on old material before he can have sufficient mastery of it and confidence in it to enable him to use it easily and naturally on the new material. In the treatment of the conics, either of two methods is possible. According to one, the discussion of conics in general precedes the briefer consideration of the particular properties of the different species. In the other method the order of presentation is reversed. There are, of course, arguments for and against each method. The second method is that employed in the present book, where Chapters IV, V, and VI deal with the parabola, ellipse, and hyperbola respectively, while conics in general are not discussed till Chapter VIII.

The treatment of tangents and poles and polars is always an interesting subject for discussion. One criticism that might be made on this book is that there is too much repetition in the subject of tangents. In the cases of the parabola and ellipse, three methods for finding the equation of the tangent in terms of the coordinates of the point of contact are worked out in detail. For the hyperbola the corresponding equation is derived from that for the ellipse. Would it not have been wiser for the authors to have selected one of these methods and omitted the other two? The teacher whose chief interest is in engineering students would probably wish the calculus introduced in the derivation of the tangent to the first conic. This, of course, brings up the ever old and ever new question of whether analytics shall be treated without the aid of calculus, or whether an attempt shall be made to combine the two subjects. As the present text-book is evidently intended for students who expect to have a course in calculus later, there is little use made of its symbols here. Nevertheless, they are not entirely neglected. In one of the chapters on solid geometry, after the equation of the tangent plane to a conicoid in terms of the coordinates of the point of contact has been obtained, the statement is made that the coefficients may be represented by the partial derivative symbols (provided the variables are replaced by the coordinates

of the point of contact). Again Table C at the end of the book is a one page explanation of derivatives and partial derivatives.

Poles and polars are not discussed in the separate chapters on the circle, parabola, ellipse, and hyperbola, nor in the chapter on the general equation of the second degree. They appear for the first time in Chapter IX, on tangents and polars of the conic. After the equation of the tangent to any conic in terms of the coordinates x', y' of the point of contact has been obtained, the statement is made that the straight line represented by this equation is called the polar of the point (x', y') with reference to that conic, whether point (x', y') lies on the conic or not. For the circle, it is shown that if the polar of P_1 contains P_2 then the polar of P_2 contains P_1. A solution is given for finding the pole of a given line with reference to a circle. It is also noted that the polar of a point outside a circle is the chord of contact for the tangents from that point. The corresponding fact if the point is inside the circle is stated. No mention is made of harmonic properties. It is to be noted that poles and polars hold a very insignificant place in the plan of this book.

In view of the recent discussions on the question of the place of loci in college entrance examinations, it is interesting to notice the treatment of this subject in an elementary text intended for the early part of a college course. In addition to the illustrations of loci found in the several conics and in their diameters and in certain of the special curves, the last chapter in the plane geometry is devoted to this subject. Nine examples are fully explained and then, after some general remarks on loci, fifty carefully graded problems are given. The idea of loci is certainly made prominent.

Solid geometry occupies more than one third of the volume. This is a more extended treatment than is usually found in the elementary texts. In the present book the first two chapters, on coordinates and direction cosines and on planes and straight lines, are very similar to corresponding chapters in C. Smith's Elementary Treatise on Solid Geometry. These two chapters occupy slightly less than one half of this part of the book. After a discussion of the general shape and the sections of the different species of conicoids and a brief survey of polar coordinates and transformation of coordinates, a chapter is devoted to the general equation of the second degree. Centers and diame-

tral and principal planes and the classification of conicoids are then briefly treated. There is a short discussion of the invariants of the general equation under a transformation from one orthogonal system of axes to another orthogonal system.

The text proper is followed by six short tables, which deal with algebraic and trigonometric formulas, derivatives and partial derivatives, four-place table of logarithms of a few numbers, lengths of arcs in radians, and the letters of the Greek alphabet. The book is concluded by a set of nine very good plates showing the silk thread figures of the ruled surfaces of the second order, as well as the usual plaster models of the conicoids. Indeed, one of the best features of the book is to be found in the excellence of the numerous figures. The young man in whose hands this text is placed will probably note first of all that it is small enough to fit into his pocket. By employing rather thin backs and paper that is not too heavy and by lessening the margins, the size and weight of the volume have been reduced to a minimum. Perhaps the strongest feature of the book is to be found in the abundant supply of examples. After each bit of theory there are some exercises, and at the end of each of the longer chapters there is a set of about fifty carefully graded problems.

E. B. COWLEY.

Leçons sur les Fonctions définies par les Equations différentielles du premier Ordre. Par PIERRE BOUTROUX. Paris, Gauthier-Villars, 1908. 190 pp.

THE little volume bearing the above title is one of the series of monographs on the theory of functions published under the editorship of E. Borel. The author's aim is to set forth the theory of functions defined by a differential equation as based on the work of Painlevé. He abandons the "local point of view" of Cauchy and studies the ensemble and form of the integral not only in the neighborhood of a point but in general. The particular question discussed is one raised by Painlevé, viz., how does the solution behave when the initial point x_0 at which it is considered varies from point to point in an arbitrary manner.

The book is divided into five chapters. Chapter I presents the fundamental notions. After a review of the usual theory of singular points the following theorem of Painlevé's is dem-

onstrated: As x varies from x_0 to $\bar{x}$ along a curve L joining x_0 to $\bar{x}$ an integral Y of the equation

$$(1) \qquad \frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}$$

(P and Q being polynomials in y) which takes in x_0 the value C will approach a definite value.

From this theorem the following one is deduced at once: Aside from fixed singular points, an integral of (1) can have only poles or algebraic critical points for singularities.

Another important theorem also due to Painlevé is given in this chapter: If P and Q are polynomials in y and algebraic in x and if the integrals are multiform functions of n branches then equation (1) can be reduced to Riccati's equation by a rational change of variable. One can always determine by a finite number of operations if this change of variable is possible. A discussion of multiform functions of an infinite number of branches and the corresponding Riemann surfaces closes the chapter.

Chapter II deals with the growth and behavior of a branch of an integral. The subject here discussed is the manner in which an integral increases as x increases. The problem is to determine whether an integral increases more slowly or more rapidly than a power of $|x|$ or than an exponential function. The subject is treated in quite a fascinating manner, but the presentation must be read in order to be appreciated.

Chapter III is on classification of singularities. Transcendental singular points are divided into two classes, those directly critical and those indirectly critical. A point x is directly critical if when an elementary circuit is described about x an infinity of branches are permuted. It is indirectly critical if an infinity of branches are permuted when an elementary circuit is described not about x properly speaking but about an infinity of critical points which are converging toward x as a limit. Following the above definitions the singular points of the equation

$$(2) \qquad 2z\frac{dz}{dx} = \alpha x + \beta z$$

are discussed in considerable detail under various assumptions, and the manner in which the branches permute themselves is shown.

After a discussion of other particular examples a few pages are devoted to a more minute classification of transcendental singularities.

Chapter IV considers the singular points of Briot and Bouquet. The equation which has singularities of this type is connected with the equation above by introducing a parameter μ which put equal to zero gives equation (2) and put equal to unity gives the equation under consideration. By allowing μ to vary, the intimate relation between the singularities of equation (2) and those of Briot and Bouquet is established.

Chapter V discusses some of the relations which exist between the singularities of the same equation.

The volume closes with a note of fifty pages by Painlevé: "On the differential equations of the first order whose general integral has only a finite number of branches."

C. L. E. Moore.

Sur les premiers Principes des Sciences mathématiques. Par P. Worms de Romilly. Paris, A. Hermann, 1908. 8vo. 51 pp. 2.50 fr.

This essay undertakes to give an account of the recent work on the foundations of mathematics. The author concludes that the only branch of mathematics completely applicable to natural phenomena is arithmetic, since it depends solely upon the numeration of objects, and makes no hypothesis regarding their nature. Geometry, on the other hand, imposes upon them certain purely ideal hypotheses which indeed may differ so as to produce at least three systems of geometry, the system which nature is built upon being possibly that of Euclid, possibly otherwise. The contrast drawn here between the external validity of arithmetic as over against that of geometry is a little difficult to reconcile with the explanations devoted by the author to the varying systems of axioms on which arithmetic may be based. In fact he distinctly speaks of diverse systems of numeration. We might inquire, for example, are objects subject to the archimedean axiom or not?

A disproportionate amount of space is devoted to the setting forth of some seven foundations upon which geometry may be based, and not quite so much to mechanics. The reason for this is the underlying thesis which the author seeks to prove. He examines the different modes of grounding geometry and concludes they are all à priori and inapplicable to real objects

until certain other unprovable results of intuition are brought into play. Exactly what an external object consists of aside from its being a projection of an internal idea is not shown. And if the world as we conceive it is merely a projection of that which is wholly mental, then why so much struggling to prove the geometrical character of the world as we geometrize it? Or on the other hand, why such a certainty of its arithmetic as we arithmetize it?

The definition given by C. S. Peirce for mathematics has not been surpassed: "The study of ideal constructions (often applicable to real problems), and the discovery thereby of relations between the parts of these constructions before unknown." This implies the rôle of logic and of intuition in the architecture of this vast structure. And in a projection of two figures is A the projection of B, or B of A? Is the world framed according to the architecture or the architecture according to the world? Qui sait!

James Byrnie Shaw.

Taschenbuch für Mathematiker und Physiker. 1909. Von Felix Auerbach. Leipzig, Teubner. 1909. xliv + 450 pp. 6 Marks.

This little pocket manual initiates a series of year books to be issued by the firm publishing it. They are to be congratulated upon their enterprise in furnishing the mathematical public what it has long needed. The engineer has his Trautwine, Kent, Kidder, or Foster, but so far the mathematician has had only collections of integrals, or small collections of trigonometric formulas. This volume, on thin but opaque paper, with typography which is delightfully clear, contains not only an excellent summary of the whole field of mathematics, but also a resumé of mechanics, physics, and physical chemistry. One is much surprised and pleased at the amount of valuable material compressed into so small a space, yet so easily found. The chief formulas and definitions are to be found here for arithmetic, algebra, group theory, combinatory analysis, determinants, series, differential calculus, integral calculus, definite integrals, calculus of variations, differential equations, transformation groups, functions of a real variable, functions of a complex variable, gamma function, elliptic integrals and functions; principles of geometry, topology, planimetry, stereometry, goniometry, plane trigonometry, spherical trigonometry; coordinate

geometry, lines in a plane, general plane curves, conics, cubics, general space geometry, algebraic surfaces, families of surfaces, quadrics, twisted curves ; line geometry, transformations in a plane ; differential geometry of the plane and of space; probabilities, errors, numerical calculations, graphics, vector analysis and quaternions. The other headings mentioned are equally fully treated. Many topics have a place assigned for future exposition. In a few years this small encyclopedia will be almost a necessity for student, teacher, and investigator.

The book is illustrated with a portrait of Lord Kelvin, and his biography opens the introduction. The properly "yearbook" topics are a calendar, astronomical data, lists of journals and proceedings and transactions of learned societies, new books, necrology for 1908, lists of teachers of mathematics and physics in Germany. The errors in the book are few, so far as the reviewer noticed in his reading; those existing are noticeable at once and doubtless will disappear in the next volume.

JAMES BYRNIE SHAW.

Statistique Mathématique. Par H. LAURENT. Paris, Octave Doin, 1908. vi + 272 + xii pp.

THE author states that, for him, the object of mathematical statistics is to indicate and investigate methods of making good observations, when the point in question is to make numerical estimates concerning matters which interest economists. He has thus limited his purposes to matters which relate to specific applications. This fact may account, in part, for the entire omission of that important body of mathematical statistics which has been developed in close connection with applications to biology. However, these methods have been applied by others to problems of economics.*

It is well stated in the preface that it is a very common error to suppose that those who direct statistical investigations do not need to know mathematics. The author remarks that official statistics are not good, in general, because those who direct statistical investigations are not prepared for the work, and that if it is not necessary to exact of the statistician that he have a command of universal science, it is necessary, at least, that he should have surveyed the field of scientific knowledge.

*See Yule, *Journal of the Royal Statistical Society*, vol. 60, pp. 812–854. Norton, Statistical Studies in the New York Money Market.

The book begins with the elements of probability — leading soon to the principle of Bayes on the probability of causes. In the proof of the inverse of the theorem of Bernoulli, it may be worth while to call attention to a few details which may confuse the reader: If l is to be used as indicated on page 28 as a deviation from a/s, instead of using l/s as on page 26 for this purpose, the expression $P=\theta(ls/\sqrt{2\alpha\beta s})$ on page 30 becomes $P=\theta(l\sqrt{s^3/2\alpha\beta})$. The expression $P=\theta(ls/\sqrt{2\alpha\beta s})$ is correct if l/s is used throughout as a deviation from a/s. On page 29, in the formula for P, the radical should be $\sqrt{s/2\alpha\beta\pi}$ instead of $\sqrt{\alpha\beta/2\pi s}$.

On page 37 there appears to be a numerical error such that we should read $\theta(l/628)$ instead of $\theta(l/6280)$ but the conclusion is all the more valid if this change is made.

In developing what he calls a general theory of errors, the author follows the well-known method of obtaining the normal curve from the hypothesis of Gauss in regard to the arithmetic mean. In presenting the method of least squares, the analysis of Laplace is followed rather closely, and the approximation consists in neglecting

$$\int_{-\infty}^{\infty}\phi(\epsilon)\,\epsilon^h d\epsilon \quad (h>2),$$

where $\phi(\epsilon)d\epsilon$ is the probability (facilité) of an error between ϵ and $\epsilon+d\epsilon$, but no further assumption is made in regard to the form of the function $\phi(\epsilon)$. The integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{\mu(t-a)\sqrt{-1}}\,\frac{\sin\mu l}{\mu}\,d\mu$$

is used as an auxiliary function to evaluate the integral which expresses the total probability of the concurrence of errors.

The greater portion of the book is given to applications rather than to pure theory. The various sources of statistics in France are indicated, and conditions are discussed which must be satisfied in order that an observation which is known only by description may be of value. Agricultural statistics which report crops of grain to seven significant figures are severely criticised in the following terms: "As there is no cause to suspect the honesty of those who publish these figures, it is necessary to infer their ignorance in scientific matters or their native sim-

plicity, and then only natural to doubt the exactness of the first figures."

It is stated that of all statistics relative to man the best and perhaps the most important deal with the duration of human life and with immigration. The book contains the well known methods (aside from the method of moments) of graduating mortality statistics, with the somewhat reasonable à priori considerations which lead to the various types of functions used in such graduations.

An argument is given to establish that part of the law of Malthus

$$p = p_0 e^{k(t-t_0)}$$

which states that populations have a tendency to increase in geometrical progression, but of the part stating that subsistence tends to increase in arithmetical progression, the author says he does not see how this can be established.

In studying the assessment of taxes for Prussia, and also for other countries that have adopted the same method of collection, M. Pareto arrived at the following law: If N represents the number of individuals who have an income over x, then

$$N = Ax^{-\alpha}e^{-\beta x},$$

or less exactly (β being very small),

$$N = Ax^{-\alpha},$$

A, α, β being independent of x and of N.

It is stated that it is probable that this formula is applicable to all countries, and some arguments are given in favor of the law.

To the reviewer one of the most interesting sections of the book consists in a mathematical argument from which it results that under free competition prices are determined to the greatest satisfaction of all concerned in the transactions. This argument is based on the assumption that an individual retires from a market satisfied when a certain mathematical relation

$$\phi(q_1, q_2, \cdots, q_s) = a$$

exists between the quantities of merchandise $q_1, q_2, \cdots, q_s$ which he is engaged in exchanging. The value of this assumption might be questioned. The function $\phi(q_1, q_2, \cdots, q_s)$ is called

by M. Pareto the ophélimité of the individual considered. Under a monopoly, the prices are again mathematically determined but not to the greatest satisfaction of all engaged in the transaction, but to the advantage of the proprietors of the monopoly. If there is a maximum of "ophelimité," the cost of production is equal to services rendered in production.

Many other applications are given in the book. Among these should be mentioned Cournot's theory of exchange, the rôle of the theory of games of chance in statistics, which includes the questions of annuities and insurance.

Taken as a whole, the book is useful for the clearness of presentation as well as for the numerous applications to economic theory. While the reviewer would expect a treatise on statistics to contain more recognition of the recent work of Karl Pearson and those associated with him, the present book contains much valuable material for the student of mathematical statistics.

H. L. RIETZ.

Σώζειν τὰ Φαινόμενα. Essai sur la Notion de Théorie physique de Platon à Galilée. Par P. DUHEM. Paris, A. Hermann et Fils, 1908. 144 pp.

OFTEN, when fatigued with the perplexities of modern physics or the intricacies of modern mathematics, it is a pleasant change to take a dilettante interest in the science of the ancients, to draw an optimistic courage from the progress twenty centuries have made or a pessimistic cheer from the little that so long a time has won. Then a volume of Pliny or parts of Plutarch's works suggest themselves — in a translation, alas! despite or to spite eight years of Latin and six of Greek. There we can find a dissertation on flesh eating which reads like some of all too recent date or a disquisition on the moon and her inhabitants that seems quite modern Martian. The philosophers who live much by and with and for the Greeks have collected, collated, and translated the words of philosophic wisdom of these ancients. If such a collection should be made for science with some appropriate comments relative to our present point of view, a highly entertaining book could be printed. Perhaps Duhem will sometime get to this; his present work with its Greek title and French subtitle is merely an essay on the conception of physical theory from Plato to Galileo — and by physical theory is apparently meant only such as regards astronomy, the best developed of the Greek physical sciences.

Duhem points out that at the very outset in the days of Plato and Aristotle there were two distinct and largely contrary attitudes toward astronomical science. The first was the mathematical (or astronomical) point of view that the test of a hypothesis relative to the motion of the planets was merely whether or not that hypothesis enabled one to describe the facts — *σώζειν τὰ φαινόμενα.* The second was the metaphysical (physical) point of view which called upon a hypothesis to conform to the real nature of things — *κατὰ φύσιν* — whatever that may mean. Of course it is at once apparent to everybody that there probably always have been, surely are now, and very likely always will be these two points of view; that the scientist will incline to the first and the metaphysician to the second; that some will be content to make use of nature as best they can, while others will not be satisfied unless they make nature according to their own best ideas. Not only do the two points of view correspond to two different types of mind; they appeal to different moods of the same mind. A new theory seems naturally a convention and an old theory equally a reality. Osiander might well maintain in his preface to Copernicus's work that the Copernican view was merely a clever but unreal way to explain appearances. Yet it was not long before the adherents of the theory were willing to make martyrs of themselves for its reality and still to maintain: E pur si muove! It probably takes almost as much courage now-a-days to maintain that "the earth moves" means merely that "it is more convenient to assume that the earth moves."

Throughout the essay Duhem carefully traces the history of the conflict between the scientific and the metaphysical view. That the text is readable and entertaining may be taken for granted when it is written by the author of such a variety of well known works. It is interesting to note that during the fourteenth, fifteenth, and sixteenth centuries the masters at the Sorbonne set forth views on physical theory which were better and deeper than any heard up to the middle of the last century. The author's conclusion is also noteworthy, namely: En dépit de Kepler et de Galilée, nous croyons aujourd'hui, avec Osiander et Bellarmin, que les hypothèses de la Physique ne sont que des artifices mathématiques destinées à sauver les phénomènes; mais grace à Kepler et à Galilée, nous leur demandons de sauver à la fois tous les phénomènes de l' Univers inanimé. Perhaps there are still a number of unenlightened

physicists who can not take quite this view with respect to our more firmly fixed theories of physics; probably the majority of metaphysicians would not acquiesce. It may be that some day reality will consist merely of conventions, or it may be that the pendulum will again swing to the other side and make the conventions real. At any rate much will still be spoken and written on both sides of the question.

E. B. WILSON.

The Slide Rule. An Elementary Treatise. By J. J. CLARK. Technical Supply Company, Scranton, Pa., and New York. 62 pp.

ONE familiar with algebra and logarithms usually needs little instruction in the use of the slide rule. He needs only practice in reading results accurately and expeditiously. On the other hand a person ignorant of logarithms finds the mastery of the instrument a more difficult task. The author of this booklet directs his attention to the wants of the latter class. His aim is to give directions for the use of the slide rule, so simple and explicit that any pupil with a fair knowledge of arithmetic can understand them. In this laudable purpose the author has been eminently successful. The booklet is a model of clear exposition.

The author confines his attention to two slide rules, the Mannheim rule and the Rietz rule. The term "Mannheim rule" has become generic. The Mannheim type is now used more than any other for ordinary purposes, and is manufactured by many firms in different countries. The name Rietz is attached to a specific rule, manufactured by the firm of Albert Nestler in Lahr, Baden. The Rietz rule is one of the very numerous rules with the Mannheim arrangement of the lines A, B, C, D, to which one or more other lines are added (in this case the E line for cube root, etc.). Just why this Rietz rule should have been selected out of a very large number of similar domestic and foreign makes is not quite evident.

The author gives nothing on the history of the slide rule. It is perhaps just as well that no attempt should have been made in this line. Only very recently have I been able to settle the long-disputed question as to the inventor of the straight-edge slide rule.* Mr. Clark gives in his book just

* "History of the logarithmic slide rule," Engineering News Publ. Co., New York, 1909.

four words of biographical information, but in so doing he comes to grief. Mannheim was not "a German army officer." As a young man he was a French army officer, and for about half a century he was professor in the Polytechnic School in Paris.

FLORIAN CAJORI.

Annuaire du Bureau des Longitudes pour l'An 1910, Paris. Gauthier-Villars.

THIS handy little volume makes its annual appearance as usual in good time. So far as we know there is nothing approaching to it in completeness for giving the astronomical, physical, electrical, meteorological, magnetic, and chemical constants. And what is of even more importance, everything is easy to find and the matter is constantly kept up to date. For the construction of problems in which one wishes to insert numerical data, the hunting in various books can be avoided and much time saved by having the *Annuaire* at hand. Two consecutive volumes give everything likely to be needed.

The principal appendix this year is an article on earth tides and the elasticity of the globe by M. Ch. Lallemand. A full historical survey of our knowledge up to the present time forms one of its most useful features. M. B. Baillaud tells of the work done at the last international conference on the photographic chart of the sky, and M. G. Bigourdan appends a list of all the Notices which have appeared in the *Annuaire* since its foundation in 1796.

ERNEST W. BROWN.

NOTES ON THE INSTITUT DE FRANCE AND THE ANNUAL MEETING OF THE ACADÉMIE DES SCIENCES.

IT is not generally known in America that the Institut de France is made up of five distinct bodies, known as the Académie Française, the Académie des Sciences, the Académie des Inscriptions et Belles-Lettres, the Académie des Sciences Morales et Politiques and the Académie des Beaux-Arts. Each of these academies has 40 members, with the exception of the Académie des Sciences which — in addition to its two permanent secretaries, one in each of the departments, physics and

mathematics — has 66 members, six in each of the eleven sections: geometry, mechanics, astronomy, geography and navigation, general physics, chemistry, mineralogy, botany, agriculture, anatomy and zoology, medicine and surgery. Each academy appoints its own officers, elects its own members and each, with the exception of the Académie Française has, in addition to its regular members, "académiciens libres," "associés étrangers," and "correspondants." The term "forty immortals" is exclusively applied to the members of the Académie Française, which however is much less eminent in the matter of erudition than its less widely celebrated fellows. Its chief official function is to prepare a dictionary, for which it has no exceptional philological equipment. The other academies are extremely active and have great influence on the progress of learning. The weekly *Comptes Rendus* edited by the secrétaires perpétuels and published by the Académie des Sciences is known to all mathematicians.

The following are the regular academicians in the sections of geometry, mechanics, and astronomy:

Geometry: Appell, Humbert, Jordan, Painlevé, Picard, Poincaré.

Mechanics: Boussinesq, Deprez, Léauté, Levy, Sebert, Vieille.

Astronomy: Baillaud, Bigourdan, Deslandres, Hamy, Radau, Wolf.

Mathematics gets another representative through the fact that Darboux is a secrétaire perpétuel.

By way of closing this introductory note, it may be pointed out that the term Académie des Sciences is also used in quite a different sense, namely, as one section of the Académie de Paris or the Académie de Bordeaux or any other of the sixteen groups of educational institutions into which practically all schools (primary, secondary, and superior) in France are divided.

The annual meeting of the Académie des Sciences occurred on December 20, in the ancient circular assembly hall adorned with statues of Fenelon, Bossuet, Descartes, Sully, and Henri d'Orleans, Duc d'Aumale. An audience of several hundred had assembled when on the stroke of one o'clock the drums began to roll and the four gorgeously arrayed officers of the Académie (President Bouchard, Vice-President Émile Picard, Secrétaires perpétuels Darboux and Van Tieghem), followed by some fifty of the academicians, took their places. Throughout the two-hour programme, the officers remained seated behind

the table on the raised platform and occasionally refreshed themselves from the conspicuous carafe of water and bowl of sugar. With the words "La séance est ouverte," the president, who is a member of the Faculty of Medicine in the University of Paris, commenced to read his "allocution." Before proceeding to his main theme, which was a discussion of the decrease in birth rate of the French population, he announced that the only break in the ranks of the academy during the year was caused by the death of their illustrious "associé étranger," Simon Newcomb. Drawing freely from his "Reminiscences of an Astronomer," which he compared with the autobiographies of J. J. Rousseau and Franklin, M. Bouchard sketched Newcomb's early career down to the age of 23 when he was made Docteur ès Sciénces. He then concluded as follows: "I can not follow him in his glorious career as astronomer. His discoveries of the retrograde movement of Hyperion, his theories of the four inner planets suffice to place him in the first rank. He is considered as the follower of Leverrier . . . . We elected him our associé étranger in 1903." From the representative of one of the greatest scientific societies in the world, one might well expect greater accuracy of statement. Newcomb received the degree of *Bachelor* of Science from Harvard in 1858, and was elected one of the eight associés étrangers on June 17, 1895, in place of von Helmholtz. Franklin was the only American to be so honored previously, and Agassiz alone remains now. By a decree of December 1, 1909, the number of associés etrangers was increased from eight to twelve.

M. Van Tieghem now announced the award of medals and money prizes totalling close upon 235,000 francs. The awards as far as they were of interest to mathematicians are as follows: Prix Francoeur (3,000 fr.) to M. Émile Lemoine for his mathematical works.— Prix Bordin (3,000 fr.). The problem proposed for the prize was: "The absolute invariant which represents the number of double integrals of the second kind on an algebraic surface depends upon a relative invariant ρ, which plays an important part in the theory of integrals of total differentials of the third kind and in the theory of algebraic curves traced upon the surface. It is proposed to make an exhaustive study of this invariant and to determine in particular how its exact value can be found, at least for an extended category of surfaces." Competing memoirs were to be in the hands of the secretary before January 1, 1909, but the prize was awarded to

Professors Guiseppe Bagnera, of the University of Palermo, and Michele de Franchis, of the University of Catania, for their memoir entitled "Le nombre ρ de M. Picard pour les surfaces hyperelliptiques et pour les surfaces irregulières de genre zéro," which did not arrive until January 15. It was stated that this case must not be thought to create a precedent, the circumstances being as follows: "The memoir was sent from Palermo on the evening of December 27, 1908, and it was in the following night that the awful catastrophe overtook Messina. The postal car containing the manuscript was in the station of that city at the moment of the earthquake, and nearly three weeks elapsed before it was despatched further." — Prix Montyon (700 fr.) to M. Lecornu, chief engineer of mines, professor in the École Polytechnique, for his work entitled Dynamique appliquée. — Prix Poncelet (2,000 fr.) to M. de Sparre for his studies relating to gunnery and his works on mechanics. — Prix Vaillant (4,000 fr.) for the subject: "To perfect in an important point the application of the principles of the dynamics of the fluids to the theory of the helix" was not awarded. — Prix Boileau (1,300 fr.) to M. Boulanger, assistant professor of mechanics, University of Lille, for his work entitled Hydraulique générale. — Prix Lalande (1,000 fr.) to M. Borrelly, assistant astronomer, observatory of Marseilles, for his discoveries of small planets and comets. — Prix Valz (460 fr.) to M. de la Baume-Pluvinel for his astronomical works. — Prix G. de Pontécoulant (700 fr.) to Professor E. W. Brown, of Yale University, for his researches relative to the theory of the moon. — Prix Binoux (2,000 fr.) to M. Pierre Duhem, correspondent of the academy, for his works relative to the history of the sciences. — Fondation Leconte. 1,200 fr. to M. Ritz for his works in mathematical physics and mechanics; 2,000 fr. to M. Lebeuf, director of the Besançon observatory for his chronometric and astronomical works and in particular for his part in the publication of the works of Laplace. — Prix Laplace (Laplace's works) to M. A.V.É. Vaucheret, leader of the graduating class at the École Polytechnique. — From the Bonaparte Fund 2,000 francs were granted to M. Estanave, D. Sc. Math., secretary of the Société mathématique de France, and author of several bibliographical and mathematical works, for continuing his researches "sur la projection stéréoscopique à vision directe, sur la stéréo-radiographie et sur l'autostéréoscopie;" and 2,000 francs to M. Mathias, professor

of physics at the University of Toulouse, to permit him to pursue, in the Leyden laboratory, with its eminent director, M. Kammerlingh Onnes, researches on the rectilinear diameter of liquids and on the law of state at very low temperatures.

The following prizes to be awarded in 1910 and 1911 were announced:

1910. Grand Prix (3,000 fr.). "It is known how to find all systems of two meromorphic functions of one variable, connected by an algebraic relation. An analogous problem is proposed for three uniform functions of two complex variables having for finite regions the character of rational functions and being connected by an algebraic relation. In case a complete solution is not obtained, the Academy demands at least some illustrative examples which lead to classes of new transcendental functions." — Prix Poncelet (2,000 fr.) for a work in pure mathematics. — Prix Francoeur (1,000 fr.) for a work of merit in pure or applied mathematics. — Prix Fourneyron (1,000 fr.) for an experimental and theoretical study, in hydraulics, of the effects of the shocks (coups de bélier) given to elastic tubes. — Prix Janssen, biennial gold medal for the discoverer or the work making important advancement in physical astronomy.

1911. Prix Bordin (3,000 fr.). "To perfect in an important point the theory of triply orthogonal surfaces." The Academy desires to add to the list of systems already known, and it attaches a particular prize to the discovery of the triply algebraic systems which are most simple. — Prix Vaillant (4,000 fr.). To perfect in some important point the study of the movement of an ellipsoid in an infinite viscous liquid. — Prix Damoisea To perfect Leverrier's Tables of Jupiter.

Darboux now read his long but interesting "historical notice" of General Jean Baptiste Marie Charles Meusnier (1754–1793), member of the Academy of Sciences. His scientific and military career were traced in detail and the celebrated theorem concerning the curvature of oblique sections of surfaces, with which his name is always associated, was assigned to the year 1776. The s in Meusnier's name is frequently omitted by English writers. May it now be correctly preserved.

With the President's announcement "La séance est levée" the meeting came to an informal ending.

R. C. ARCHIBALD.

PARIS, *December* 24, 1909.

NOTES.

THE opening (January) number of volume 32 of the *American Journal of Mathematics* contains the following papers: "The complementary theorem," by J. C. FIELDS; "The twelve surfaces of Darboux and the transformation of Moutard," by L. P. EISENHART; "On the problem of the spherical representation and the characteristic equations of certain classes of surfaces," by A. E. YOUNG; "The general circulation of the atmosphere," by F. R. SHARPE; "Generalizations of the tetrahedral and octahedral groups," by G. A. MILLER; "The theory of degenerate curves and surfaces," by O. E. GLENN.

THE January number (volume 11, number 2) of the *Annals of Mathematics* contains the following papers: "Necessary and sufficient conditions that ordinary differential equations shall admit a conformal group," by L. I. HEWES; "The three-space projective geometry (3, 2) and its group," by G. M. CONWELL; "The geodesic lines on the helicoid," by S. E. RASOR: "Cubic congruences with three real roots," by E. B. ESCOTT.

AT the meeting of the London mathematical society held on January 13 the following papers were read: By H. BATEMAN, "The transformations of coordinates which can be used to transform one physical problem into another"; by W. H. YOUNG, "On homogeneous oscillation"; by W. H. and Mrs. G. C. YOUNG, "On the determination of a semicontinuous function for a countable set of values"; by G. H. HARDY, "Note on a former paper on the theory of divergent series"; by H. F. BAKER, "On the expression of a certain function by means of a series of polynomials," and "On the theory of the cubic surface"; by G. N. WATSON, "The harmonic functions associated with the parabolic cylinder."

THE British mathematical association and the Association of public school science masters have appointed a joint committee to consider the possibility of correlating the teaching of mathematics and the sciences. The committee made its report on January 12, but the associations wish to consider it further before the contents are made public.

THE Euler commission announces that the editorial committee will consist of Professor F. RUDIO, of the technical school at Zürich, chairman, and Professors A. KRAZER and P. STAECKEL, of the technical school at Karlsruhe. The first volume of the works will be devoted to algebra and will be edited by Professor H. WEBER, of the University of Strassburg. As now planned, the entire series will consist of 43 quarto volumes of about 500 pages each. The 18 volumes in pure mathematics are distributed as follows: arithmetic and algebra, 5 volumes; analysis, 11 volumes; geometry, 2 volumes. Mechanics occupies 11 volumes.

THE list of German doctorates in mathematics for the years 1906–1908 which appeared in the February number of the BULLETIN was compiled by Mr. DUNHAM JACKSON from the Verzeichnis der an den deutschen Universitäten erschienenen Doktor-Dissertationen und Habilitationsschriften. A continuation of the list will appear in an early number of the BULLETIN.

THE Central Committee of the International Commission on the teaching of mathematics, consisting of Professors Klein, Greenhill, and Fehr, held a meeting at Basel on December 28, 1909. Reports received on the present state of the work in the eighteen participating countries showed that the investigation was proceeding with much energy and that it had enlisted the services of a large number of prominent teachers and mathematicians. The Central Committee will hold its next meeting at Brussels about the middle of August, 1910. It is hoped that the international exhibition to be held there at that time will bring together a considerable number of official delegates of the commission.

THE annual meeting of the council of the American Federation of teachers of the mathematical and natural sciences was held on Monday, December 27, 1909, at the Massachusetts Institute of Technology, Boston, Mass. Twenty-two representatives of eight associations were present. The report of the executive committee showed that six associations had joined the Federation during the preceding year. The total number of paid up members in the associations belonging to the Federation is now 2040. Reports were presented from the local associations showing activity and progressive work in all. The committee on a syllabus in geometry reported that work was well

under way. The committee has been divided into three subcommittees, one on logical considerations, one on lists of basal theorems, and one on exercises and applications. It expects to have its work completed during the present year.

The committee on college entrance requirements had gathered a large amount of information which showed a great variation in the requirements of the different colleges, making it impossible for any school to meet them all. The committee recommended that the Federation take up this matter with the College entrance examination board and see what can be done toward bringing about uniformity. The report was accepted and the committee continued.

It was urged that reports of local meetings be published in such periodicals as would best serve the associations concerned.

A committee was appointed to cooperate with the College entrance examination board to determine the best forms of logarithmic tables to be used at examinations.

The question of the publication of a journal for mathematics alone was discussed at some length, and it was voted that a committee be appointed to consider this question and report at the next meeting.

Informal reports of progress were presented by members of the International commission on the teaching of mathematics.

The following officers were elected for the present year: President, Professor C. R. MANN, of the University of Chicago; secretary and treasurer, Professor E. R. SMITH, of the Brooklyn Polytechnic Institute; and three other members of the executive committee. This committee will determine the time and place of the next meeting.

STATISTICS presented by Professor A. SCHOENFLIES in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* show that the number of students of mathematics (excluding foreigners) in the Prussian universities has increased from 1,440 in 1907 to 1,730 in 1910, the latter number exceeding all previous records. At present the supply of available graduates meets about one-half the demand for teachers, but warning is given that less favorable conditions are likely to result from a continued increase in the number of students.

PROFESSOR R. DEDEKIND, of the technical school at Braunschweig, has received the honorary degree of doctor of mathematics from the technical school at Zürich.

Professor E. Schmidt, of the University of Zurich, has been appointed to a full professorship of mathematics at the University of Erlangen.

Dr. I. Schur, of the University of Berlin, has been promoted to an associate professorship of mathematics.

Miss E. Greene has been appointed mathematical tutor at Bedford College, of the University of London.

Mr. A. J. Kenny has been appointed assistant lecturer in mathematics at the University of Birmingham.

Mr. J. H. Sleeman has been appointed lecturer in mathematics at the University of Sheffield.

Mr. M. J. Conran has been appointed lecturer in mathematical physics at the University College of Cork.

Professor W. W. Campbell, of the Lick Observatory, delivered at Yale University, on the Silliman foundation, in the week beginning January 24, a course of eight lectures on "Stellar motions."

During the week February 4–10, Professor Carl Runge, of the University of Göttingen, delivered a course of five lectures on graphical methods at the University of Michigan.

Professor C. J. Keyser, of Columbia University, is spending a half-year abroad, on leave of absence.

Dr. E. G. Bill, of Yale University, has received leave of absence for the coming academic year, which he will spend in the study of geometry at the University of Turin.

Professor L. A. Wait, head of the department of mathematics at Cornell University, will retire from active service at the close of the present academic year.

Edward A. Bowser, professor emeritus of mathematics and engineering at Rutgers College, died at Honolulu on February 29 at the age of sixty-five years.

Professor H. B. Newson, of the University of Kansas, died on February 18, at the age of fifty years. Professor Newson graduated from Ohio Wesleyan University in 1883. Later he studied at Johns Hopkins, Heidelberg, and Leipzig. He became associate professor of mathematics at the University

of Kansas in 1890, and was promoted to a full professorship in 1905. He had been a member of the American Mathematical Society since 1895.

PROFESSOR J. E. WRIGHT, of Bryn Mawr College, died on February 20 after an illness of several months. Professor Wright was senior wrangler of the University of Cambridge in 1900, first in the second part of the mathematical tripos in 1901, Smith's prizeman in 1902, and fellow of Trinity College since 1903. He was called to Bryn Mawr in 1903. His treatise on Invariants of Quadratic Forms was published by the Cambridge University Press in 1908.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

BERWALD (L.). Krümmungseigenschaften der Brennflächen eines geradlinigen Strahlensystems und der in ihm enthaltenen Regelflächen. (Diss.) München, 1909. 8vo. 67 pp.

BÖHM (F.). Parabolische Metrik im hyperbolischen Raum. (Diss.) München, 1908. 8vo. 62 pp.

BOJKO (J.). Neue Tafel der Viertelquadrate aller natürlichen Zahlen von 1–20000 zur Bildung aller möglichen Produkte im Bereiche 1×1 — 10000×10000. Zürich, Speidel, 1909. 8vo. 20 pp. M. 1.50

DEGENHART (H.). Ueber einige zu zwei ternären quadratischen Formen in Beziehung stehende Konnexe. (Diss.) München, 1909. 8vo. 55 pp.

DENJOY (A.). Sur les produits canoniques d'ordre infini. (Thèse.) Paris, Gauthier-Villars, 1909. 4to. 141 pp.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I (volume 1): Arithmétique. Fascicule 4: Schoenflies et Baire, Théorie des ensembles; Burkhardt et Vogt, Sur les groupes finis discontinus. Paris, Gauthier-Villars; Leipzig, Teubner, 1909. 8vo. Pp. 489–616. Fr. 5.00

—— Tome I (vol. 4): Calcul des probabilités; théorie des erreurs; applications diverses. Fascicule 4: Mehmke et d'Ocagne, Calculs numériques; Bortkiewicz et Oltramare, Statistique. Paris, Gauthier-Villars; Leipzig, Teubner, 1909. 8vo. Pp. 321–480. Fr. 6.00

GRAND (J.). Anwendung der Lindstedtschen Methode auf die Integration der Differentialgleichung für hin- und hergehende Bewegungen eines zwangläufigen Mechanismus. (Diss.) Zürich, 1908. 8vo. 52 pp.

HAAR (A.). Zur Theorie der orthogonalen Funktionensysteme. (Diss.) Göttingen, 1909. 8vo. 48 pp.

HAWLITSCHEK (K.). Kreismessung. (Progr.) Prag, 1909. 8vo. 20 pp.

HOWLAND (L. A.). Anwendung binärer Invarianten zur Bestimmung der Wendetangenten einer Kurve dritter Ordnung. (Diss.) München, 1908. 8vo. 93 pp.

JURISCH (K. W.). Beweis des Fermatschen Satzes. Berlin, Heymann, 1909. 8vo. 12 pp. M. 1.60

KOCH (H.). Ueber die praktische Anwendung der Runge-Kuttaschen Methode zur numerischen Integration von Differentialgleichungen. (Diss.) Göttingen, 1909. 8vo. 37 pp.

MEYER (C.). Zur Theorie des logarithmischen Potentials. (Diss.) Leipzig, 1909. 8vo. 67 pp.

NADLER (C.). Ueber den Zusammenhang der Raumkurve vierter Ordnung erster Spezies mit ihrem Polartetraeder. (Diss.) Rostock, 1909. 8vo. 38 pp.

PLANCHEREL (M.). Sur les congruences (mod. 2^m) relatives au nombre des classes des formes quadratiques binaires aux coefficients entiers et à discriminant négatif. (Diss.) Freiburg, 1909. 8vo. 93 pp.

POINCARÉ (H.). Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik, auf Einladung der Wolfskehl-Kommission der königlichen Gesellschaft der Wissenschaften gehalten zu Göttingen vom 22. bis 28. IV. 1909. Leipzig, Teubner, 1910. 8vo. 4 + 60 pp. M. 2.40

RENNER (J.). Konforme Abbildungen. (Progr.) Gray, 1909. 8vo. 31 pp.

SCHMID (A.). Anwendung der Cauchy-Lipschitzschen Methode auf lineare partielle Differentialgleichungen. (Diss.) München, 1908. 8vo. 30 pp.

SPEISER (A.). Die Theorie der binären quadratischen Formen mit Koeffizienten und Unbestimmten in einem beliebigen Zahlkörper. (Diss.) Göttingen, 1909. 8vo. 33 pp.

THIELE (T. N.). Interpolationsrechnung. Leipzig, 1909. 4to. 187 pp. M. 10.00

II. ELEMENTARY MATHEMATICS.

BARBISCH (H.). See JAHNE (J.).

BAUER (W.) und HANZLEDEN (E. VON). Lehrbuch der Mathematik, zum Gebrauche an höheren Mädchenschulen bearbeitet. 1ter Band. Planimetrie und Arithmetik: Pensum von Klasse IV und Klasse III. Braunschweig, Vieweg, 1910. 8vo. 11 + 147 pp. M. 2.40

BESSON (L.). Algèbre, troisième année (candidats aux arts et métiers). Macon, Perraux, 1909. 8vo. 120 pp.

BROOKS (E.). The normal standard algebra; designed for public and private schools, normal schools, academies, etc. Philadelphia, Lower, 1909. 12vo. 8 + 431 pp. Cloth. $1.22

DREYFUS (L.). See FORT (L.).

FORT (L.) et DREYFUS (L.). Eléments de géométrie, conformes aux programmes du 27 juillet 1905. Classes de quatrième B et de troisième A. 2e édition, revue et corrigée. Paris, Paulin, 1910. 18mo. Pp. 217–417. Fr. 2.50

GARÍ-MONTLLOR (T.). Primeras nociones de geometría elemental. Primer grado. Barcelona, Elzeviriana, 1909. 75 pp. P. 0.50

GUBLER (S. E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. Methodisch bearbeitet. 4tes Heft. Zürich, 1909. 8vo. 64 pp. M. 1.20

HAECKEL (P.). Sammlung gelöster und ungelöster Aufgaben aus der ebenen Trigonometrie. Leipzig, Dürr, 1909. 8vo. 159 pp. M. 2.40

HANZLEDEN (E. VON). See BAUER (W.).

HOLLMAN (T.). Mathematik für höhere Mädchenschulen. I. Geometrie. Düsseldorf, Schwann, 1909. 8vo. 6 + 132 pp. M. 2.00

JAHNE (J.) und BARBISCH (H.). Leitfaden der Geometrie und des geometrischen Zeichnens für Mädchenbürgerschulen. Auf Grundlage der Normallehrpläne. 3 Stufen. Wien, Manz, 1909. 8vo. 5 + 113 pp. M. 1.70

MAHLERT (A.). See MÜLLER (H.).

MARTÍNEZ GARCÍA (M.). Nociones y ejercicios de aritmética y geometría. 3a edición. Guadalajara, 1909. 203 pp. P. 3.00

MATHEMATICS papers. Containing the papers in first stage mathematics set by the board of education in the evening examinations during the years 1900–09, with numerical answers to all the problems, and full solutions to the last paper. London, Clive, 1909. 8vo. 44 pp. 3d.

MÜLLER (H.) und MAHLERT (A.). Lehr- und Uebungsbuch der Geometrie für Studienanstalten. Ausgabe A: Für gymnasiale Kurse. 2ter Teil. Für die oberen 3 Klassen. Leipzig, Teubner, 1910. 8vo. 8 + 228 pp. M. 3.00

ROBBINS (E. R.). Plane trigonometry. New York, American Book Co., 1909. 8vo. 153 + 13 pp. Cloth. $0.60

SCHANZ (J. A.). Lehrbuch der Mathematik, zum Gebrauch in höheren Mädchenschulen, Lyceen und Studienanstalten bearbeitet im Anschluss an den Ministerialerlass vom 18. VIII. 1908. In 2 Teilen. 1ter Teil. Für die Klassen IV–I der höheren Mädchenschulen und Klasse VI und V der Studienanstalten. Berlin, Oehmigke, 1910. 8vo. 4 + 176 pp. M. 2.80

SMITH (E. R.). Plane geometry developed by the syllabus method. New York, American Book Co., 1909. 12mo. 14 + 178 pp. Cloth. $0.75

SPIEKER. See WINKLER.

TESSITORE (L.). Teoria e pratica delle equazioni di primo e secondo ordine. Venezia, 1909. 8vo. L. 3.00

WINKLER. Lehrbuch der Geometrie für höhere Mädchenschulen. Auf Grund der ministeriellen Bestimmungen über die Neuordnung des höheren Mädchenschulwesens vom 8. VIII. 1908 nach Spieker's "Lehrbuch der Geometrie für höhere Lehranstalten" bearbeitet. 1ter Teil. Klasse IV der höheren Mädchenschule. Potsdam, Stein, 1909. 8vo. 4 + 28 pp. M. 0.50

WOODWARD (C. J.). A B C of five-figure logarithms for general use. 2nd edition, with thumb index. New York, Spon, 1909. 12mo. 166 pp. Limp cloth. $1.25

III. APPLIED MATHEMATICS.

BARKHAUSEN (G.) und OTZEN (R.). Zahlenbeispiele zur statischen Berechnung von Brücken und Dächern. 2te Auflage. Anhang: Entwickelung und Erläuterung der benutzten Gleichungen. Wiesbaden, Kreidel, 1909. 8vo. 4 + 119 pp. M. 3.60

Bozali (M.). Die Berechnung der Betonkanäle. Vereinfachte Formeln zur Berechnung der Kanalweite, Abflussmenge, Wassergeschwindigkeit und des Gefälles. Ein Hilfsbuch für technische und Stadtbaubeamte. Glauchau, Peschke, 1909. 8vo. 31 pp. M. 2.00

Coffin (J. G.). Vector analysis. An introduction to vector-methods and their various applications to physics and mathematics. New York, Wiley, 1909. 12mo. 19 + 248 pp. Cloth. $2.50

Comstock (G. C.). A textbook of field astronomy for engineers. New York, Wiley, 1909. 8vo. 12 + 218 pp. Cloth. $2.50

Crabtree (H.). An elementary treatment of the theory of spinning tops and gyroscopic motion. London, Longmans, 1909.

Frederick (F. F.). Simplified mechanical perspective; for the use of high schools, technical and manual training schools and art schools. Peoria, Ill., Manual Arts Press, 1909. 4to. 56 pp. Cloth. $0.75

Handke (F.). Untersuchungen im Gebiete der Schumann-Strahlen. (Diss.) Berlin, 1909. 8vo. 41 pp.

Jäger (G.). Theoretische Physik, II: Licht und Wärme. (Sammlung Göschen, 77.) 4te Auflage. Leipzig, Göschen, 1909. 8vo. 153 pp. Cloth. M. 0.80

Jouguet (E.). Lectures de mécanique. La mécanique enseignée par les auteurs originaux. Partie II: Organisation de la mécanique. Paris, 1909. 8vo. 294. Fr. 10.00

Lange (M.). Vereinfachte Formeln für die trigonometrische Durchrechnung optischer Systeme. (Diss.) Rostock, 1909. 8vo. 36 pp.

Loney (S. L.). An elementary treatise on the dynamics of a particle and of rigid bodies. Cambridge, University Press, 1909. 8vo. 384 pp. Cloth. 12s.

Malcolm (C. W.). A text-book on graphic statics. New York, Clark, 1909. 8vo. 12 + 316 pp. Cloth.

Möhrle (T.). Das Fördergerüst, seine Entwickelung, Berechnung und Konstruktion. Kattowitz, Phönix, 1909. 8vo. 6 + 99 pp. M. 10.00

Noether (F.). Ueber rollende Bewegung einer Kugel auf Rotationsflächen. (Diss.) München, 1909. 8vo. 56 pp.

Otzen (R.). See Barkhausen (G.).

Pilkington (W.). Coördinate geometry applied to land-surveying. New York, Spon, 1909. 16mo. 44 pp. Cloth. $0.60

Russell (G. E.). Text-book on hydraulics. New York, Holt, 1909. 8vo. 7 + 183 pp. Cloth. $2.50

Waterbury (L. A.). A vestpocket hand-book of mathematics for engineers. New York, Wiley, 1909. 197 pp. Morocco. $1.50

SIMON NEWCOMB.

Simon Newcomb was born on March 12, 1835, at Wallace, Cumberland County, Nova Scotia. His parents were of New England descent temporarily settled there. He was educated by his father, and at the age of sixteen was apprenticed to a local herb doctor, but left him after two years and made his way to Maryland, in which State he spent four years teaching in various schools. An appointment on the staff of the Nautical Almanac took him to Cambridge, the headquarters of the office at that time, and he found opportunities to attend the Lawrence Scientific School, where he took the degree of B.Sc. in 1858. In 1861 he was appointed professor in the United States Navy and was detailed for work at the Naval Observatory in Washington. In 1877 he was made Superintendent of the American Ephemeris and Nautical Almanac office, and occupied this post for the next twenty years until his compulsory retirement under the regulations at the age of sixty-two. During this period he lectured for one year on political economy at Harvard, and from 1884 to 1895 was professor of mathematics and astronomy at Johns Hopkins University, and editor of the *American Journal of Mathematics*. After his retirement he continued to live and work in Washington until his death on July 11, 1909.

In 1863 he married Mary Caroline Hassler, daughter of Dr. Charles A. Hassler, U. S. Navy. His widow and three daughters survive him.

Newcomb's boyhood in a village community where life is hard and strenuous must have been full of the minor privations of life characteristic of early New England and Canadian traditions, but modified by the conditions under which he was brought up. His father was a school teacher, frequently changing from one place to another, and evidently a man of ideas more advanced than those of his generation with respect to education. Although he had to conform to current practices in the country schools where he taught, he seems to have felt that his own boy should have more mental freedom. He must have studied carefully the capacities of the young Simon and noted the direction which they took, for he made no effort to force him into work for which he was not adapted. But few books were obtainable, and those which were read were of little

intrinsic value. In his Reminiscences Newcomb says that his earliest ideas were chiefly influenced by Fowler's Phrenology and Combe's Constitution of Man. We can only judge from this that Newcomb was one of those lonely spirits that will achieve their development under any conditions, provided sufficient freedom of thought is allowed.

His somewhat erratic school education was closed at the age of sixteen by his apprenticeship to a man whose influence was of the worst possible kind for a young lad just growing into manhood. The picture which Newcomb draws of Dr. Foshay, a medical Squeers, would be notable as fiction; as a description of real life it gives one cause to wonder at the strength of character of the boy who could withstand the insidious council: "This world is all a humbug, and the biggest humbug is the best man." Fortunately, the boy had one keen desire, to learn and to know, and the unsatisfied longing finally drove him to run away from his legal master and return to his father in Maryland. He abandoned finally the idea of becoming a doctor and while in doubt as to his future career earned his living by teaching.

He evidently found time for a good deal of study in the few and second-grade books that he was able to find. Perhaps the very poverty of their treatment stimulated him to make efforts of his own, for we find him shortly afterwards sending a new proof of the binomial theorem to Professor Henry, and later answering a "crank" theorist on the Copernican doctrine in the columns of the *National Intelligencer* — his first publication. These efforts brought him to the notice of one or two scientific men, and he was not slow to avail himself of the advantages for the further pursuit of knowledge which were to be obtained through them. He discovered Bowditch's translation of Laplace and obtained a Nautical Almanac; from that time on, the main direction of his studies was never in doubt.

Although he was nominally a student of the Lawrence Scientific School for a year, and received his degree from that institution, there can be no doubt that Newcomb was, like the majority of men of his generation in America who have attained scientific fame beyond the shores of the continent, essentially self-taught. His opportunities could scarcely have been more meagre in quality or quantity, but he let none of them slip by unutilized in his desire to learn. During his four years of experience in Cambridge as a computer on the Nautical

Almanac, he came into contact with Benjamin Peirce, B. A. Gould, Runkle, and Safford, and from them he doubtless learned much, directly and indirectly; his early tendencies there received a definite scientific trend.

It was during this period that he began an investigation which resulted in the publication of one of his most important papers. At that time only a few of the minor planets had been discovered, and several theories had been put forward to account for their existence. Newcomb, collecting the various elements of these small bodies, applied to them the theory of perturbations, and showed that the distribution of their nodes and perihelia is quite arbitrary within the limits of error, and further that there was no point at which all the orbits could have intersected at any time in the past. This was generally considered as disposing of the explosion theory of Olbers. Newcomb recognized, however, that it was by no means certain that the criterion applied is sufficient. The mechanical conditions of such a swarm, under collisions and the disturbances caused by other planets, may ultimately result in an apparently arbitrary distribution. What is more interesting is the fact that even at this early date Newcomb shows in these papers that grasp of the general principles of celestial mechanics and the methods of dealing with observations, which has always been such a marked feature of all his researches. It is, in fact, a key-note to his work.

The small planets continued to interest him for several years, both in bulk and individually. We find several papers on the subject, even up to the year 1900, and he had the management of the asteroids which had been "endowed" by Watson.

His appointment as professor in the United States Navy and removal to the Observatory at Washington gave him his first opportunity to become acquainted with the practical side of astronomy. The somewhat chaotic state of the administration during the war, and for some time afterwards, again gave him the freedom to develop the subjects which interested him. Although he was making routine observations during this period, the papers which were published under his name show that his interests lay in other directions. Positional astronomy was always connected in his mind with some theory. His published papers are rarely mere sets of observations, and more generally were deductions made from the observations of others. As his knowledge gradually extended in range from the fixed

stars to the moon, he was becoming more and more impressed with the confusion creeping into exact astronomy by the different values of the constants used by different observers. He was ambitious in his programme for the future, and two plans were maturing in his mind. The first was a determination of all the constants of astronomy and their reduction to a homogeneous system. He soon saw that this would involve extended work on the theories of the planets and satellites in order to make the comparisons with observations free from empirical terms. The second problem was more special, and though not likely to demand so much time, more difficult, — the resolution of the lunar motions and the test of the law of gravitation which a comparison with the lunar theory would involve. All of his best scientific work was directed towards these two ends.

In the period between 1861 and 1877, when he became Director of the Nautical Almanac and was in a position to carry out his larger projects, his mind was active in many directions. We find short notes on a variety of subjects, — optics, finance, taxation, social science, the labor question, copyright, political economy, non-euclidean geometry. He wrote numerous reviews and popular articles, chiefly on astronomical matters. Whatever may be the ultimate fate of the views which he expressed, one feature of his writings could not fail to be valuable: he never left the reader in doubt as to his meaning. Any difficulty would be stated with that clearness and freedom from unessential detail which characterized everything he said or wrote.

By the age of thirty, and within a very few years after his arrival in Washington, Newcomb had "found himself." He had definite plans for his future, and these were more than sufficient to occupy the life-time of one man. His more important contributions to celestial mechanics were thought out before 1870, and he had a sufficiently wide knowledge of the problems of astronomy to pursue his work steadily. In fact, one sees only occasional flashes of originality after this time; he seemed to be too busy on his chosen plans to give deep thought to novelties. It is difficult to notice much difference between his earlier and his later papers, either in power or breadth of treatment. Knowledge naturally increased and he kept fully abreast of contemporary work in the astronomy of position, so that at the time of his death he was undoubtedly its chief exponent, and he may perhaps, without injustice to his predecessors, be given the foremost place amongst those who have labored for the development of this subject.

From this point on it seems proper to speak of the development of his ideas with reference to subject matter rather than in chronological order.

The position of Mars in 1862 suggested a redetermination of the solar parallax, both from the observations of that year and from all previous observations and methods. His concluded value, $8''.848$, was larger than the latest determination of that much discussed constant, and in the final summary of his life-work contained in the Constants of Astronomy he abandoned it for the value $8''.790$, which is quite close to the latest value $8''.806$, derived by Hinks from observations of Eros. The question depends mainly on the weights to be attributed to the results obtained by different methods.

In the same line of thought is his reduction of star places in different catalogues to a homogeneous system. Each observer appears to have had his own method for reducing his observations. Any one undertaking an investigation which involved the use of different catalogues of stars was obliged to find out, first of all, how the observations had been treated. "Newcomb's aim," says Dr. Hill,* "was to eliminate as far as possible systematic errors of a personal or local nature and thus obtain a homogeneous system. This is an admirably conducted investigation and has served as a foundation for whatever has been since accomplished in this subject."

Another problem, necessary for knowledge of the constant of aberration, was a fresh determination of the velocity of light. This had been undertaken by Michelson, and though Newcomb carried through a separate investigation, partly with the help of Michelson, the results were not entirely satisfactory, and the final determination rested mainly on the former investigation. The constants of precession and nutation were also obtained. With the former appeared a catalogue of fundamental stars. The methods caused a somewhat lively discussion.

Newcomb's plans for new tables of the motions of the planets involved a recalculation of their orbits. He had already investigated those of Uranus and Neptune. What had to be done was to calculate afresh the orbits of the other planets, gather all known observations together, compare them with his preliminary tables of the planets so as to correct the constants and show any deviation not accounted for by theory, and then make final tables on a uniform system of constants. This work

* *Science*, September 17, 1909.

occupied more than twenty years. Nearly all of it was published in a series of volumes founded by Newcomb and entitled: Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac.

In starting to compute the planetary perturbations Newcomb had a choice of three methods. There was the direct method of obtaining a solution of the equations in polar coordinates, by which Laplace had treated the planetary theory; there was the method known as the variation of arbitrary constants, which had been used by Leverrier; and there was the peculiar method of Hansen. Newcomb seems to have been troubled all through his life by the difficulty of mastering complicated analysis, and always turned by preference to a simple theory which could without much trouble be prepared for computers. He thus used, at the start, Laplace's method, and we find it adopted in all his planetary theories. But he was not satisfied with Leverrier's development of the disturbing function, and some of his earliest attempts were directed towards improvements in this direction. He saw a simple method of starting by means of the eccentric anomalies, and one of his long papers contains a complete development on these lines. But he afterwards found that it did not work very well, and later he returned to the Leverrier method, with certain improvements which rendered the computations more simple.

This concluded, he was able to undertake the orbits of the four inner planets. Approximate elements only are needed in finding the perturbations, since any changes in the constants can be made easily. The elements of Leverrier served for this purpose. The investigation is published in detail in Volume III of the Astronomical Papers, and it is, at the present time, the best theory of these planets that we have. With somewhat more care it might have been made final, but knowing that his term of office was drawing to a close Newcomb seems to have been in somewhat of a hurry to finish. The calculations were executed in duplicate by computers, and the two sets of results, together with those of Leverrier, are printed side by side. In glancing over them one notices here and there considerable differences between the duplicate calculations. It would have been more satisfactory if the sources of these differences had been traced down and corrected. Duplicate calculations are indeed almost essential in such work, but they lose much of

their value if the errors revealed are not removed. There are also occasional misprints which give a feeling of insecurity to one using any special numbers for other purposes. In a later paper the secular variations of the four planets are computed by Gauss's method.

The theory was not, however, the heaviest part of the work. All known observations had to be collected and compared. It was here that Newcomb's particular genius for the organization of huge masses of material, and his firm grasp of the facts which could be deduced from them, was given free play. Over 62,000 meridian observations of Mercury, Venus, and Mars were employed for those planets alone. The work had to be done with but little addition to the ordinary force of computers employed on the Nautical Almanac. Besides the planetary comparisons, it was necessary to find the corrections to the general constants of the solar system, and also to those of the earth, such as precession and nutation, which affect every observation. All these had to be fused into a homogeneous whole so that the motions of the solar system should be made to depend on one and the same system of constants. This immense mass of work is summarized in a little volume which was published as a Supplement to the Nautical Almanac for 1897, and is entitled: Elements of the Four Inner Planets and the Fundamental Constants of Astronomy, or more briefly, Astronomical Constants. This volume gathers together Newcomb's life-work and constitutes his most enduring memorial.

The mutual perturbations of Jupiter and Saturn are so large that the problem of unravelling their motions is a much more difficult application of the planetary theory than the other six large planets require. For the successful completion of Newcomb's scheme within his life time it was necessary that these two planets should be treated by another hand and he was able to enlist the services of G. W. Hill for this purpose. Hill, indeed not only worked out the theories in full but compared them with observation and formed the tables which were the necessary complement if his labors were to be utilized to their full extent.

The publication of the tables of the planets followed rapidly. All of them bear Newcomb's name, except those of Jupiter and Saturn, for which, as we have just seen, G. W. Hill was entirely responsible. Newcomb was fortunate in securing assistance, not only from a man like Hill with a world-wide reputation, but

also from others who were far above the grade of ordinary computers, and to whom he was able to explain his methods so that they could carry the work forward with an understanding of what they were doing. Indeed, it would have been impossible for him to have covered the ground effectually without such assistance. The tables themselves contain no very notable features; a few slight improvements may be noted on the general methods for tabulating introduced by Hansen in his tables of the moon.

While carrying forward to successful completion his schemes for the planets, Newcomb continually returned to the subject which interested him above all others — the motion of the moon. As early as 1859 we find him comparing the observed places of the moon with those given in the Ephemeris, and his last paper, not yet published, deals with the same subject. Indeed, he always hoped to be able to add a set of new tables of the moon's motion to his already long list of achievements. That this did not become possible was undoubtedly a source of regret to him. But posterity will accord greater praise for what he did achieve than it would for the mere fulfilment of the ambition of a life-time.

It was not long before Newcomb's advent into the astronomical world that Hansen's tables of the moon had been introduced for the purpose of computing the moon's ephemeris in the Nautical Almanac. It was then thought that they would satisfy all the needs of astronomy. Hansen had said that they satisfied all the observations for a century back, and that the eclipses of ancient times were fairly well acounted for. Shortly before the publication of Hansen's tables, Adams had shown that the change in the mean length of the month was different from that given by Hansen, so different, indeed, that the ancient eclipses could not be reconciled if the new theoretical value were used. Newcomb, always interested in the comparison between theory and observation, wondered if Hansen's tables would agree with observations before 1750, and he started to look for data which should test the question. Early meridian observations could not be relied on, but occultations are phenomena which even the earliest scientific observers would be apt to note and which are free from many of the errors to which meridian observations are liable. At the first opportunity he went on a search over Europe for such observations. His labor was well rewarded. He was able to find sufficient ma-

terial for comparison over another century. On working it up he discovered that there was a large periodic change in the moon's longitude not indicated by theory and running through its phases in some three hundred years. The term is still unexplained.

These discoveries necessitated a more careful study of the ancient eclipses in order to see how they were affected. The new periodic term had but little influence on them; the difference between Adams's and Hansen's values for the secular change was the point at issue. In 1878 Newcomb was able to publish the results of his investigations and to give additional tables by means of which the moon would keep more nearly to its predicted place. It is true that the new terms were empirical, but that mattered little for the purposes of the Nautical Almanac. His last memoir, and one of his best, consists of a reexamination of all the known occultations since the beginning of the seventeenth century, including fresh material that he had gathered in the last twenty years. The earlier conclusions are confirmed and one or two new terms of shorter period are brought to light.

It had long been known that Hansen's theoretical investigations of the effects of the planets on the moon's motion were of doubtful accuracy. Newcomb must have had the problem of their determination, and, indeed, the whole question of Hansen's lunar theory in his mind quite early in his career. In 1868 he made a partial comparison with the only other theory approaching it in degree of accuracy, that of Delaunay. The comparison was completed in 1882 by himself and J. Meier, and it showed that the errors of the solar terms were not serious enough to affect the places of the moon when averaged over long periods of time. But Delaunay had not computed the planetary terms. In 1871 Newcomb published a method for finding them. It finally turns out to be practically a continuation of Delaunay's theory but it was far from the stage of numerical application. It was not until twenty-three years later that he continued the idea of that memoir sufficiently far to show its possibilities, though even then nothing like an extensive investigation of the whole subject was attained. Nevertheless, this second memoir is one of his ablest investigations, from the point of view of theory. In it he finds some remarkable relations that arise between the constants in the problem of four bodies. When finally developed, these rela-

tions reduce the computation of the secular accelerations to a comparatively simple problem. He himself never thought very highly of this paper, mainly because it did not fulfill the object for which it was undertaken.

His work on the subject was only completed in a long paper published a couple of years ago. Although before the appearance of Newcomb's second memoir Hill had given a practical and thoroughly effective method for computing all small perturbations, and in 1891 Radau had developed Hill's idea in considerable detail, Newcomb still clings to his own methods. He felt afterwards that this proceeding had been a mistake. In a letter to the writer he says:

"I now see had I devoted a few days or weeks to carefully studying the phases of the problem as presented not only by yourself, but Delaunay, Radau, and Hill, I might have profited immensely and been able to do my work much more easily as well as to put it into more condensed shape. But I am always repelled by intricate algebraic expressions stringing on without end without any well marked division into parts. Any attempt to manipulate them simply tires me out before I get to the end; so, however complex may be the expressions I have to deal with, I like to cut them up into parts and condense them to handle them. So I went to work trying my system without having sufficiently learned what could be done from the work of others." The whole trouble really consisted in his attempt to develop the disturbing functions by special values instead of by expansion; in spite of the help of computers such a method gives enormous calculations from which comparatively few results are obtained. The final numerical values were unfortunately erroneous in many cases, owing to a slip at the end of the work.

Almost the only excursion that Newcomb made into pure theory is contained in a paper entitled "The general integrals of planetary motion," and it consists of an attempt to show how the coordinates of a planet, under the attraction of any finite number of planets, may be represented by trigonometric series. Poincaré has used this paper as a text for his investigations into the possibilities of such developments. It is, at bottom, an extension of Delaunay's methods to any number of bodies. The various relations between the constants of integration and those present in the developments are worked out with great skill by means of Lagrange's methods for the variation of arbitrary constants. All through his theoretical work Newcomb

seems to cling to Lagrange's equations and to reach the canonical forms through them rather than to obtain the latter directly through the brief methods of Jacobi. There were, however, certain advantages in so doing. He was able to deduce, in the natural course of the work, the final canonical constants L, G, H, of Delaunay, directly from the final expressions of the rectangular coordinates in terms of the time. These are important when other perturbations of the moon, beyond those produced by the sun, have to be obtained.

A curious anomaly in the motion of Hyperion, the seventh satellite of Saturn, is satisfactorily explained in Volume III of the Astronomical Papers. Professor Hall had deduced from observation that the point of nearest approach to the planet possesses a large retrograde motion, while the ordinary theories indicate a forward motion. Newcomb traced the difficulty to an approximate commensurability between the mean motions of Hyperion and the disturbing satellite Titan. It is a case of "libration," in which an angular element oscillates instead of making complete revolutions.

In fact, Newcomb frequently had a happy faculty of hitting on the explanation of some curious anomaly and in quickly publishing a brief note on the subject. His suggestion that the difference between the observed period of the motion of the earth's pole and the period demanded by theory is due to the elasticity of the earth, has been fully confirmed by subsequent investigations. In quite another direction was the theorem that "if a fourth dimension were added to space, a closed shell could be turned inside out by flexure without stretching or tearing."

Although the paper just mentioned is perhaps Newcomb's only notable contribution to pure mathematics, he was familiar with the general directions which investigation had taken during the nineteenth century. In his presidential address before the American Mathemetical Society in 1893, he begins with a disclaimer of any right to be considered a mathematician in the modern sense of the word. But from the remarks which follow, it is evident that he had not only read but had devoted some thought to modern ideas on hyperspace, group theory, projective geometry, and the like. What is more interesting is his analysis of the manner in which the thought of the modern mathematician differs from that of the ancients, and his general conclusion that one of the chief advantages of present-day methods is their

economy in the reduction of the number of mental concepts required to express the same ideas.

Of his numerous contributions to economics I quote the statements of Professor Irving Fisher in the *British Royal Economic Journal:*

"Personally I would rank Professor Newcomb high as an economist, and am glad to find that in this judgment President Hadley heartily concurs. Doubtless Professor Newcomb leaves a large number of other admirers, yet it would seem that his economic writings did not attract the attention among economists which they deserved. . . .

"Perhaps his chief and most fruitful contribution to economic science was the distinction he showed so clearly between a 'fund' and a 'flow,' a 'fund' relating to a point of time, and a 'flow' relating to a period of time. This distinction he applied in particular to what he called 'societary circulation,' or the equation of exchange between money and goods. So far as I am aware, he was the first definitely to enunciate this equation expressing the fact that quantity of money multiplied by its velocity of circulation is equal to price level multiplied by volume of business transactions. This equation, with due amplification, represents the so-called 'quantity theory of money' in its highest form. He alone employed this same distinction between a 'fund' and a 'flow' to expose the fallacy of the 'wage-fund.' . . .

"Professor Newcomb's economic writings include many of a controversial nature. He took a vigorous part, for instance, in the discussion some years ago, as to the best standard of deferred payments, and still earlier, in the question of labor and capital. He was an advocate in general of the 'let alone' policy, though he distinguished it sharply from what he called the 'keep out' policy. In other words, he believed in free economic activity of individuals, but did not advocate the exclusion of government from economic activity.

"As to methodology, he believed that economics is a science, and consequently that the method to be followed by economic science should be the same as the method followed by science in general. He emphasized the fact that 'A law of nature can only be expressed in the form of a conditional proposition.' . . .

"In applying this idea he pointed out the fallacy of judging the effect of a tariff by the subsequent rise or fall of prices. The imposition of the tariff, as he pointed out, will make prices

higher than they *otherwise would be,* but not necessarily higher than they *had been.*"

He wrote many text-books on astronomy and cognate subjects. His Popular Astronomy ran through many editions and was translated into several languages. A number of elementary text-books on mathematics were not the least of his contributions to education and they exerted considerable influence on the study of mathematics in their time. The long list of nearly four hundred publications, gathered together by Dr. Archibald in Volume XI (1905) of the *Transactions of the Royal Society of Canada,* shows titles of published addresses, reviews, numerous popular articles, several short stories, and a novel. The last he did not care to discuss, owing to some difficulties which had resulted in its publication before it was in the form in which he would have desired it to come before the public.

Education, its theory and practice, has always been a matter of interest to the people at large in the United States, and Newcomb as a professor at Johns Hopkins University and a writer of text-books naturally spoke on the subject with some authority. He was chairman of the conference on mathematics appointed by the committee on secondary school education in 1892, generally known as the Committee of Ten. The report adopted by this conference has had a large influence on the teaching of mathematics and his share in its preparation was undoubtedly considerable.

If we attempt to judge Newcomb's work, it is necessary to consider the three sides of astronomy in which he was chiefly interested. The first, purely theoretical, consists practically of mathematical investigations; the third, the purely observational side, is the function of the observatory; the second is the combination of these two, the comparison between theory and observation. There can be no doubt that Newcomb's chief title to lasting fame lies in the second of these three directions. He was a master, perhaps as great as any that the world has known, in deducing from large masses of observations the results which he needed and which would form a basis for comparison with theory. Only in his earlier years had he made any observations, and the impression one gets in reading his Reminiscences, conveys the feeling that he never cared much for that department of astronomy. On the purely mathematical side of the problems of celestial mechanics he

cannot be said to have been thoroughly at home. We have from his pen no new method for dealing with the motions of the bodies in our solar system. On the other hand, he had a thorough grasp of their mechanical relations and this nearly always served his purposes. He had too a remarkable facility for seizing on the essential point of some known method, provided it was not too complicated, improving and then adapting it so that computers could perform nearly all the numerical work. As stated above, Newcomb's predilections alway tended towards some practical result, even when he wrote a purely theoretical paper; but curiously enough, the two or three papers which represent this side of his work never reached fully successful application at his hands.

In reading Newcomb's papers one becomes impressed with his geometrical point of view of almost every subject. He seemed doubtful of analysis by means of symbols unless it were possible to get a bird's-eye view of the process and of the manner in which physical principles led up to it. Consequently, errors of theory were unusual in his publications. It has always been a matter of astonishment to the writer how easily either theoretical or numerical errors, when they did occur, could be traced down and corrected. His clear mental view of the problem was reflected in his prose, which, without graces of style, always expressed exactly what he meant.

Newcomb's influence on the thought and men of his times must be judged mainly by the condition in which he found the subjects over which his mind ranged and by his life in a land whose people were passing through the transition stage from a nation of pioneers to a firmly seated world power. Possessed of a broad mind, a wide range of knowledge, and a capacity for taking an interest in almost all matters brought to his attention, he is peculiarly well qualified to stand as a representative of the mental attitude of his generation towards the problems which confronted it. His great power of organization was applied to astronomy at a time when the subject needed just such a man to prevent confusion from falling into chaos. Deep insight was less to be desired than broad comprehension. If it cannot be said of Newcomb that he gave a new set of ideas to the world, there can be no doubt as to the value of his contributions to almost every department of his own science and to many portions of other sciences. Again, he was essentially practical in every undertaking, even to the extent of devoting

himself during his last years to a problem which he might hope to solve and in avoiding sole responsibility for any large scheme involving many years of labor. Less characteristic of his times was the impersonal attitude which he appeared to have and to expect others to have towards the objects of science. Whether this be judged as a defect or not, the task of a future historian will be rendered more simple by the absence of those qualities which would, if prominent, demand a careful investigation into the unofficial relations with the men of his time in order to obtain an accurate estimate of his influence. He seemed unable or unwilling to give that personal touch to a discussion which has played so large a part in the management of the general affairs of the nation.

Newcomb was indefatigable in his attendance at congresses, scientific meetings, and academic functions. Although he never indulged much in the lighter sides of conversation he was always ready to talk on any subject of real interest. On such occasions he would not hesitate to express any ideas which might occur to him, and sometimes mistook reticence in others for absence of ideas. He was fond of presiding over the deliberations of scientific bodies, and by his own attitude always imparted additional dignity to the proceedings. His great reputation amongst astronomers was combined with extended popular knowledge of his fame, at least on the American continent, and he received unusually numerous evidences of recognition from all parts of the world.

His naïve enjoyment of the honors which he received shows real modesty in his estimate of his own work. In correspondence he would frequently regret its defects and express the hope that others might succeed in improving it. The successful completion of a plan or an investigation seemed to call for far more praise, in his mind, than originality in conception or ability in treatment.

Newcomb's last year was spent in a courageous struggle to work in spite of much physical suffering caused by an incurable disease. As soon as he knew that time was limited for him, almost all his remaining energy was devoted to the completion of an investigation which had been in his mind for many years. It was sent to the printer towards the end of June, and he died three weeks later. Thus passed away one of the great pioneers of American science, and one who also attained a high place in the ranks of those whose merits are judged to be worthy of honor apart from time or place.

E. W. Brown.

A NEW PROOF OF WEIERSTRASS'S THEOREM CONCERNING THE FACTORIZATION OF A POWER SERIES.

BY PROFESSOR GILBERT AMES BLISS.

THE theorem which is to be proved here may be stated in the following form:

Let $f(x_1, x_2, \cdots, x_p, y)$ *be a convergent series in* $x_1, x_2, \cdots, x_p, y$, *and such that the series* $f(0, 0, \cdots, 0, y)$ *begins with the term of degree* n. *Then* $f(x_1, x_2, \cdots, x_p, y)$ *is factorable in the form*

$$f(x_1, x_2, \cdots, x_p, y) = (a_0 + a_1 y + a_2 y^2 + \cdots + a_{n-1} y^{n-1} + y^n)\phi(x_1, x_2, \cdots, x_p, y),$$

where $a_0, a_1, \cdots, a_{n-1}$ *are convergent power series in* $x_1, x_2, \cdots, x_p$ *which vanish for* $x_1 = x_2 = \cdots = x_p = 0$, *and* ϕ *is a power series in* $x_1, x_2, \cdots, x_p, y$ *which has a constant term different from zero.*

In the *Bulletin de la Société Mathématique de France* * Goursat has called attention to the fact that the proof which Weierstrass gave of this important theorem, as well as the later proofs which occur in the literature † make use of the notions of the function theory, while the theorem itself is essentially of an algebraic character. In the paper referred to he has given an elegant and elementary proof of the theorem which is in outline as follows:

By means of the substitution

$$y^n = -a_0 - a_1 y - a_2 y^2 - \cdots - a_{n-1} y^{n-1}$$

the series f can be reduced to a polynomial P of degree $n - 1$ in y, whose n coefficients are convergent series in $a_0, a_1, \cdots, a_{n-1}, x_1, x_2, \cdots, x_p$. By the usual theorems in implicit function theory it is shown that the n equations found by putting these coefficients equal to zero have unique solutions for $a_0, a_1, \cdots, a_{n-1}$ as power series in $x_1, x_2, \cdots, x_p$ which vanish with $x_1, x_2,$

* "Démonstration élémentaire d'un théorème de Weierstrass," vol. 36 (1908), p. 209.

† Picard, Traité d'Analyse, vol. II, p. 243; Goursat, Cours d'Analyse, vol. II, p. 284.

$\cdots, x_p$. If the values so found are substituted in the formula

$$y^n = -a_0 - a_1 y - a_2 y^2 - \cdots - a_{n-1} y^{n-1} + \mu$$

and the series f again reduced, a polynomial P_1 of degree $n-1$ in y will be found whose coefficients are series in $x_1, x_2, \cdots, x_p, \mu$. On account of the way in which the functions $a_0, a_1, \cdots, a_{n-1}$ were determined, this polynomial P_1 has a factor μ and hence f has a factor $(a_0 + a_1 y + \cdots + a_{n-1} y^{n-1} + y^n)$.

The proof which is given below seems to the writer even more direct than that of Goursat, and it furnishes besides convenient formulas for the determination of the coefficients of the series $a_0, a_1, \cdots, a_{n-1}$.

The series f can evidently be written in the form

$$(1) \qquad f = -y^n + f_0 + f_1 y + f_2 y^2 + \cdots + f_n y^n + \cdots,$$

where the coefficients f_k are power series in $x_1, x_2, \cdots, x_p$, and $f_1, f_2, \cdots, f_n$ have no constant terms. If a convergent series

$$b = b_0 + b_1 y + b_2 y^2 + \cdots$$

having its constant term different from zero, together with n other convergent series $\mu_0, \mu_1, \cdots, \mu_{n-1}$ in $x_1, x_2, \cdots, x_p$, can be determined so that the identity

$$(2)\ b(-y^n + f_0 + f_1 y + \cdots) \equiv \mu_0 + \mu_1 y + \cdots + \mu_{n-1} y^{n-1} - y^n$$

is true, then the series ϕ of the theorem will be $1/b$ and the theorem will be proved. By comparison of the coefficients of the different powers of y in (2), the equations

$$\begin{aligned}
\mu_0 &= b_0 f_0,\\
\mu_1 &= b_0 f_1 + b_1 f_0,\\
&\cdots\cdots\cdots\\
(3) \qquad \mu_{n-1} &= b_0 f_{n-1} + b_1 f_{n-2} + \cdots + b_{n-1} f_0,\\
b_0 - 1 &= b_0 f_n + b_1 f_{n-1} + \cdots + b_{n-1} f_1 + b_n f_0,\\
&\cdots\cdots\cdots\\
b_k &= b_0 f_{n+k} + b_1 f_{n+k-1} + \cdots + b_{k-1} f_{n+1} + b_k f_n + \cdots + b_{n+k} f_0,\\
&\cdots\cdots\cdots
\end{aligned}$$

are found. These equations determine uniquely the coefficients of the series b_0, b_1, $\cdots$; μ_0, μ_1, $\cdots$, μ_{n-1} as rational integral functions with positive coefficients of the coefficients of the series $f_0, f_1, f_2, \cdots$. For on account of the fact that $f_0, f_1, \cdots, f_n$ have no constant terms, the terms of order m in b_k can be determined from the last equation in the form just described as soon as the terms of order m and less in b_0, b_1, $\cdots$, b_{k-1}, and those of order $m-1$ and less in b_k, b_{k+1}, $\cdots$, b_{k+n}, are known. Suppose, for example, that the terms of order zero of all the b's up to and including b_{k+mn} have been computed. From them the terms of order one of b_0, b_1, $\cdots$, $b_{k+(m-1)n}$ can be found; then the terms of order two of b_0, b_1, $\cdots$, $b_{k+(m-2)n}$; and so on, until the terms of order m of b_0, b_1, $\cdots$, b_k are obtained. Hence step by step the terms of the different orders can be determined for all of the series b_k, and hence for all the series μ. It is evident, therefore, that *if there exist convergent series* b, μ_0, μ_1, $\cdots$, μ_{n-1} *satisfying identically the relation* (2), *then those series have coefficients which are uniquely determined by the relations* (3). *Furthermore if a function* F *of the form* (1) *can be found for which the coefficients in* F_0, F_1, $\cdots$ *are positive and greater in numerical value, respectively, than those of* f, *and such that the corresponding series* B, M_0, M_1, $\cdots$, M_{n-1} *for* F *are convergent, then the series* b, μ_0, μ_1, $\cdots$, μ_{n-1} *for* f *will also be convergent.*

A function F of the type desired can readily be found. The series f can be supposed without loss of generality to be convergent for $x_1 = x_2 = \cdots = x_p = y = 1$. For if it were convergent for $|x_i| \leqq \rho_i$, $|y| \leqq \rho$, it would only be necessary to make the transformation $x_i = \rho_i x_i'$, $y = \rho y'$ in order to have a series with the desired property. If the values $x_1 = x_2 = \cdots = x_p = y = 1$ are substituted in f, the resulting series is the series of the coefficients of f and is absolutely convergent. Hence each coefficient of f is in absolute value less than a certain positive constant N. For the function F, then, let

$$F_0 = F_1 = \cdots = F_n = N\left[\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_p)} - 1\right],$$

$$F_{n+k} = N\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_p)},$$

where $k = 1, 2, \cdots, \infty$. Every coefficient of F is positive and greater in absolute value than the corresponding coefficient of f.

For the function F the relation (2) after a simple transformation takes the form

$$\frac{B}{1-y}\left[-y^n+\frac{N}{(1-x_1)(1-x_2)\cdots(1-x_p)}-N\right.$$
$$\left.+(1+N)y^{n+1}\right]\equiv M_0+M_1y+\cdots+M_{n-1}y^{n-1}-y^n.$$

Series $B, M_0, M_1, \cdots, M_{n-1}$ of the type desired can be found satisfying this identity provided that series $C, M_0, M_1, \cdots, M_{n-1}$ of similar type can be found satisfying the identity

$$-y^n+F_0+cy^{n+1}\equiv(M_0+M_1y+\cdots+M_{n-1}y^{n-1}-y^n)C,$$

where $c=1+N$. The latter, however, can be satisfied by a linear function $C=M-cy$. For by comparing the coefficients of y^n it is found that $M=1-cM_{n-1}$, and by comparing the coefficients of the lower powers

$$\begin{aligned}
M_0 \qquad\qquad &= cM_0M_{n-1}+F_0,\\
-cM_0+M_1 \qquad &= cM_1M_{n-1},\\
-cM_1+M_2 &= cM_2M_{n-1},\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
-cM_{n-2}+M_{n-1} &= cM_{n-1}M_{n-1}.
\end{aligned}$$

From the well known theorems concerning implicit functions it follows that these equations have solutions $M_0, M_1, \cdots, M_{n-1}$ which are power series in $x_1, x_2, \cdots, x_p$, vanishing when $x_1=x_2=\cdots=x_p=0$. Hence the theorem is proved.

It should be noted that the dominant function F and the implicit function theory have been used only in the proof of the convergence of the series b and $\mu_1, \mu_2, \cdots, \mu_{n-1}$. The computation of the coefficients of these series is effected by means of the equations (3).

THE UNIVERSITY OF CHICAGO,
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ON SOME THEOREMS IN THE LIE THEORY.

BY PROFESSOR L. D. AMES.

(Read before the Southwestern Section of the American Mathematical Society, November 27, 1909.)

THE well-known treatment of the infinitesimal transformation in the Lie theory makes use of power series, assuming that the functions involved are analytic. The treatment here given demands at most the existence of second partial derivatives. Since no use is made of the group theory, the proofs here given may be used either in connection with the group theory or apart from it.

Three theorems are stated in this paper, of which Theorem I. was proved in a previous paper.* Of the three, any one is an immediate result of the other two taken together. They may be compactly stated as follows:

Given any differential equation of the form

$$\Omega(x, y, y') = 0 \tag{1}$$

which, in a given region R, can be written in the form linear in y'

$$\Omega_1 \equiv X(x, y)y' - Y(x, y) = 0, \tag{2}$$

and of which $\omega(x, y) = c$ is an integral; let $\xi(x, y)$ and $\eta(x, y)$ be such functions that

$$X\eta - Y\xi \not\equiv 0.$$

Consider the statements

A. *The equation*

$$\frac{X}{X\eta - Y\xi} y' - \frac{Y}{X\eta - Y\xi} = 0 \tag{3}$$

is exact. That is, $1/(X\eta - Y\xi)$ is an integrating factor of (2).

B.

$$\xi \frac{\partial \Omega}{\partial x} + \eta \frac{\partial \Omega}{\partial y} + \eta' \frac{\partial \Omega}{\partial y'} = 0,\dagger \tag{4}$$

* BULLETIN, vol. 15, no. 8 (May, 1909).

† In the Lie theory this is the condition for invariance under the extended transformation

$$U'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}.$$

where

$$\eta' = \frac{d\eta}{dx} - y'\frac{d\xi}{dx}.$$

C.

$$\xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial y} = F(\omega).\text{*} \tag{5}$$

THEOREM I. B *is a necessary and sufficient condition for* A.
THEOREM II. C *is a necessary and sufficient condition for* B.
THEOREM III. C *is a necessary and sufficient condition for* A.

The proofs are in outline as follows:

Theorem I. The familiar condition that (3) be exact, after obvious simplifications, is (4).

Theorem II. Differentiate (5) totally along $\omega = c$. The result, after obvious substitutions, is (4).

Theorem III. Assume that M is the unknown integrating factor of (2). Writing the exact equation in two forms, we obtain an equation which we can solve for M. This proof does not assume a previous knowledge of the form of the integrating factor.

More detailed proofs of Theorems II. and III. follow.

Proof of Theorem II. First to prove C sufficient. Since $\omega = c$ is an integral of (2) we may write (2) in the form

$$\Omega_1 \equiv M\frac{\partial \omega}{\partial x} + My'\frac{\partial \omega}{\partial y} = 0. \tag{6}$$

As in the previous paper, the partial derivatives of Ω with respect to x, y and y' are proportional to those of Ω_1. Form these and substitute in the equation obtained by differentiating (5) totally along the curve $\omega = c$. The result is (4).

To prove C necessary we assume (4), and retrace the steps of

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* In the Lie theory this is the condition that the family of integral curves $\omega(x, y) = c$ is invariant under the transformation

$$Uf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}.$$

Throughout this paper $F(\omega)$ is to be read "some function of ω" when (5) is a necessary condition, and "any function of ω" when (5) is a sufficient condition.

the above proof, obtaining

$$(7) \qquad \frac{d}{dx}\left[\xi \frac{\partial \omega}{\partial x} + \eta \frac{\partial \omega}{\partial y}\right] = 0.$$

Noting that y and y' are functions of x and c, the integration of (7) gives

$$(5) \qquad \xi \frac{\partial \omega}{\partial x} + \eta \frac{\partial \omega}{\partial y} = F(c) = F(\omega).$$

Proof of Theorem III. First to prove C sufficient. Equation (2) has an integrating factor which we will call M. Then the equation

$$MXy' - MY = 0$$

is exact. It is therefore of the form

$$\frac{\partial \omega_1}{\partial y} y' + \frac{\partial \omega_1}{\partial x} = 0,$$

where ω is some function of ω_1, $\omega = \phi(\omega_1)$. Hence

$$\frac{\partial \omega}{\partial x} = \phi'(\omega_1) \frac{\partial \omega_1}{\partial x} = - \phi'(\omega_1) MY,$$

$$\frac{\partial \omega}{\partial y} = \phi'(\omega_1) \frac{\partial \omega_1}{\partial y} = \quad \phi'(\omega_1) MX.$$

Putting these values in (5) and solving for M, we obtain

$$M = \psi(\omega_1)/(X\eta - Y\xi),$$

where ψ is some function of ω_1. Hence $1/(X\eta - Y\xi)$ is also an integrating factor.

To prove that (5) is necessary, let $1/(X\eta - Y\xi)$ be a particular integrating factor, and $\omega_1(x, y) = c$ the corresponding integral. Let $\omega(x, y) = c$ be any integral. Then the corresponding integrating factor is $\chi(\omega_1)/(X\eta - Y\xi)$, where χ is some function of ω_1. Hence the equations

$$\frac{X\chi}{X\eta - Y\xi} y' - \frac{Y\chi}{X\eta - Y\xi} = 0$$

and

$$\frac{\partial \omega}{\partial y} y' + \frac{\partial \omega}{\partial x} = 0$$

are identical. Hence

$$(5) \qquad \xi \frac{\partial \omega}{\partial x} + \eta \frac{\partial \omega}{\partial y} = \chi(\omega_1) = F(\omega).$$

UNIVERSITY OF MISSOURI,
November, 1909.

ON THE DISCONTINUOUS ζ-GROUPS DEFINED BY RATIONAL NORMAL CURVES IN A SPACE OF n DIMENSIONS.

BY PROFESSOR J. W. YOUNG.

(Read before the Chicago Section of the American Mathematical Society, January 1, 1910.)

THE present note completes in an important particular a paper* which I presented to the Society some two years ago. I there considered the discontinuous groups Γ_n of linear fractional transformations

$$(1) \qquad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

on the complex variable ζ, defined as follows by a rational normal curve C_n in a space S_n of n dimensions: The given C_n is transformed into itself by a group of ∞^3 collineations

$$(2) \qquad z_i' = \sum_{k=0}^{n} a_{ik} z_k \qquad (i = 0, 1, \cdots, n)$$

in S_n. Each of these collineations subjects the parameter ζ of the points of C_n to a substitution (1), so that the continuous three-parameter groups of transformations (1) and (2) are simply isomorphic. If now the transformations (2) be restricted to those whose coefficients a_{ik} are rational integers with determinant $|a_{ik}| = 1$, the resulting subgroup of the three-parameter group of transformations (2) will be properly discontinuous.

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* "A fundamental invariant of the discontinuous ζ-groups defined by the normal curves of order n in a space of n dimensions," BULLETIN, vol. 14 (1908), pp. 363–367.

The same is then true of the group of transformations (1) which corresponds in the isomorphism already noted to the discontinuous subgroup just defined.

In the paper cited above I proved the following theorem:

In the discontinuous ζ-groups defined as indicated by any normal curve C_n in a space S_n the period ω of any elliptic substitution must satisfy a relation of the form

$$\sin\frac{(n+1)\pi}{\omega} = J \sin\frac{\pi}{\omega},$$

where J is an integer (positive, negative, or zero).

This theorem gives a necessary condition on the positive integers ω that may be periods of elliptic substitutions in any Γ_n. In particular, the numbers n, $n+1$, $n+2$ are all possible values of ω under this condition. It has not been shown hitherto, however, that groups Γ_n really exist containing substitutions of these periods. In the present note it is shown how to construct a C_n which will define in the manner indicated a group Γ_n containing any given substitution of period $n+1$ or $n+2$. The solution $\omega = n$ is not in general possible when $n > 2$.

We recall first the necessary and sufficient condition that a collineation in S_n leave invariant a normal C_n.* We will suppose that the collineation produces on the C_n a projectivity with two distinct double points, P_0 and P_n. The osculating k-spaces ($k = 1, \cdots, n-1$) at P_0 and P_n are then invariant; and the osculating k-space at P_0 determines with the osculating $(n-k)$-space at P_n a double point P_k of the collineation.

The invariant configuration of the collineation is then a complete $(n+1)$-point in S_n,† two of whose vertices are on the curve and which is completely determined by these two points and the curve. An $(n+1)$-point thus situated with reference to a C_n will be said to *osculate* the C_n at the two points. If this invariant $(n+1)$-point is taken as the fundamental $(n+1)$-point of the coordinate system, the points $(0, 0, \cdots, 1)$ and $(1, 0, \cdots, 0)$ being taken as P_0 and P_n respectively, the equations of the C_n are of the form

* Gino Loria, "Sulle curve razionali normali in un spazio an n dimensioni," *Giornale di Matematiche*, vol. 26 (1888), p. 345.

† It may, of course, happen that more than $n+1$ points are left invariant by the collineation. But in any case the invariant configuration must include a complete $(n+1)$-point of the specified kind.

(3) $$z_i = c_i \zeta_i^{n-i} \qquad (i = 0, 1, \cdots, n),$$

where the c_i are any constants.

It now follows readily that the roots of the characteristic equation of any collineation leaving the C_n invariant must be proportional to the powers $\mu^n, \mu^{n-1}, \cdots, \mu, 1$ of a number μ. But this condition with the preceding is at once seen to be also sufficient. We have then

THEOREM 1. *The necessary and sufficient condition that a collineation in S_n (with at least two distinct double points *) leave a rational normal curve C_n invariant is that its invariant configuration contain a complete $(n+1)$-point and that the roots λ of its characteristic equation be proportional to the powers $\mu^n, \mu^{n-1}, \cdots, \mu, 1$ of a number μ. A collineation satisfying this condition leaves every normal curve invariant which is osculated by the invariant $(n+1)$-point at the points $\lambda = \rho\mu^n$ and $\lambda = \rho$.*

We are now in a position to write down the equations of any C_n which is left invariant by a given collineation, if such C_n's exist. Let π be any collineation satisfying the condition of Theorem 1. Let

$$Z_0 = 0, \quad Z_1 = 0, \quad \cdots, \quad Z_n = 0$$

be the equations the $n+1$ invariant $(n-1)$-spaces of the invariant $(n+1)$-point of π. The collineation may then be written

$$\rho Z_i' = \mu^{n-i} Z_i \qquad (i = 0, 1, \cdots, n).$$

Any C_n left invariant by π is then given by the equations

(4) $$Z_i = c_i \zeta^{n-i} \qquad (i = 0, 1, \cdots, n).$$

To complete the proof of the existence theorem which we have in view it is necessary only to show that collineations of periods $n+2$ and $n+1$ with integral coefficients and determinant 1 exist in S_n which satisfy the conditions of Theorem 1. Representing a collineation by the matrix of its coefficients we find readily that the two collineations

* It is readily seen that, if the collineation has two distinct double points, it must have at least $n+1$, two of which are on the curve. Collineations with only a single double point and leaving a C_n invariant exist, moreover; a fact which was overlooked by Loria in the paper quoted above. Cf. in this connection the latter part of the present paper.

$$\pi_1 = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

and

$$\pi_2 = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

are respectively of periods $n + 2$ and $n + 1$. Indeed their characteristic equations are respectively

for π_1: $$\lambda^{n+1} + \lambda^n + \cdots + \lambda + 1 = 0;$$

for π_2: $$\lambda^{n+1} - 1 = 0.$$

The collineations π_1, π_2 are then readily seen to satisfy the conditions of Theorem 1. This completes the proof of

THEOREM 2. *Rational normal curves C_n exist which define in the way indicated discontinuous ζ-groups containing elliptic substitutions of period $n + 1$ or period $n + 2$.*

This result is of considerable interest on account of the fact that it proves for the first time that the class of ζ-groups Γ_n $(n > 2)$ is indeed more extensive than the class of groups Γ_2. In particular the classes of groups Γ_3 and Γ_4 contain groups with elliptic substitutions of period 5; no extensive class of such groups have as yet been defined arithmetically. The detailed discussion of the groups defined by normal curves C_3 and C_4 is, therefore, greatly to be desired. This discussion doubtless would lead to results of much interest in themselves, and might point the way for a more detailed treatment of the general case, which at present seems to be of too great complexity for immediate treatment. In this connection it may be noted that the curves C_n defined by equations (4) will lead to groups in which the

elliptic substitutions of periods $n+1$ or $n+2$ have double points at $\zeta=0$ and $\zeta=\infty$. Their coefficients will therefore be imaginary. If, however, the curve C_n is written

$$Z_i = c_i(\zeta - \eta)^{n-i}(\zeta - \bar{\eta})^i \quad (i = 0, 1, \cdots, n),$$

these double points will be at $\zeta = \eta$, $\zeta = \bar{\eta}$ respectively. It is, therefore, possible to write down at once the equations of a C_n for which the corresponding Γ_n contains any given elliptic substitution of period $n + 1$ or $n + 2$ of form (1).

It may also be noted that *the period $\omega = n$, which satisfies the necessary condition of the earlier paper, is not in general possible if $n > 2$.* For $n = 3$, for example, there exist no collineations in S_3 of period 3 which satisfy the conditions of Theorem 1.

Finally we enquire regarding the existence of parabolic substitutions in a Γ_n. Supposing the coordinate system so chosen as to render the equations of the C_n of the form

$$(5) \qquad z_i = \zeta^{n-i} \qquad (i = 0, 1, \cdots, n),$$

any parabolic substitution may be taken to be

$$\zeta' = \zeta + 1.$$

The corresponding collineation leaving c_n invariant is then readily calculated from the equations

$$z_i' = (\zeta + 1)^{n-i} \qquad (i = 0, 1, \cdots, n),$$

and substituting from (5). The characteristic equation of this collineation is found to be

$$(\lambda - 1)^{n+1} = 0.$$

This shows first that *a collineation in S_n which produces on an invariant C_n a parabolic projectivity has only a single double point.**

It shows, furthermore, that any C_n defining a Γ_n with parabolic substitutions must contain a point with integral coordinates. This is a generalization of the corresponding fact for the case $n = 2$. Fricke † used this in the case $n = 2$ to show that every Γ_2 with parabolic substitutions must be commensurable with the elliptic modular group. His argument, however, does

* Cf. in this connection the footnote on p. 365.

† Fricke-Klein, Automorphe Functionen, vol. 1, p. 518.

not seem to apply when $n > 2$. This suggests the important question: Can any fundamental circle group (discontinuous ζ-group that is representable with real coefficients) be defined arithmetically as a Γ_n for a sufficiently high value of n?

UNIVERSITY OF ILLINOIS,
December, 1909.

A NEW ANALYTICAL EXPRESSION FOR THE NUMBER π, AND SOME HISTORICAL CONSIDERATIONS.

BY DR. G. VACCA.

IN the common exposition of the history of mathematics, more attention is given to results than to methods, and it is only rarely that old theorems and demonstrations are translated into the modern and living mathematical language. The great mathematicians of the eighteenth century (Euler, Lagrange, and others) have made important contributions to this difficult work. It seems to me that even to-day something can be done in this direction, and many new results can be obtained by a careful reading of the ancient classics of mathematics. The following is an example:

Let $f(x)$ be a function defined by the relation

$$f(x) = \tfrac{1}{2}(x + |x|)$$

and suppose that $f^n(x)$ means the result of the operation f applied n times to the number x. Then we have, if $i = \sqrt{-1}$,

$$\frac{2}{\pi} = \lim_{n=\infty} f^n(i). \tag{1}$$

This elegant theorem can be easily proved. It gives a possible analytical definition of the number π.

It is now to be observed that this *new* formula is only the analytical expression (using the geometrical representation of complex quantities of Gauss) of a series of points (approaching the vertex) of the quadratrix ($\tau\epsilon\tau\rho\alpha\gamma\omega\nu\acute{\iota}\zeta o\upsilon\sigma\alpha$) of Deinostratos.*

We may try to transform the second member of (1) into a real expression. This can be done by elementary methods. But the result is nothing else than the well-known infinite product of complicated quadratic radicals first given by Vieta.†

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* Pappi Coll. Math., lib. IV, prop. 25.

† Vieta, Opera, ed. Schooten, Lugd. Batav., 1646, p. 400.

But with the use of trigonometric functions the reduction of the formula (1) is immediate. We find

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cdots,$$

an elegant expression given by Euler.*

Note. We can avoid in the formula (1) the use of the intermediate symbol f by employing the notations followed by the Formulaire of Peano.

First, the parentheses in the expression $f(x)$ are useless, and were never employed by classic writers (Lagrange, Gauss).

Secondly, given an expression A containing the variable x, it is useful to denote by $A|x$ the operator or functional symbol f such that $fx = A$. The operation | called *inversion*, was first suggested by Eisenstein.†

Thirdly, instead of the modern symbol $|x|$ for the modulus of the quantity x, it is more convenient to use the classical symbol of Cauchy ‡ mod x, where mod is an operator prefixed to the variable, like sin, cos, $\cdots$.

Lastly, the ordinary symbol lim has some indeterminacy: it does not express in what field the variable should vary while tending to ∞. A complete expression § to indicate the limiting value of the function f when the variable, varying in the field u, tends to the value a should be of the form

$$\lim\,(f, u, a).$$

Then, if N denotes the whole class of the positive integers, we have

$$\frac{2}{\pi} = \lim\left\{\left[\left(\frac{x+\operatorname{mod} x}{2}\,\middle|\,x\right)^{n} i\right] n, N, \infty\right\},$$

and this formula is complete in itself, and denotes a truth of the same kind as the formula $7 + 5 = 12$.

GENOA, ITALY,
December, 1909.

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* *Comm. Petropol.*, vol. 9 (1737).

† Eisenstein, Math. Abhandl., Berlin, 1847, pp. 71, 91.

‡ Cauchy, Exercices, vol. 4 (1829), p. 47; Oeuvres ser. 2, vol. 9, p. 95.

§ Cf. G. Peano "Sur la définition de la limite d'une fonction," *American Journal of Mathematics*, vol. 17 (1894).

HERMITE'S WORKS.

Oeuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. II. Paris, Gauthier-Villars, 1908. 8vo. 520 pp.

IN a former number of the BULLETIN * we have given a biographical sketch of Hermite and endeavored to set forth the commanding position he occupied among French mathematicians during the last century. We shall therefore give only a brief notice of the present volume. Its 37 papers bring his publication down to the year 1872 and their chief interest centers about the elliptic functions, the solution of algebraic equations, and the theory of numbers. The reader will perhaps best obtain an idea of the rich contents of this volume if we number its papers in the order in which they occur, arrange them in groups, and indicate briefly their nature. A few of the more important papers will however be analysed at length.

Elementary Algebra.

28) A note of 28 pages written for Gerono and Roguet's Cours, edition of 1856, on homogeneous forms of the second degree in n variables.

Differential Calculus.

37) Elimination of arbitrary functions. Extracted from Hermite's Cours at the Ecole Polytechnique, 1873.

29) Short proof of $1/\rho = d\phi/ds$.

Integral Calculus.

30), 33), 34), 36) Short notes on some integrals, such as

$$\int \frac{x^m dx}{\sqrt{1-x^2}}.$$

Theory of Substitution Groups.

19) Discussion of the nature of $\theta(i)$ in order that

$$\begin{bmatrix} z_i \\ z_{\theta(i)} \end{bmatrix}$$

* 2d ser., vol. 13, p. 182.

may represent a substitution of $z_0, z_1 \cdots, z_{n-1}$. Application to the group of order 168 on 7 letters.

Invariants.

1), 20) Short notes on cubic ternary forms.
9) Reduction of the binary cubic.
10) Resultant of three ternary quadratic forms.
26) The invariants of the binary quintic.
27) The skew invariant of the binary sextic.

Theory of Numbers.

8) A modification of Dirichlet's method of determining the number of classes of binary quadratic forms having a given determinant.

Theory of Functions.

8) Determination of a polynomial $P(x)$ of degree m such that the sum of the squares of the differences between $P(x)$ and a given function for n values of x, $n \geqq m$, shall, when each difference is multiplied by a constant, be a minimum. A different solution from that given by Tchebychef.

21), 22), 23), 24) These four papers occupy 54 pages and constitute one of the most interesting features of this volume. They are a study of different functions of more than one variable which have properties in common with Legendre's function X_n in one variable.

Hermite begins by observing the fundamental nature of the θ's in the elliptic and abelian functions. These θ's are sums of terms of the type $e^{-\phi(x+h, y+h_1, \cdots)}$, ϕ being a quadratic form. The development of this term gives the series

$$\sum \frac{h^n h_1^{n'} \cdots}{n!\ n'! \cdots} U_{n, n' \cdots},$$

where the U's are polynomials in $x, y \cdots$ of degree $n + n' + \cdots$. These U's and a set of associate polynomials V are the first functions which Hermite considered. Their importance in analysis is due, so Hermite thought, partly to the fact that they have many properties of the X_n function and partly to their intimate relation with the abelian functions. After some intermediate steps Hermite arrives at another set of functions which

seem to please him better. Let us restrict ourselves to two variables, and set

$$L(a,\ b) = 1 - 2ax - 2by + a^2 + b^2,$$

$$Q(a,\ b) = 1 - 2ax - 2by + a^2(1 - y^2) + 2abxy + b^2(1 - x^2).$$

On developing we get

$$\frac{1}{Q} = \sum a^m b^n U_{m,\,n}, \qquad \frac{1}{L} = \sum a^m b^n V_{m,\,n},$$

where $U_{m,\,n}$, $V_{m,\,n}$ are polynomials in x, y of degree $m + n$. The analogy between $U_{m,\,n}$ and X_n is most perfect. We note that the integral relations analogous to

$$\int_{-1}^{1} X_m X_n dx = 0, \quad \text{etc.}$$

depend on the evaluation of integrals such as

$$\iint \frac{dxdy}{L(ab)L(a'b')} = \frac{\pi}{ab' - ba'} \arctan \frac{ab' - ba'}{1 - aa' - bb'},$$

$$\iint \frac{dxdy}{L(ab)Q(a'b')} = \frac{\pi}{aa' + bb'} \log \frac{1}{1 - aa' - bb'},$$

the field of integration being the interior of the unit circle $x^2 + y^2 \leq 1$. We recommend the evaluation of these integrals as an excellent exercise of the reader's ingenuity.

Elementary Elliptic Functions.

13) An elementary presentation of this theory occupying 114 pages. It is taken from a note in Lacroix's Calculus, edition of 1862. This little gem will always have a historical value as coming from the hand of a great master of this theory ; but even to-day one finds points of view which are suggestive and worthy of thought.

14) A transformation of the third order by means of invariants.

17) A method for rapidly computing the coefficients of x^n in the development of sn, cn, dn.

18) Development of $\sqrt[4]{k}$, $\sqrt[4]{k'}$ in series involving q.

31) Development of elliptic integrals of the first and second species.

32) Development of k^2 in powers of $(\omega - i)/(\omega + i)$.
33) Geometric demonstration of the addition theorem.

Application of Elliptic Functions to the Solution of Equations.

2) This is the world-famous solution of the quintic. For three centuries mathematicians had tried in vain to effect its solution.

3) A delightful letter written by the aged Hermite in 1900 to Tannery, giving a proof of the formulas relating to the linear transformation of the modular functions

$$\phi(\omega) = \sqrt[4]{k}, \quad \psi(\omega) = \sqrt[4]{k'}$$

employed in 2). How touching is the close of this letter. "De ma proximitée de l'Espagne je rapporte des cigarettes d'Espagnoles; si vous ne venez pas (after his return to Paris) en fumer avec votre collaborateur d'aujourd'hui, votre professeur d'autrefois, c'est que vous avez le coeur d'un tigre. Totus tuus et toto corde."

4), 5) Application of the invariant theory and the elliptic functions to the solution of the biquadratic. Here is also introduced Hermite's χ function with a table of its linear transformations.

6) Elliptic modular equations. This is a famous memoir, published serially in the *Comptes Rendus* for 1859 and then as a monograph. The point of departure of Hermite's researches on this subject is the actual determination of the resolvents of 7th and 11th degrees of the modular equations $M = 0$ of 8th and 12th degrees, whose existence Galois first pointed out in 1832. The coefficients of all the terms except the absolute term may be found without excessive labor; the last term turns out to be the square root of the discriminant $\sqrt{D}$ of $M = 0$. The calculation of D for equations of so high degree by general methods would be utterly impracticable; it is therefore necessary to make use of the peculiar properties of $M = 0$. Doing this, it is easy to show that D has the form for a prime degree n

$$D = u^{n+1}(1 - u^8)^{n+\epsilon} P^2(u),$$

where u is the function $\phi(\omega)$ above, ϵ the Legendrian symbol $\left(\frac{2}{n}\right)$ and P a polynomial with distinct roots of degree

$$\nu = \frac{n^2 - 1}{8} - \frac{n + \epsilon}{2}$$

in u^8. The determination of the coefficients of P requires now a stroke of genius. Hermite observes that the roots of $P(u)$ correspond to values of u or, what is the same thing, of ω for which $M=0$ has equal roots. His transformation theory of $\phi(\omega)$ shows him at once that this requires that ω shall satisfy the quadratic equation

$$(1) \qquad P\omega^2 + 2Q\omega + R = 0$$

with negative determinant $-\Delta$ of the type

$$(2) \qquad \Delta = (8\delta - 3n)(n - 2\delta) \text{ or } \Delta = 8\delta(n - 8\delta),$$

where δ is any integer such that $\Delta > 0$. Thus for $n = 7$ and 11, and these are the two cases Hermite is especially interested in as we observed above, we have : $n = 7$, only one determinant, viz., $\Delta = 3$; $n=11$, only two, viz., $\Delta = 7$ and $\Delta = 24$. But conversely, with certain easily determined exceptions, for each of the classes of binary forms corresponding to the determinants (2) there exist two or six quadratic forms (P, Q, R) whose coefficients in (1) give values of ω for which two of the roots of $M = 0$ are equal. Thus with hardly any calculation Herm te finds that for

$$n = 7, \quad P = 1 - u^8 + u^{16}.$$

The case $n = 11$ is more difficult. To determine P in this case Hermite observes that the values of ω given by (1) correspond to complex multiplication.* The resulting modular equations must therefore split up into rational factors. Applying this fact to the modular equation of 12th degree which Sohnke had calculated in 1837, Hermite was able to compute P for this case with relatively little labor.

7) Resolvent of the 7th degree of the modular equation ot the 8th degree.

11) Employs the skew invariant of the quintic to find a quintic resolvent of the general equation of 5th degree.

25) A long memoir of 87 pages devoted to a study of the quintic. The invariant theory is freely used. An especially interesting result is a set of criteria for the reality of the roots.

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* For a glimpse of this fascinating subject, all too little known in this country, the reader is referred to a review by the author in the BULLETIN, 2d ser., vol. 6 (1900), p. 460.

Application of Elliptic Functions to Quadratic Forms.

12), 15) To make clear to the reader the import of these two papers, in which Hermite's genius shines with such lustre, it will be necessary to go back a little. Abel stated without proof that the equation of transformation of the elliptic functions could be solved by radicals in the case of complex multiplication. This precious fact lay long buried and forgotten until Kronecker resurrected it and made it the point of departure of a long series of brilliant discoveries. He found that the equations in these singular cases split up into rational factors which stand in the most intimate relation with the number of classes of quadratic forms with negative determinants. By this means he found eight fundamental relations which being published (1857–1861) with scarcely any proof were long the wonder and admiration (or envy) of his fellow workers in this field. As an illustration let us cite one of them.

$$F(2m) + 2F(2m - 1^2) + 2F(2m - 2^2) + \cdots = 2\Phi(m),$$

where $F(m)$ is the number of classes of determinant $-m$ in which at least one of the outer coefficients is odd and $\Phi(m)$ is the sum of the divisors of m.

Right well might the mathematical world be astonished when Hermite showed how some at least of them might be obtained by elementary means, using a method already employed by Jacobi. Before explaining how Hermite did this, let us illustrate the method by a simple example which the reader can follow in detail. We know that

$$(3) \qquad \frac{4k^2}{\pi^2} = (1 + 2q + 2q^4 + 2q^9 + \cdots)^4$$

and also that this is equal to

$$(4) \quad 1 + 8\sum_n \frac{nq^n}{1 + (-1)^n q^n} = 1 + 8\sum_{m,n} n(-)^{m(n+1)} q^{(m+1)n}.$$

Now the exponent of q in the last member of (4) can take any integral value ν, whereas when we raise the parenthesis to the fourth power in (3) it has the form

$$n_1^4 + n_2^4 + n_3^4 + n_4^4.$$

This gives at once the celebrated theorem of Fermat; every

integer can be represented as the sum of four squares. If we consider the coefficient of q^ν in (4) more carefully, we observe that it is eight times the sum of the divisors of ν, if ν is odd; and 24 times the sum of the odd divisors of ν, if ν is even. Thus we get with no difficulty a celebrated theorem due to Eisenstein: the number of representations of the integer ν is eight times the sum of its divisors, when ν is odd; and 24 times the sum of its odd divisor when ν is even. This is the theorem whose proof as given by Eisenstein requires an elaborate knowledge of quaternary quadratic forms.* Let us now consider one of the cases considered by Hermite. Letting $\Theta(z)$, $H(z)$, $\Theta_1(z)$ and $H_1(z)$ be the functions of Jacobi, setting $z = 2Kx/\pi$,

$$\Theta(z) = 1 - 2q\cos 2x + \cdots, \quad \Theta_1(z) = 1 + 2q\cos 2x + \cdots,$$

$$H(z) = 2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + \cdots,$$

$$H_1(z) = 2\sqrt[4]{q}\cos x + 2\sqrt[4]{q^9}\cos 3x + \cdots,$$

Hermite finds

$$(5)\quad \frac{K}{2\pi}\sqrt{\frac{2kK}{\pi}}\frac{H^2(z)\Theta_1(z)}{\Theta^2(z)} = A\Theta_1(z) - q\sqrt[4]{q^{-1}}\cos 2x$$
$$- q^4(\sqrt[4]{q^{-1}} + 3\sqrt[4]{q^{-9}})\cos 4x - \cdot$$

On the other hand if we write

$$\frac{H^2(z)\Theta_1(z)}{\Theta^2(z)} = \frac{H(z)\Theta_1(z)}{\Theta(z)} \cdot \frac{H(z)}{\Theta(z)},$$

we have

$$\sqrt{\frac{K}{2\pi}}\frac{H(z)\Theta_1(z)}{\Theta(z)} = \sqrt{q}\sin x + \sqrt[4]{q^9}(1 + 2q^{-1})\sin 3x + \cdots,$$

$$\frac{\sqrt{kK}}{\pi}\frac{H(z)}{\Theta(z)} = \frac{2\sqrt{q}}{1-q}\sin x + \frac{2\sqrt{q^3}}{1-q^3}\sin 3x + \cdots.$$

If we multiply these two series, we get another expression for the left side of (5). Comparison of these two developments gives

$$A = \sum F(N)q^{\frac{1}{4}N}, \quad N \equiv 3 \mod 4,$$

* In our review of vol. 1 we noticed, l. c., p. 185, another demonstration by Hermite of this theorem. Cf. Oeuvres, vol. 1, p. 260.

where $F(N)$ is the number of solutions of

$$N = (2n+1)(2n+4b+3) - 4a^2 \qquad (a = 0, \quad \pm 1, \cdots \pm n)$$

and n, b are positive integers. On the other hand Hermite shows that $F(N)$ is nothing but Kronecker's function F defined above. Let us now set $x = 0$ in (5). The left side vanishes, and if we arrange the right side according to powers of q, Hermite finds, letting d', d'' be divisors of N such that $d' > \sqrt{N}$, and $d'' < \sqrt{N}$, that

$$A\Theta(0) = \tfrac{1}{2}\Sigma q^{\frac{1}{4}N}(\Sigma d' - \Sigma d'').$$

The coefficient of $q^{\frac{1}{4}N}$ on the right Kronecker calls $\Psi(N)$; the left side we see is the product of two infinite series in q. Performing the multiplication and equating coefficients of like powers of q gives finally

$$F(N) + 2F(N-2^2) + 2F(N-4^2) + \cdots$$
$$+ 2F(N-4k^2) = \tfrac{1}{2}\Psi(N),$$

a relation between the number of properly primitive quadratic forms with the determinants $-N$, $-(N-4)$, $-(N-16)$, $\cdots$

If we have gone into some details in speaking of the papers 6), 12), and 15), it is partly because their importance demands more than a passing notice and partly with the hope that our remarks may awaken the interest of some reader of this BULLETIN to look farther into these matters.

JAMES PIERPONT.

SHORTER NOTICES.

Serret's Lehrbuch der Differential- und Integralrechnung. Dritte Auflage, dritter Band,* neu bearbeitet von GEORG SCHEFFERS. Leipzig, Teubner, 1909. xii + 658 pp.

THIS book on differential equations is the third and last volume of Scheffer's "Umarbeitung" of the second edition of Serret's Lehrbuch. In comparison with the first two volumes, there are many more alterations made in this third edition of the third volume. In fact one can hardly recognize any traces

*The first two volumes of this work were reviewed in the BULLETIN, vol. 15 (1908–09), p. 140.

of the old book in the new. The arrangement of material has been changed, some subjects have been omitted entirely, some have been developed from mere paragraphs into chapters, the real variable has been separated from the complex, the figures have been redrawn, and although the new edition contains fewer general topics than the old yet it is nearly one half larger. The worked out illustrative examples are left practically unchanged, but many of them have been shifted to more fitting sections.

The first chapter is a thirty-page general discussion of differential equations, ordinary, total, and partial, with a clear statement of the two directions which a course in differential equations might take:

I. "Die Bestimmung aller Lösungen bzw. Lösungensysteme soweit wie mittels Elimination, Substitution und Differentiation möglich auf blosze Quadraturen zurückzuführen."

II. "Unmittelbar aus den vorgelegten Gleichungen oder Systemen von Gleichungen die Eigenschaften der Lösungensysteme zu erkennen."

A broad interpretation of the first point of view dominates the book, but the second is not neglected, about a quarter of the volume being devoted to the function theory side of the subject.

The second chapter consists of the derivation of existence theorems for ordinary differential equations, systems of ordinary differential equations, and implicit functions of one or more variables. This chapter is parenthetical in character and "upon first reading" may be omitted. The real variable alone is considered. The third, fourth, and fifth chapters treat of ordinary differential equations of the first order, systems of ordinary differential equations of the first order, and ordinary differential equations of higher orders respectively. The rather brief discussion of Lie's integration methods based upon infinitesimal transformations which appeared in the second edition has been greatly expanded in these chapters and has been made one of the features of the book, notwithstanding the detailed treatment of the older methods. The sixth chapter, also parenthetical in character, extends the theory of the second chapter into the domain of the complex variable.

Partial differential equations are taken up in the next two chapters. The seventh treats of the linear partial differential equation of the first order, first reducing the problem to that of the homogeneous partial differential equation. A few pages at the end are devoted to Pfaff's equation in three variables. The

eighth chapter takes up the general partial differential equation of the first order by two methods. First the emphasis is laid on the geometrical methods of Lagrange and Monge, which are made to serve as an introduction to the analytical method of Cauchy given in the second part. Partial differential equations of higher orders are not discussed in this edition.

Zermelo's chapter on the calculus of variations in the second edition has been replaced by a shorter chapter with practically the same content. The notation has been changed to conform to the notation of the more recent articles and treatises. There is no treatment of sufficient conditions, merely a discussion of Euler's equations for the simplest problem of the calculus of variations, with some extensions to isoperimetric problems and problems in three variables. The subject matter is illustrated by the usual examples, the catenary, brachistochrone, etc. We note that in this edition, "Die Eulersche Differentialgleichung" replaces "Die Lagrangesche Differentialgleichung," a result probably of Professor Bolza's championship of Euler's claim to priority.

Harnack's appendix on the integration of partial differential equations and the few pages of "Bemerkungen" have been omitted in this edition.

As a third volume in a course in calculus, intended for students in their first three semesters, the present volume will be found rather advanced, notwithstanding the footnotes pointing out paragraphs and chapters which may be omitted. Scientifically, however, the new edition is a vast improvement over the old. The arrangement of the subject matter, the clearness of the language, the precise statement of definition and theorem, the copious index, and the typography place this work among the best reference text-books on differential equations in German or English.

A. R. CRATHORNE.

Kreis und Kugel in senkrechter Projektion, für den Unterricht und zum Selbststudium. Von Dr. OTTO RICHTER. Leipzig und Berlin, Teubner, 1908. x + 187 pp., with 147 figures.

THE author's aim as set forth in the preface is to furnish a supplement to the numerous elementary treatises on descriptive geometry. He proposes to give general solutions of certain fundamental problems which are studied for special cases only in books on descriptive geometry, and to give the student a

knowledge of the principles of stereometry which he has a right to know. The hope is expressed that the book may serve to promote skill in draughting, to increase the reader's ability to see things in space, and to arouse and develop in the youthful mind a sense of accuracy and beauty.

The first chapter is devoted to the ellipse, beginning in a most elementary fashion with the ellipse as an oblique section of a right circular cylinder. The geometrical properties of the curve are developed in detail, as well as all construction problems involved. This chapter may seem unnecessarily drawn out, but the author justifies it on the ground that the beginner with little or no knowledge of conic sections is enabled thus to collect his tools and become acquainted with much which will be of value in subsequent work.

The second chapter treats the sphere in like detail. This chapter introduces the idea of a non-euclidean (elliptic) geometry on a sphere with arcs of great circles as "straight lines," the object then being to represent on a plane the configurations of this geometry. Thus, in analogy to plane geometry, we erect the perpendicular to a straight line at a given point, bisect lines, bisect angles formed by two straight lines, extend two segments of straight lines and find their point of intersection and draw "tangents" to circles. Such problems as the construction of the poles of a plane and a general discussion of rectangular axonometry are included in this chapter.

The third and last chapter is devoted to applications, and forms more than one half of the text. This chapter is less detailed and more suggestive than the preceding ones. Many problems are left for the reader to complete or investigate entirely. The chapter is divided into five sections. The first treats of prisms, pyramids, cylinders, cones, and spheres. Two classes of problems are discussed, those involving relative position only, as inscribing a pyramid in a sphere; and those involving relative magnitudes, as drawing the beam of greatest strength in a given cylindrical log, the section to be rectangular. The drawing of the intersections of these solids is eventually included, but the cones and cylinders are only of the second degree. The second section is devoted to spherical wedges and allied forms. The third includes a classification and general method for representing Archimedean and Platonic bodies, and a "free" drawing of regular polyhedra including the Kepler and Poinsot star polyhedra. The fourth deals with bodies of

revolution and spiral forms, a greater part of the space being devoted to the torus and the spiral stairway. The closing section is concerned with the problem of representing cylinders, cones, and spheres having an orthogonal network of lines on their surface. This leads at once to drawing maps of the earth's surface and of the celestial regions. Throughout the last chapter numerous and really wonderful examples of regular and semiregular forms occurring in nature are pointed out, and in a few cases the principles of the text are employed in drawing a flower or plant.

While a portion of the book could be read with advantage by the beginner, it would seem to the reviewer to be most useful to the teacher in suggesting methods and examples. It is much more mathematical than American text-books on descriptive geometry, and less extensive than many of the German books on that subject, such as Geyger. On the other hand it does not cover the field of stereometry in general on such an ambitious scale as does Dr. Holzmüller in his four-volume Elemente der Stereometrie. In many cases the figures are rendered unnecessarily complicated by the attempt to use the same figure for a number of examples, often separated by many pages. The author has undoubtedly succeeded in thoroughly discussing the limited field he selects, and the reader who covers the book carefully will be certain to gratify the author's desire to increase his power of space perception.

D. D. LEIB.

Plane and Spherical Trigonometry and Four-Place Tables of Logarithms. By W. A. GRANVILLE. Boston, Ginn and Company, 1909. xi + 264 + 38 pp.

THIS is one of the excellent series of elementary mathematical text-books which are published under the supervision of Professor P. F. Smith, of the Sheffield Scientific School. The present volume is up to the standard for which the series has already earned a reputation.

The number of text-books in trigonometry is growing larger every day, but there always seems to be room for one more, provided it is written in a modern spirit so as to satisfy the needs of the present day. This book covers the usual topics and contains all the trigonometry that is usually taught in the undergraduate classes of colleges and technical schools. The demonstrations are simple and exceedingly clear, and the book

is abundantly rich in exercises. With regard to the numerical work it is evident that the author believes in the pedagogic value of accuracy and the proper arrangement of computation. Great prominence has also been given to graphical methods.

The book differs from the general type of text-books in trigonometry in several points. Under each case in the solution of triangles there are two sets of examples, one in which the angles are given in degrees and minutes, and another in which they are expressed in degrees and the decimal part of a degree. Another feature is the treatment of spherical trigonometry, in deriving the formulas of which the author makes use of the principle of duality, which is stated in the following form: "If the sides of a general spherical triangle are denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters, α, β, γ, then, from any given formula involving any of these six parts, we may write down a dual formula by simply interchanging the corresponding Greek and Roman letters." This method has many advantages, a great part of the work required in deriving formulas being done away with.

The general appearance of the book is very attractive. The cuts, the typography, and the arrangement of matter on the page are excellent.

JACOB WESTLUND.

Dynamique Appliquée. Par L. LECORNU. Paris, Octave Doin, 1908. 534 pp. 5 francs.

Hydraulique Générale. Par A. BOULANGER. Paris, Octave Doin, 1908. Vol. 1, xvi + 382 pp. Vol. 2, vii + 299 pp. 5 francs each.

(Encyclopédie Scientifique. Publiée sous la direction du Dr. Toulouse.)

M. LECORNU's textbook on applied mechanics is divided into four parts: Résumé of the chief results of rational mechanics; Mechanical properties of materials; Applications of dynamics; Theory of machines.

The first part occupies seventy-one pages, and furnishes the theoretical foundation for the rest of the book. It is a summary of the well-known equations of translations, moments; kinematics of a point, a set of points, and a solid; dynamics of particles, momentum, lifting force; statics of systems, virtual work, restraints, equilibrium of solids and of jointed systems,

funiculars, statics of fluids; dynamics of free systems; dynamics of restrained systems; dynamics of rigid bodies; dynamics of fluids. The subject is evidently sufficiently vast, and one could scarcely expect to find in the small space assigned even a complete résumé. The author has succeeded however in including the chief results of a course in theoretical mechanics, rather as a review to the student than as a formulary for reference.

In the second part three topics are considered: friction, which receives very careful attention, resistance of the air, and impact. Under friction we find discussed the friction of journals, gears, screws, the universal joint, sliding pieces, friction clutches, skidding, cord on cylinder, cylinder on inclined plane, billiard ball, the tendency to extinguish the friction. These topics give some idea of the completeness of the discussion. The next chapters treat rolling friction, pivotal friction, and the stiffness of ropes.

The third part considers applications of theoretical dynamics to springs, to Watt's indicator, and to pendular movements — these rather fully. More briefly, the top, the hoop, the bicycle, the balancing of wheels, the equilibration of turbines. The fourth part discusses machines: in particular, motors, governors, brakes, and transmission of power.

The object of the book is to exemplify the great practical advantage in attacking problems of applied mechanics directly by the methods of rational mechanics. "It has for principal object the study of the general properties of machines, excepting questions concerning thermodynamics or electricity." The author has succeeded in his object and the illustrations he uses are very well chosen. It would be well if some such exemplification as this could be introduced into our textbooks on theoretical mechanics.

The latter of the two books under review is avowedly a systematic résumé of the results and methods of Professor J. Boussinesq, first presented in various memoirs treating of the movements of water. The problems discussed are well classified and treated with much completeness. The introduction establishes the fundamental laws of hydrodynamics; the first section discusses phenomena in which friction is negligible; the second section, phenomena in which friction has a sensible influence; the third section, the phenomena of turbulent movements. In

the second volume, which is devoted to singularities and applications, there are also three sections with the same headings as in the first volume.

The swell on the ocean is first discussed, followed by the solitary wave or wave of translation, and waves of emersion and impulsion. The flow of water through fine tubes, the phenomena of filtration, and rotatory flow are taken up in the next section. Under turbulent movement we find the uniform and the gradually varied flow in conduits and canals.

In the second volume the topics are: flow through orifices, flow over weirs, the extinction of the ocean swell and waves, resistance to immersed solids, motion in conduits and canals with singularities such as enlargements, contractions, bends, etc., natural water courses, motion in rectangular canals with eddies, waves in elastic conduits, and the water hammer.

The treatment of the subject is deep enough for the usual student and tends to incite reading of the original memoirs of investigators in this field. It is also calculated to interest the student of physics on the one hand and the scientific engineer on the other. If such courses as this were introduced into the senior year of our engineering colleges in place of certain customary "practical" courses which could just as well be read up outside by the intelligent student (and no other should be considered!), it would tend to improve the scientific study of many engineering problems now worked out only empirically and approximately. Particularly as the laws developed here are in general experimentally verifiable. In fact the title might be translated into Theoretical Hydraulics in contradistinction to Practical Hydraulics. With the definiteness found in the treatment throughout, the engineering student can scarcely fail to look upon his problems from a much higher point of view than if he has only the usual course in hydraulics.

These two treatises exemplify very well the scientific character of the Encyclopédie Scientifique. If all the projected volumes are kept up to the same level, the whole will undoubtedly be a collection which every student should possess. At least it would serve in America as a corrective to too much practicalism, and would encourage the scientific spirit in engineering. It is needless to say that courses in "engineering mathematics" would not suffice for the reading of such works as these.

JAMES BYRNIE SHAW.

Die Theorie der Besselschen Funktionen. Von PAUL SCHAFHEITLIN. Leipzig, Teubner, 1908. v + 129 pp.

B. G. TEUBNER is publishing a series of mathematical and physical monographs for engineers. Each volume treats some topic in mathematics or in physics and in a hundred pages gives the busy engineer a general idea of the subject and a short account of its history, proves some theorems and gives others unproved, tells him something of the applications and gives him references to more complete works.

The volume under review is number four of the series. It is designed to give a reader who knows only the elements of the calculus a working knowledge of the theory of Bessel's functions. The introduction gives a short history of the functions, a list of important articles and treatises, and mentions the applications to hydrodynamics, wireless telegraphy, vibrations of a membrane, etc. Then comes a collection of some thirty unproved theorems and formulas, chiefly in hypergeometric series and gamma functions, to which the author refers in the course of the book. The subject matter proper begins with a solution of Bessel's differential equation in series by undetermined coefficients, and a definition of Bessel's function of the first kind. Since no knowledge of differential equations is assumed on the part of the reader, a short discussion of linear homogeneous differential equations and their solutions is given. Then follow with proofs the usual formulas giving the relations between Bessel's functions of different orders and their derivatives and the different expressions for the Bessel's functions of the first kind. In particular the case in which the parameter is an integer and the case in which it is the half of an odd number are discussed. The remaining chapters take up Bessel's functions of the second kind, the representation of arbitrary functions by Bessel's functions, the addition and multiplication theorems and the variation, graphical representation, and roots of the functions for various orders. Throughout the book important theorems and formulas are numbered in heavy type and these collected together form an appendix to the volume. The book is a mine of information about Bessel's functions, but the engineer will wish for a fuller discussion of the applications.

A. R. CRATHORNE.

CORRECTION.

In Miss Noether's abstract on page 116 of the January BULLETIN the last sentence should read: The two fundamental theorems of the symbolic method, that concerning invariant processes and that regarding the totality of the invariant identities, readily appear by this method.

In the February BULLETIN, page 274, the amount of the prize granted by the Göttingen academy to Dr. Wieferich should have been stated as 1,000 Marks. Dr. Wieferich's paper appeared in volume 136 of *Crelle*.

NOTES.

THE twenty-sixth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday and Saturday, April 8, 9, 1910, the first session opening at 10 o'clock a. m. in Room 32, Ryerson Physical Laboratory.

THE lectures delivered before the New Haven Colloquium of the American Mathematical Society in September, 1906, by Professors E. H. MOORE, E. J. WILCZYNSKI, and MAX MASON are now in press and will soon be issued in book form. The lectures are published by Yale University.

AT the meeting of the London mathematical society held on February 10, the following papers were read: By H. W. RICHMOND, "A note on double series of lines"; by H. LAMB, "On the diffraction of a solitary wave"; by H. F. BAKER, "Notes on various points in the theory of functions."

THE valuable library on mathematics and science of the late OREN ROOT, for many years professor of mathematics at Hamilton College, has been presented to the college by his son, Senator ELIHU ROOT.

UNDER the auspices of the Italian ministry of public instruction and the editorial direction of Professors V. Volterra, G. Loria and D. Gambioli the first complete edition of the mathematical works of Count JULIUS CHARLES DI FAGNANO will be

published in the course of the year. The edition will include his papers of a controversial character and scientific correspondence, several valuable documents of which have been recently discovered in the Oliveriana library in Pesaro. In order to make this collection as complete as possible, the editors request the coöperation of any persons who can supply information as to the existence in public or private libraries of manuscripts regarding Count Fagnano.

THE mathematical papers of the late Professor H. MINKOWSKI are being collected and edited by Professor D. HILBERT. They will be published in two octavo volumes by Teubner during the summer, and will include all his writings except the two books, Geometrie der Zahlen and Diophantische Approximationen. The memoirs are arranged in four groups, comprising the theory of numbers, the geometry of numbers, geometry, and physics. An extensive manuscript concerning the theory of convex bodies will appear for the first time in the forthcoming publication.

THE firm of Macmillan and Company of London announce the publication of a two-volume biography of Sir William Thomson by Professor S. P. THOMPSON. While the present work also considers his early life, it does not encroach upon the family narrative which has just been edited by Lord Kelvin's nieces.

THE prize of 1,200 francs announced by the Istituto Lombardo (at Milan) for an important contribution to the theory of continuous groups has been awarded to Professor U. AMALDI, of the University of Modena, for a paper entitled "I gruppi continui infiniti di transformazioni puntuali della spazio a tre dimensioni."

THE next summer meeting of the British association for the advancement of science will be held at Sheffield beginning August 31, under the presidency of Professor T. G. BONNEY. Professor E. W. HOBSON is chairman of section A, mathematics and physics.

DURING the interval from April 11 to April 22 a vacation course in mathematics for teachers of secondary schools will be held at the University of Göttingen. The following programme is announced: By Professor H. BEHRENDSEN: Geo-

metric methods in instruction in arithmetic, four hours.—By Professor F. KLEIN: Elementary mechanics, six hours.—By Professor L. PRANDTL: Aerial dynamics and ballooning, four hours.—By Professor E. LANDAU: Theory of aggregates, four hours.

THE following university courses are announced for the summer semester:

UNIVERSITY OF BONN.—By Professor E. STUDY: Theory of elliptic functions, four hours; Seminar, two hours.—By Professor F. LONDON: Descriptive geometry with exercises, four hours; Selected chapters of analytic geometry, two hours: Seminar, two hours.—By Professor S. HESSENBERG: Foundations of geometry, two hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Theory of numbers, three hours.—By ———: Differential and integral calculus, four hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Analytic geometry of the plane, four hours; Selected portions of the theory of functions, two hours; Seminar.—By Professor H. GRASSMANN: Ordinary differential equations, four hours; Descriptive geometry, II, five hours; Seminar.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Applications of the calculus to geometry, four hours; Seminar on the boundary between mathematics and philosophy (with Professor Zermelo), two hours.—By Professor D. HILBERT: Principles of mathematics, four hours; Selected chapters in the theory of partial differential equations, two hours; Seminar, two hours.—By Professor E. LANDAU: Differential and integral calculus, with exercises, five hours; Selected chapters in the theory of functions, two hours; Concerning the Fermat theorem, one hour; Seminar, two hours.—By Professor C. RUNGE: Analytic geometry with exercises, four hours; Seminar, two hours.—By Professor L. PRANDTL: Introduction to thermodynamics, three hours; Seminar, two hours.—By Professor E. ZERMELO: Differential equations, three hours; Seminar (with Dr. Toeplitz and Dr. Born), two hours.—By Dr. P. KOEBE: Introduction to synthetic geometry, two hours; Riemann's theory of functions, four hours.—By Dr. O. TOEPLITZ: Introduction to the theory of integral equations, and of an infinite number of variables, four hours.—By Dr. F. BERNSTEIN: Quadratic and relative quadratic number fields, two hours;

Calculation of insurance, one hour.—By Dr. C. H. Müller: Mathematics of Archimedes, one hour; Proseminar in mechanics of rigid bodies, two hours.

University of Greifswald. — By Professor W. Thomé: Theory of analytic and particularly elliptic functions I, four hours; Algebraic surfaces and space curves, two hours; Seminar. — By Professor F. Engel: Differential and integral calculus I, four hours; Analytic geometry of space, two hours; Seminar. — By Professor K. T. Vahlen: Differential geometry, three hours, with exercises, one hour.

University of Munich. — By Professor F. Lindemann: Integral calculus, five hours; Theory of substitutions and higher algebraic equations, four hours; Seminar, two hours. — By Professor A. Voss: Analytic geometry of space, four hours; Calculus of variations, three hours; Seminar, two hours. — By Professor A. Pringsheim: Selected chapters in the theory of analytic functions, four hours. — By Professor A. Sommerfeld: Partial differential equations of physics for beginners, three hours; with exercises, one hour. — By Professor H. Brunn: Elements of higher mathematics for students of all faculties, four hours. — By Professor K. Doehlemann: Descriptive geometry, II, four hours; with exercises, 2 hours; Synthetic geometry, II, two hours; with exercises, two hours. By Dr. H. Hartogs: Theory of abelian functions, II, with applications to the theory of algebraic curves, three hours. — By Dr. O. Perron: Selected chapters of elementary geometry, five hours; Theory of linear differential equations, four hours. By Dr. R. Wagner: Mathematical supplement to the lectures on experimental physics, one hour.

Professor E. Zermelo, of Göttingen, has been appointed professor of mathematics at the University of Zurich.

Dr. A. Buhl has been appointed professor of general mathematics at the University of Toulouse.

Professor A. R. Forsyth, of the University of Cambridge, has resigned the Sadlerian professorship of mathematics. The vacancy has been filled by the election of Dr. E. W. Hobson, of Christ's College.

At the University of Berlin, Professor G. Frobenius and Professor F. Schottky have been given the title of Geheimer Regierungsrat.

DR. —. HELLEBRAND, of the agricultural institute of Vienna, has been promoted to an associate professorship of mathematics.

THE mathematical staff of Columbia University has been enlarged by the appointment of Professor W. B. FITE, of Cornell University, and H. E. HAWKES, of Yale University, as professors of mathematics.

MR. H. H. MITCHELL, of Princeton University, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

PROFESSOR A. CAPELLI, of the University of Naples, died January 28 at the age of 55 years. He was a member of the Italian Society of Sciences, of the Accademia dei Lincei, etc.

PROFESSOR F. PURSER, of the University of Dublin, author of several memoirs on the Bessel functions, died January 28 at the age of 70 years.

BOOK catalogues: A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogue for 1910, about 150 titles in mathematics. — H. Sotheran and Company, 140 Strand, London, price current No. 702, about 200 titles in mathematics.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor*, PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

BACHMANN (P.). Niedere Zahlentheorie. Zweiter Teil: Additive Zahlentheorie. Leipzig, Teubner, 1910. 8vo. 10 + 480 pp. Cloth. M. 17.

BAUER (G.). Vorlesungen über Algebra. 2te Auflage, herausgegeben vom Mathematischen Verein München. Leipzig, Teubner, 1910. 8vo. 6 + 366 pp. Cloth. M. 12.

DANIELSEN (O.). Nogle bemærkninger om en gruppe algebraiske flader, der kunne bringes til at svare entydig til en plan punkt for punkt. Kjöbnhavn, 1909. 8vo. 92 pp. M. 3.60

FABRY (E.). Problèmes et exercices de mathématiques générales. Paris, Hermann, 1900. 8vo. 424 pp. Fr. 10.00

FRÉCHET. See HADAMARD (J.).

FREUND (E.). Entwicklung willkürlicher Funktionen vermittelst meromorpher. (Diss.) Breslau, 1909. 8vo. 60 pp.

GEUS (A.). Die eindeutigen Transformationen der ebenen Kurve dritter Ordnung in sich, invarianten- und funktionentheoretisch behandelt. (Diss.) Erlangen, 1909. 8vo. 43 pp.

HADAMARD (J.). Leçons sur le calcul des variations. Recueillies par M. Fréchet. Vol. I: La variation première et les conditions du premier ordre; les conditions de l'extremum libre. Paris, Hermann, 1910. 8vo. $8 + 520$ pp. Fr. 18.00

HÖEGH (E. VON). Elementarer Beweis des Fermatschen Satzes. Rostock, 1909. M. 1.50

IHLENBURG (W.). Ueber die geometrischen Eigenschaften der Kreisbogenvierecke. Leipzig, 1909. M. 8.50

KOWALEWSKI (G.). Die klassischen Probleme der Analysis des Unendlichen. Ein Lehr- und Uebungsbuch für Studierende zur Einführung in die Infinitesimalrechnung. Leipzig, Engelmann, 1910. 8vo. $8 + 383$ pp. Cloth. M. 16.50

LACHELIER (H.). See LEIBNIZ.

LA COUR (P.). Historisk Matematik. 3. udgave. III: Bogstavregning. Kjöbnhavn, 1909. 8vo. 72 pp. M. 1.50

LEHMER (D. N.). Factor table for the first ten millions, containing the smallest factor of every number not divisible by 2, 3, 5, or 7 between the limits 0 and 10,017,000. Washington, Carnegie Institution, 1909. Folio. 476 pp.

LEIBNIZ. La monadologie. Publiée d'après les manuscrits de la bibliothèque de Hanovre, avec introduction, notes, et suppléments, par H. Lachelier. 7e tirage. Paris, Hachette, 1909. 16mo. 103 pp. Fr. 1.00

MANNING (H. P.). The fourth dimension simply explained. A collection of popular essays, with an introduction and editorial notes by Henry P. Manning. New York, Munn, 1910. 8vo. 251 pp. Cloth. $1.50

MEYER (W. F.). Allgemeine Formen- und Invariantentheorie. Vol. I: Binäre Formen. Leipzig, 1909. See M. 9.60

NIEWIADOMSKI (R.). Analyse de l'équation $z^n = x^n + y^n$, avec une résolution du grand théorème de Fermat. Varsovie, 1909. 8vo. 40 pp. M. 2.50

SCHOY (C.). Beiträge zur konstruktiven Lösung sphärisch-astronomischer Aufgaben. Leipzig, Teubner, 1910. 8vo. $7 + 40$ pp. M. 1.60

SYLVESTER (J. J.). Collected mathematical papers. Vol. 3. Cambridge, University Press, 1909. 8vo. Cloth. 18s.

ZIEMKE (E.). Ueber partielle Differentialgleichungen erster Ordnung mit Integralvereinen, die als Punktmannigfaltigkeiten zweifach ausgedehnt sind. (Diss.) Greifswald, 1909. 8vo. 73 pp.

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BELLENGER (H.). See NEVEU (H.).

BÖGER (R.). Elemente der Geometrie der Lage, für den Schulunterricht bearbeitet. 2te Auflage. Leipzig, Göschen, 1910. 16mo. 45 pp. M. 0.90

—— Projektive und analytische Schulgeometrie. Ein Lehr- und Uebungsbuch für die Oberklassen. Leipzig, Göschen, 1910. 8vo. 8 + 211 pp. Cloth. M. 3.60

BONET Y GARCÍA (J.). Apuntes de algebra elemental. 3a edición corregida y aumentada. Cuaderno 1°: Pizarras. Cuaderno 2°: Aclaraciones. Madrid, 1909. 2 vols., 104 and 137 pp. P. 8.50

BRIOT (C.) et VACQUANT (C.). Elementos de geometría aplicáda. Paris, Hachette, 1909. 16mo. 436 pp. Fr. 3.50

BÜTZBERGER (F.). Lehrbuch der ebenen Trigonometrie. Resultate der Aufgaben. Zürich, 1910. 8vo. 35 pp. M. 2.00

CAMINATI (P.). La somma degli angoli d'ogni triangolo rettilineo è sempre precisamente equivalente alla somma di due angoli retti: dimostrazione elementare indipendente dal V postulato e dall' XI assioma di Euclide. Parma, Zerbini, 1909. 4to. 4 pp.

CANNAVIELLO (M.). Corso die geometria elementare, per le scuole medie. Parte II: Stereometria. Napoli, Simeone, 1909. 8vo. Pp. 323–469. L. 1.30

COHN (B.). Tafeln der Additions- und Subtraktions-Logarithmen auf sechs Dezimalen. Leipzig, 1909. M. 4.00

COMBET (E.). Tables abrégées de logarithmes, de lignes trigonométriques naturelles, de nombres usuels et de constantes physiques. Paris, Belin, 1909. 16mo. 15 pp. Fr. 0.50

ENRIQUES (F.) et AMALDI (U.). Elementi di geometria, ad uso delle scuole tecniche. Bologna, Zanichelli, 1909. 16mo. 7 + 254 pp. L. 2.00

GODFREY (C.) and SIDDONS (A. W.). Elementary geometry. Part 2, theoretical. Cambridge, University Press, 1910. 8vo. 344 pp. Cloth. 3s.

GRÉVY (A.). Géométrie à l'usage des élèves des classes de cinquième B, troisième B (programmes du 31 mai 1902). 7e édition. Paris, Vuibert, 1910. 16mo. 323 pp.

INSKIP (G. D.). A new manual of squares and logarithms from 0 feet to 100 feet, advancing by 32nds of inches. Logarithms of numbers from 0 to 10000, logarithmic trigonometric functions for each minute and the auxiliary trigonometric functions S and T for the calculation of the logarithms of sines, tangents and cotangents. London, Wesley, 1910. 8vo. 546 pp. Cloth. 12s.

INTERMEDIATE applied mathematics papers. Questions set at the University of London from 1892 to 1909. London, Clive, 1909. 8vo. 88 pp. 2s. 6d.

KAMBLY und ROEDER. Trigonometrische und stereometrische Lehraufgabe der Untersekunda. Sonderdruck aus der 148–151ten Auflage der Planimetrie, neu bearbeitet von A. Thaer. Breslau, Hirt, 1909. 8vo. Pp. 177–240. M. 0.80

MAHLERT (A.). See MÜLLER (H.).

MARTÍN MENGOD (A.). Elementos de geometría, y trigonometría rectilínea y esférica. Málaga, La Española, 1909. 161 pp. P. 4 00

—— Programa de aritmética y algebra. Málaga, La Española, 1909. 20 pp. P. 1.00

MÉNDEZ SORET (L.). Programa de nociones y ejercicios de geometría. Málaga, La Española, 1909. 15 pp. P. 1.00

MÜLLER (H.) und MAHLERT (A.). Lehr- und Uebungsbuch der Arithmetik und Algebra für Studienanstalten. Ausgabe A: Für gymnasiale Kurse. 1ter Teil: Bis zur Lehraufgabe der Klasse IV. Leipzig, Teubner, 1909. 8vo. 6 + 181 pp M. 2 40

NEVEU (H.) et BELLENGER (H.). Cours de géométrie théorique et pratique à l'usage des élèves des écoles primaires supérieures et des candidats aux Ecoles d'arts et métiers. 1re partie. Géométrie plane. 2e édition. Paris, Masson, 1909. 16mo. 470 pp. Fr. 3.50

PAHL (F.). See SCHULZE (F.).

PETERS (J.). New calculating tables for multiplication and division by all numbers from 1 to 10,000. Berlin, Reimer, 1909. 6 + 500 pp. Cloth. M. 15.00

POTTER (W. J.). Concurrent, practical and theoretical geometry. Parts 1–3. London, Holland, 1910. 8vo. 704 pp. 4s. 6d.

ROEDER. See KAMBLY.

SAINT-PAUL (H. DE). Tables des lignes trigonométriques naturelles des angles et des arcs, variant de minute en minute, depuis 0° jusqu'à 90°. Paris, Gauthier-Villars, 1910. 8vo. 32 pp. Fr. 1.50

SALOMON (MME. A.). Leçons d'algèbre à l'usage de l'enseignement secondaire des jeunes filles (classes de quatrième et cinquième années) et des aspirantes au brevet supérieur. Paris, Vuibert, 1910. 16mo. 6 + 229 pp. Fr. 2.00

SCHULZE (E.) und PAHL (F.). Mathematische Aufgaben. Ausgabe für Realgymnasien und Oberrealschulen. Ergebnisse. 2ter Teil. Leipzig, Dürr, 1909. 8vo. 112 pp. M. 5.00

SCOTTI (G.). Elementi di geometria intuitiva, ad uso del ginnasio inferiore e dei corsi complementari. Tredicesima edizione. Torino, Salesiana, 1910. 8vo. 139 pp. L. 1.00

SIDDONS (A. W.). See GODFREY (C.).

THAER (A.). See KAMBLY.

TUÑÓN DE LARA (M.). Programa de geometría. Madrid, Ducazcal, 1910. 32 pp. P. 1.00

VACQUANT (C.). See BRIOT (C.).

III. APPLIED MATHEMATICS.

BIRCK (O.). See SCHWARZSCHILD (K.).

BOREL (E.). See VESSILLIER (G.).

BOWLEY (A. L.). An elementary manual of statistics. London, Macdonald, 1910. 8vo. 224 pp. Cloth. 5s.

COOPS (G. H.). Uebersichtliche Darstellung des 2ten Hauptsatzes der Thermodynamik und der daraus herzuleitenden Folgen. Groningen, 1909. M. 0.75

COTTERILL (J. H.) and SLADE (J. H.). Lessons in applied mechanics. Vols. I and II. London, Macmillan, 1910. Each volume. 3s.

DARWIN (G. H.). Scientific papers. Vol. 3. Cambridge, University Press, 1910. 8vo. 15s.

EBERHARDT (C.). Theorie und Berechnung der Luftschrauben. Mit Beispielen und Versuchsresultaten aus der Praxis. Berlin, Krayn, 1910. 8vo. 122 pp. M. 7.50

GEOMETRICAL solutions derived from mechanics; a treatise of Archimedes. Translated by J. L. Heiberg. London, Paul, 1910. 8vo. 2s. 6d.

GRETHER (H.). Ueber Potentialbewegung tropfbarer Flüssigkeiten in gekrümmten Kanälen. (Diss.) Karlsruhe, 1909. 4to. 118 pp.

GUICHARD (C.). Traité de mécanique à l'usage des élèves de mathématiques A et B et des candidats aux écoles. 6e édition. Paris, Vuibert, 1909. 8vo. 7 + 248 pp.

GUILLAUME (C. E.). Initiation à la mécanique. Ouvrage étranger à tout programme, dédié aux amis de l'enfance. 2e édition, revue. Paris, Hachette, 1909. 16mo. 14 + 214 pp. Fr. 2.00

HEIBERG (J. L.). See GEOMETRICAL.

IZART (J.). Mécanique. 32e édition. Paris, Dunod, 1910. 16mo. 451 pp. Fr. 3.50

JULLIOT DE LA MORANDIÈRE (L.). De la réserve mathématique des primes dans l'assurance. (Thèse.) Paris, Larose, 1909. 8vo. 604 pp.

POINCARÉ (H.). See VESSILLIER (G.).

SCHWARZSCHILD (K.) und BIRCK (O.). Tafeln zur astronomischen Ortsbestimmung im Luftballon bei Nacht, sowie zur leichten Bestimmung der mitteleuropäischen Zeit an jedem Orte Deutschlands. Göttingen, 1909. M. 3.80

SLADE (J. H.). See COTTERILL (J. H.).

VESSILLIER (G.). Théorie des systèmes géométriques (à masses égales) appliqués aux chances simples de la roulette. Avec des lettres de H. Poincaré et de E. Borel. Paris, Delarue, 1909. Fr. 20.00

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and forty-seventh regular meeting of the Society was held in New York City on Saturday, February 26, 1910, extending through the usual morning and afternoon sessions. The following twenty-eight members were present:

Professor G. D. Birkhoff, Professor C. L. Bouton, Professor E. W. Brown, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Professor L. P. Eisenhart, Professor Peter Field, Professor T. S. Fiske, Dr. C. C. Grove, Professor J. I. Hutchinson, Dr. L. C. Karpinski, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor J. H. Maclagan-Wedderburn, Mr. A. R. Maxson, Mr. H. H. Mitchell, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Dr. W. M. Strong, Dr. Elijah Swift, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Vice-President J. I. Hutchinson at the afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. E. S. Allen, Berkshire School, Sheffield, Mass.; Mr. B. A. Bernstein, University of California; Mr. G. W. Evans, Charlestown High School, Boston, Mass.; Mr. C. E. Flanagan, Wheeling, W. Va.; Mr. C. E. Githens, Wheeling, W. Va.; Mr. J. S. Mikesh, University of Minnesota; Professor G. P. Paine, University of Minnesota; Mr. W. L. Putnam, Boston, Mass.; Mr. V. M. Spunar, Pittsburg, Pa. Nine applications for membership in the Society were received.

Committees were appointed by the Council to arrange for the coming summer meeting and to report on the subject of the publication of the Princeton Colloquium Lectures.

The following papers were read at this meeting:

(1) Professor G. D. Birkhoff: "A simplified treatment of the regular singular point."

(2) Professor G. D. Birkhoff: "Some oscillation and comparison theorems."

(3) Professor P. F. Smith: "On osculating bands of surface-element loci."

(4) Professor EDUARD STUDY: "Die natürlichen Gleichungen der analytischen Curven im euklidischen Raume."

(5) Professor G. A. MILLER: "Addition to Sylow's theorem."

(6) Professor PETER FIELD: "On the circuits of a plane curve."

(7) Professor C. L. BOUTON: "Examples of transcendental one-to-one transformations."

(8) Professor JACOB WESTLUND: "On the fundamental number of the algebraic number field $k(\sqrt[p]{m})$."

(9) Professor L. P. EISENHART: "Surfaces with isothermal representation of their lines of curvature, and their transformations (second paper)."

(10) Professor EDWARD KASNER: "Isothermal nets."

(11) Dr. ARTHUR RANUM: "On the principle of duality in spherical geometry."

(12) Dr. O. E. GLENN: "On multiple factors of ternary and quaternary forms: applications to resolution of rational fractions."

In the absence of the authors the papers of Professor Smith, Professor Study, Professor Miller, Professor Westlund, Dr. Ranum, and Dr. Glenn were read by title. Professor Study's paper will appear in the July number of *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. This note by Professor Birkhoff is to appear in an early number of the *Transactions*. The theorems of Fuchs concerning the solutions of ordinary linear differential equations in the vicinity of a regular singular point are proved by a very short analysis which does not involve the substitution of an infinite series in the equation.

2. In a second note Professor Birkhoff derived oscillation and comparison theorems for ordinary linear equations of the third order analogous to the well-known one for second order equations. This paper will be offered to the *Annals of Mathematics*.

3. Professor Smith's paper is a continuation of that presented by the author at the summer meeting of 1909. The new matter consists chiefly in the discussion of cubic bands in space, that is, unions of ∞^1 surface elements whose points lie on a skew cubic and whose planes envelope a quadric cone. The vertex of the cone must lie on the cubic. If this cubic is

required to pass through three fixed points, then a cubic band is uniquely determined by two united surface elements. The point of contact with the theorems established in the former paper lies in this last statement. The cubic bands which osculate the characteristic bands of a partial differential equation of the first order osculate also the characteristic bands of a second differential equation. Other theorems analogous to those proved in the first paper hold for cubic bands also. Incidentally, a simple contact transformation appears, under which parabolic bands become cubic bands, vertical parabolas become four-nodal cubic surfaces, and straight lines become quadric cones.

5. According to Sylow's theorem the Sylow subgroups of order p^m are transformed according to a transitive substitution group. Professor Miller's paper deals with the question whether this transitive group is primitive or imprimitive. His main theorem may be stated as follows: If a group G contains more than one Sylow subgroup of order p^m and if one such subgroup has no more than p^a substitutions in common with any of the others, then G cannot transform these subgroups according to an imprimitive group unless their number is of the form $(1 + kp^{m-a})(1 + lp^{m-a})$, where both k and l are greater than zero. As a direct corollary we have that a group cannot transform its Sylow subgroups of order p^m according to an imprimitive group unless the number of these subgroups is of the form $(1 + kp)(1 + lp)$ where both k and l are integers greater than 0. For instance, if the order of G is divisible by 7, its Sylow subgroups of order 7^m cannot be transformed according to an imprimitive group unless their number is at least 64, and if it exceeds this number it must be at least 120, etc. This theorem is frequently very useful to determine whether a given substitution group is primitive.

6. It is known that there exist curves for every order n, $p = 0$ or 1, composed of a single circuit of index $n - 2$. In case $p = 1$ the curve may also have a simple oval (C. A. Scott, *Transactions*, volume 3). Professor Field's paper is devoted to a proof that there exist for every order n, $p = r$ $(r = 1, 2, 3, \cdots, n - 2)$ curves composed of r circuits, the sum of whose indices is $n - 2$. There may also be in addition a simple oval.

7. In this paper Professor Bouton gives a method for constructing a general class of transcendental one-to-one transfor-

mations in n variables, $n>1$. To his knowledge no examples of such transformations have heretofore been given. A simple example is

$$u = x + e^{y+\sin x}, \quad v = y + \sin x,$$

with the inverse

$$x = u - e^{v}, \quad y = v - \sin(u - e^{v}).$$

The method also gives classes of transcendental one-to-one contact transformations in n variables.

8. In Professor Westlund's paper the algebraic number field $k(\sqrt[p]{m})$ generated by the real pth root of a number m, where p is any odd prime, is discussed. An integral basis and the fundamental number of k are determined.

9. In a former paper with the same title (*Transactions*, volume 9) Professor Eisenhart established a transformation of surfaces S with isothermal spherical representation of their lines of curvature into surfaces of the same kind, such that upon a surface S and a transform S' the lines of curvature correspond, and S and S' constitute the envelope of a two-parameter family of spheres. This transformation was established by means of the Thybaut transformation of minimal surfaces and the transformation of partial differential equations of the second order due to Moutard. In the present paper transformations of this kind are established in a more simple and fundamental manner by determining the W-congruences (that is, congruences upon whose focal surfaces the asymptotic lines correspond) which are of such a character that the Lie line-sphere transformation converts the focal surfaces of the congruence into surfaces, S and S', with isothermal spherical representation of their lines of curvature. At the same time, S and S' are then necessarily the envelope of a two-parameter family of spheres. This method seems to give a more satisfactory reason for the existence of such transformations. The analysis is very much simpler in this case, and one obtains almost immediately a "theorem of permutability" for transformations of this kind.

10. Professor Kasner obtains the conditions which must be satisfied by two analytic curves intersecting orthogonally in order that they may be regarded as members of an isothermal net; that is, in order that they may be conformally transformable into orthogonal straight lines. The conditions are expressed

as an infinite set of relations between the coefficients of the power series representations of the given curves. All the coefficients of one series and half the coefficients of the other series may be taken at random, the other half then being determined by the general relations. Geometric interpretations are obtained by introducing the successive derivatives of the radii of curvature with respect to arc.

11. In the geometry of the surface of a sphere the principle of duality applies to metrical as well as projective properties. The consequences of this elementary fact do not seem to have been utilized much in the past. In this paper Dr. Ranum shows how the duality between distance and angle, between arc and area, between rolling and sliding, immediately leads to some interesting new theorems. For example, any continuous spherical movement can be regarded as due to the sliding (without rolling) of one spherical curve on another; the locus of a point lying on a variable tangent to a curve at a distance of $\frac{1}{2}\pi$ from the point of tangency is the envelope of the polar of the center of curvature at a variable point of the same curve; the area swept out by a moving tangent to one of two polar curves is equal to twice the length of the corresponding arc of the other.

12. In a paper published in the *American Journal of Mathematics*, volume 32, number 1, Dr. Glenn developed the theory of the decomposition of ternary and quaternary forms into simple quadratic factors. In the present paper this theory is extended to the cases where multiple factors occur. The multiple linear factors and multiple quadratic factors of a form are completely determined, save as to the solution of certain ordinary algebraical equations of higher degree.

In a footnote in the before-mentioned paper a new partial fraction theorem was announced, the unique feature of which was that the numerators of those partial fractions of b_x^{m-1}/a_x^m which have simple quadratic denominators were determined by ordinary differentiation and application of the Aronhold operator $(b \cdot \partial/\partial a)$. This theory is brought to a degree of completion in the present paper. Differential operators are developed by which are determined the numerators of the fractions corresponding to multiple quadratic factors of a_x^m. This theory is also extended to include the case where b_x^{m-1} and a_x^m are ternary forms.

F. N. COLE,
Secretary.

THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE seventeenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, on Saturday, February 26, 1910. The following members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor C. A. Noble, Mr. E. W. Ponzer, Professor T. M. Putnam, Dr. H. W. Stager.

Professor Blichfeldt occupied the chair. A committee consisting of Professors Allardice, Haskell, and Lehmer was appointed to draft resolutions in memory of the late Professor Irving Stringham.

The following papers were read at this meeting:

(1) Professor L. M. HOSKINS: "The strain of an elastic sphere of which the density is a function of the distance from the centre."

(2) Professor D. N. LEHMER: "A list of primes."

(3) Professor W. A. MANNING: "A note on Bochert's theorem."

(4) Professor C. A. NOBLE: "Characteristics of two partial differential equations of the first order."

(5) Mr. E. W. PONZER: "The principles of the calculus as applied in technical courses."

Abstracts of the papers are given below in order as numbered in the foregoing list:

1. The problem solved in the paper of Professor Hoskins is that of the strain of an elastic sphere of which the density is any known function of the distance from the center, the bodily forces having a potential which is a known function of the coordinates of position. The only restriction on the potential function is that it is developable in a series of spherical surface harmonics; an analogous restriction must hold regarding the surface stresses. General formulas are deduced for the elastic displacements, and the solution is given in full for the case in which the potential is a spherical solid harmonic of the second degree while the surface is free from stress. In this case the

surface values of the ellipticity e and angular displacement α are found to be expressible in simple form. Computations for the case of Laplace's law of density show that, for given values of the modulus of rigidity and total mass, e is about 11 per cent. less and α about 22 per cent. less than in the case of uniform density, these results being nearly independent of the compressibility.

2. The list of primes upon which Professor Lehmer is engaged is intended to extend to the limit of the factor tables recently published by the Carnegie Institution of Washington. The list is already completed for the primes contained in the 1st, 2d, 7th, 8th, 9th, and 10th millions. The table is to contain 133 pages, each with 5,000 entries. There are fifty columns, each containing 100 lines. As the table stands, it will be only a few moments' work to obtain the number of primes between any two given limits, or the rank of any given prime in the series of primes. The number of primes in each successive thousand is checked against the counts made by Glaisher (see his Introduction to the factor table for the sixth million) and the number in each successive fifty thousand is compared with the results of Bertelsen's computations (see *Acta Mathematica*, volume 17, 1893). No discrepancies with the latter have as yet appeared and the frequent differences with Glaisher are all due to the presence of errors in the tables from which he made his count.

3. In this note Professor Manning shows that if the substitutions of degree u in a 4-fold transitive group of class u and degree n (not alternating) are all of order 2, then $u > \frac{1}{2} n$.

4. Professor Noble deduces, after Hilbert, the ordinary differential equations of the characteristics of two partial differential equations of first order, shows how these suffice to obtain the general solution of the given equations, and constructs the corresponding problem in the calculus of variations.

5. Mr. Ponzer's paper presents a study of the various principles of the calculus as applied in the principal technical courses, both qualitatively and quantitatively; the nature and frequency of the principles applied is discussed, with suggestions as to where emphasis should be placed, and what should be omitted, in a course in the calculus.

C. A. NOBLE,
Secretary of the Section.

AN APPLICATION OF THE NOTIONS OF "GENERAL ANALYSIS" TO A PROBLEM OF THE CALCULUS OF VARIATIONS.

BY PROFESSOR OSKAR BOLZA.

(Read before the Chicago Section of the American Mathematical Society, April 8, 1910.)

THE object of the following note is to give an illustration of the unifying power of Professor E. H. Moore's methods of "General Analysis" * by showing that a certain theorem of the calculus of variations and a certain theorem of analytic geometry are special cases of one and the same theorem of general analysis.

The theorem of the calculus of variations is the so-called fundamental lemma for isoperimetric problems,† viz.,

THEOREM I. "*If*

$$(1) \qquad \mu_0(\eta) \equiv \int_{x_1}^{x_2} [M_0(x)\eta(x) + N_0(x)\eta'(x)]\,dx = 0$$

for all functions $\eta(x)$ *which are* (*a*) *of class* C' *on* $[x_1x_2]$, (*b*) *vanish at* x_1 *and* x_2, *and* (*c*) *satisfy the* m *conditions*

$$(2) \qquad \mu_i(\eta) \equiv \int_{x_1}^{x_2} [M_i(x)\eta(x) + N_i(x)\eta'(x)]\,dx = 0$$
$$(i = 1, 2, \cdots, m),$$

then there exist m *constants* $c_1, c_2, \cdots, c_m$ *such that*

$$(3) \qquad \mu_0(\eta) + c_1\mu_1(\eta) + c_2\mu_2(\eta) + \cdots + c_m\mu_m(\eta) = 0$$

for all functions $\eta(x)$ *satisfying conditions* (*a*) *and* (*b*).

The functions $M(x)$, $N(x)$ *are supposed to be continuous on* $[x_1x_2]$.

The theorem of analytic geometry is the well known

* Compare E. H. Moore, "On a form of General Analysis with applications to linear differential equations and integral equations," Atti del IV congresso internazionale dei Mathematici, vol. 2, p. 98; and "Introduction to a form of General Analysis," in The New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910.

† Compare for instance Bolza, Vorlesungen über Variationsrechnung, p. 462, footnote 1, and the references given there.

THEOREM II. "*If, in a plane and in homogeneous coordinates,*

$$(1') \qquad U_0 \equiv A_0 x + B_0 y + C_0 z = 0$$

is the equation of a straight line passing through the point of intersection of the two non-coinciding * *lines*

$$(2') \quad U_1 \equiv A_1 x + B_1 y + C_1 z = 0, \quad U_2 \equiv A_2 x + B_2 y + C_2 z = 0,$$

then there exist two constants λ_1, λ_2 *such that*

$$U_0 \equiv \lambda_1 U_1 + \lambda_2 U_2.\text{"}$$

§ 1. *The General Theorem.*

Let p be a general parameter † ranging over a set $\mathfrak{P}$ of elements; these elements may be any mathematical entities whatever: real or complex numbers, pairs, triples, etc., of such numbers, even infinite sets of numbers; functions of one or several variables; systems of functions; points, curves, surfaces; etc., etc.

Along with the set $\mathfrak{P}$ we consider the set $\mathfrak{Q}$ of all possible systems $(a_1, a_2; p_1, p_2)$ of a pair of real numbers a_1, a_2 and a pair of elements p_1, p_2 of $\mathfrak{P}$, and we suppose that a correspondence has been established by which to every element of $\mathfrak{Q}$ corresponds a unique element of $\mathfrak{P}$ which we denote by ‡

$$F(a_1, a_2; p_1, p_2).$$

We shall then say that a real single-valued function § $\mu(p)$ defined on $\mathfrak{P}$ is "*linear as to F,*" if

$$(4) \qquad \mu[F(a_1, a_2; p_1, p_2)] = a_1\mu(p_1) + a_2\mu(p_2) \quad \text{on } \mathfrak{Q},$$

i. e., for every combination $(a_1, a_2; p_1, p_2)$ of $\mathfrak{Q}$.

Then the following theorem holds: ||

THEOREM III. *If*

$$\mu_0(p), \ \mu_1(p), \ \cdots, \ \mu_m(p)$$

* We may omit the word "non-coinciding" if we replace "point of intersection of" by "point or points common to."

† Compare Moore, "Introduction etc.," § 1; I use throughout this section Moore's notation.

‡ In Moore's terminology F is a "function on $\mathfrak{Q}$ to $\mathfrak{P}$," "Introduction etc.," § 4.

§ Compare Moore, "Introduction etc.," § 5; if $\mathfrak{A}$ denotes the set of all real numbers, $\mu(p)$ is in Moore's terminology a "function on $\mathfrak{P}$ to $\mathfrak{A}$."

|| This generalization of Theorem I has been suggested to me by a remark in § 177 of Hadamard's Leçons sur le calcul des variations, Paris, 1910.

are $m+1$ *real single-valued functions of* p, *defined on* $\mathfrak{P}$, *which satisfy the following two conditions:*

A) they are linear as to F,

B) the equation

$$(1'') \qquad \mu_0(p) = 0$$

holds for every element of $\mathfrak{P}$ *which satisfies simultaneously the* m *equations*

$$(2'') \qquad \mu_1(p) = 0, \ \mu_2(p) = 0, \ \cdots, \ \mu_m(p) = 0,$$

then there exist m *real numbers* $c_1, c_2, \cdots, c_m$, *independent of* p, *such that*

$$(3'') \qquad \mu_0(p) + c_1\mu_1(p) + \cdots + c_m\mu_m(p) = 0 \quad \textit{on} \ \mathfrak{P},$$

i. e., for every element of $\mathfrak{P}$.

Proof: We notice first that there always exist elements of $\mathfrak{P}$ which do satisfy the m equations $(2'')$; for $F(0, 0; p_1, p_2)$ is an element of $\mathfrak{P}$ for any two elements p_1, p_2 of $\mathfrak{P}$, and on account of A)

$$\mu_i[F(0, 0; p_1, p_2)] = 0, \quad (i = 1, 2, \cdots, m).$$

Further we observe that if we define

$$F[1, a_3; F(a_1, a_2; p_1, p_2), p_3] = F(a_1, a_2, a_3; p_1, p_2, p_3)$$

and generally

$$(5) \quad F[1, a_n; F(a_1, a_2, \cdots, a_{n-1}; p_1, p_2, \cdots, p_{n-1}), p_n] = F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n),$$

then $F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n)$ is again an element of $\mathfrak{P}$, and, if (4) is satisfied, then also

$$(6) \quad \mu[F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n)] = a_1\mu(p_1) + a_2\mu(p_2) + \cdots + a_n\mu(p_n).$$

After these preliminary remarks we distinguish two cases:

Case I: The m equations $(2'')$ are satisfied for every p of $\mathfrak{P}$. Then according to B)

$$\mu_0(p) = 0 \ \text{ on } \ \mathfrak{P}.$$

Hence we may write

$$\mu_0(p) + 0 \cdot \mu_1(p) + 0 \cdot \mu_2(p) + \cdots + 0 \cdot \mu_m(p) = 0 \quad \text{on } \mathfrak{P},$$

and the theorem is proved with the particular values $c_1 = 0$, $c_2 = 0, \cdots, c_m = 0$.

Case II: The m equations $(2'')$ are not all satisfied for every p of $\mathfrak{P}$.

Then there exists a definite integer n $(1 \leqq n \leqq m)$ such that in the determinant

$$\Delta = |\mu_i(p_k)| \qquad (i, k = 1, 2, \cdots, m)$$

at least one minor of degree n is different from zero for some special system $p_1, p_2, \cdots, p_m$, whereas (for $n < m$) all minors of degree $n + 1$ vanish identically, that is, for every choice of the m elements $p_1, p_2, \cdots, p_m$. In order to fix the ideas we suppose that the minor

$$(7) \qquad \Delta_0 = |\mu_g(p_h)| \neq 0 \quad (g, h = 1, 2, \cdots, n).$$

Let now p be any element of $\mathfrak{P}$ and $p_1, p_2, \cdots, p_n$ the n special elements for which $\Delta_0 \neq 0$; then

$$q = F(1, a_1, a_2, \cdots, a_n; p, p_1, p_2, \cdots, p_n)$$

is an element of $\mathfrak{P}$, and according to A)

$$(8) \qquad \mu_j(q) = \mu_j(p) + a_1\mu_j(p_1) + \cdots + a_n\mu_j(p_n)$$
$$(j = 0, 1, 2, \cdots, m).$$

On account of (7) we can so determine $a_1, a_2, \cdots, a_n$ that

$$(9) \qquad \mu_1(q) = 0, \ \mu_2(q) = 0, \ \cdots, \ \mu_n(q) = 0.$$

If $n < m$, it follows from the identical vanishing of the minors of degree $n + 1$ of the determinant Δ, p taking the place of p_{n+1}, that also

$$(10) \qquad \mu_{n+1}(q) = 0, \ \mu_{n+2}(q) = 0, \ \cdots, \ \mu_m(q) = 0.$$

Hence for $n < m$ as well as for $n = m$, q is an element of $\mathfrak{P}$ which satisfies the m equations $(2'')$ and therefore it satisfies according to B) also the equation

$$(11) \qquad \mu_0(q) = 0.$$

But from the $n + 1$ equations (9) and (11) it follows, if we write the $\mu_j(q)$'s in their explicit form (8), that the determinant

$$(12) \quad |\mu_j(p), \ \mu_j(p_1), \ \cdots, \ \mu_j(p_n)| = 0 \qquad (j = 0, 1, 2, \cdots, n).$$

If now we expand this determinant according to the elements of the first column, the coefficient of $\mu_0(p)$ is the determinant Δ_0 and therefore different from zero, and this determinant as well as the remaining coefficients of the expansion is *independent of p*. Hence if we divide by Δ_0, we obtain equation (3″) with $c_{n+1}=0, c_{n+2}=0, \cdots, c_m=0$, and this equation holds on $\mathfrak{P}$, since p was *any* element of $\mathfrak{P}$. Thus our theorem is proved.*

§ 2. *Theorems I and II as Special Cases of Theorem III.*

In order to obtain Theorem I as a special case of Theorem III, we identify the set $\mathfrak{P}$ with the totality of all functions $\eta(x)$ of class C' on $[x_1x_2]$ which vanish at x_1 and x_2, and define

$$F(a_1, a_2; \eta_1, \eta_2) = a_1\eta_1 + a_2\eta_2. \tag{13}$$

If a_1, a_2 are two constants and $\eta_1(x)$, $\eta_2(x)$ two functions of $\mathfrak{P}$, $a_1\eta_1(x) + a_2\eta_2(x)$ again belongs to $\mathfrak{P}$ and the "functions"

$$\mu_j(\eta) = \int_{x_1}^{x_2} [M_j(x)\eta(x) + N_j(x)\eta'(x)]\,dx \qquad (j=0, 1, \cdots, m)$$

are "linear as to F," since

$$\mu_j(a_1\eta_1 + a_2\eta_2) = a_1\mu_j(\eta_1) + a_2\mu_j(\eta_2). \tag{14}$$

For this special choice of the set $\mathfrak{P}$, the operator F, and the functions μ_j, Theorem III becomes identical with Theorem I.

More generally we may take for $\mathfrak{P}$ the totality of all functions $\eta(x)$ of class C' on $[x_1x_2]$ which satisfy any given system of conditions provided only that these conditions are *linear*, *i. e.*, such that they are satisfied by $a_1\eta_1 + a_2\eta_2$ whenever they are satisfied by η_1 and η_2, two functions of class C' on $[x_1x_2]$. We thus obtain a generalization of Theorem I indicated by Hadamard.†

On the other hand, to obtain Theorem II as a special case of Theorem III, we identify the set $\mathfrak{P}$ with the totality of all triples $p = (x, y, z)$ formed with three independent variables x, y, z,

* I had originally thought it necessary to add to the assumptions A) and B) of the theorem the further assumption that $\Delta \neq 0$ for some system $p_1, p_2, \cdots, p_m$; I am indebted to Professor Moore for calling my attention to the fact that this assumption may be omitted, as well as for other valuable suggestions.

† loc. cit., § 176.

each ranging over all real values, and define, in Cayley's set notation,

$$(15)\qquad \begin{aligned} F(a_1, a_2; p_1, p_2) &= a_1(x_1, y_1, z_1) + a_2(x_2, y_2, z_2), \textit{ i. e.}, \\ &\equiv (a_1x_1 + a_2x_2, a_1y_1 + a_2y_2, a_1z_1 + a_2z_2). \end{aligned}$$

$F(a_1, a_2; p_1, p_2)$ belongs again to $\mathfrak{P}$, however the numbers a_1, a_2 and the triples $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ may be chosen.

With this definition of F, the functions

$$(19)\qquad \mu_j(p) = A_jx + B_jy + C_jz, \quad (j = 0, 1, 2)$$

are "linear as to F."

If $n = 2$, there exists at least one pair of triples (x_1, y_1, z_1), (x_2, y_2, z_2) for which the determinant

$$\begin{vmatrix} A_1x_1 + B_1y_1 + C_1z_1, & A_2x_1 + B_2y_1 + C_2z_1 \\ A_1x_2 + B_1y_2 + C_1z_2, & A_2x_2 + B_2y_2 + C_2z_2 \end{vmatrix} \neq 0.$$

This means geometrically, if we interpret x, y, z as homogeneous coordinates of a point in a plane, that the two lines

$$(20)\qquad A_1x + B_1y + C_1z = 0, \; A_2x + B_2y + C_2z = 0$$

do not coincide.

Theorem III then specializes into Theorem II.

The assumption $n = 1$ leads to the trivial case alluded to on page 403, footnote *.

In like manner the corresponding theorems on pencils and bundles of planes and their generalizations to spaces of higher dimensions follow immediately as special cases from Theorem III.

THE UNIVERSITY OF CHICAGO,
February, 1910.

THE INFINITESIMAL CONTACT TRANSFORMATIONS OF MECHANICS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, February 29, 1908.)

1. THE significance of contact transformations in the development of general dynamics and optics, appreciated to some extent by Hamilton, was first brought out explicitly by Lie.* A very thorough and elegant discussion of the whole subject, including a number of new results, has recently been given by Vessiot.† With a conservative dynamical system, defined by its potential energy U (a function of n generalized coordinates) and its kinetic energy T (a quadratic form in the n generalized velocity components), there is associated an infinitesimal contact transformation whose characteristic function W is of special type.‡ The main result of the present note may be stated as follows:

The alternant (Klammerausdruck) of the contact transformations associated with two dynamical systems, of the same number of degrees of freedom, will be a point transformation when, and only when, the expressions for the kinetic energies are either the same or differ merely by a factor.

2. For simplicity and clearness we shall confine ourselves to two degrees of freedom. The infinitesimal contact transformations are then defined by a characteristic function $W(x, y, p)$, each lineal element (x, y, p) being converted into a neighboring element $(x + \delta x, y + \delta y, p + \delta p)$ according to the standard formulas

$$(1) \quad \delta x = W_p \delta t, \quad \delta y = (p W_p - W)\delta t, \quad \delta p = -(W_x + p W_y)\delta t.$$

If the transformation is applied repeatedly to any given element, a series of ∞^1 elements is obtained, the locus of whose points is termed a path curve or trajectory. The direction of the path generated by any element is defined by the formula

* "Die infinitesimalen Berührungstransformationen der Mechanik," *Leipziger Berichte* (1889), pp. 145–153. Lie-Scheffers, Berührungstransformationen, p. 102.

† *Bulletin de la Societé Mathématique de France*, vol. 34 (1906), pp. 230–269.

‡ The constant of total energy h is assumed to have a given value, so the discussion is connected with the theory of *natural families*.

$$(2) \qquad m = \frac{\delta y}{\delta x} = p - \frac{W}{W_p}.$$

We shall speak of the direction m as being *transversal* * to the direction p, and shall refer to (2) as the law of transversals connected with the given transformation.

3. In the simplest case of a particle moving in a plane the kinetic energy is of the form $2T = \dot{x}^2 + \dot{y}^2$, and the associated contact transformations are of the type

$$(3) \qquad W = \Omega(x, y)\sqrt{1 + p^2}.$$

Lie showed that this type is characterized geometrically by the fact that transversality reduces to orthogonality; or, what is equivalent, each point is converted into a circle of infinitesimal radius with the given point as center. We now prove that *The alternant of any two transformations of type* (3) *is a point transformation.*

For this purpose we make use of the general formula

$$(4) \qquad W_2 = \begin{vmatrix} W_p & W_{1p} \\ W_x + pW_y & W_{1x} + pW_{1y} \end{vmatrix} - \begin{vmatrix} W & W_1 \\ W_y & W_{1y} \end{vmatrix},$$

where W and W_1 are the characteristic functions of any two given transformations and W_2 is the characteristic function of their alternant (commutator, Klammerausdruck). Substituting

$$(5) \qquad W = \Omega\sqrt{1 + p^2}, \quad W_1 = \Omega_1\sqrt{1 + p^2},$$

we find

$$(6) \qquad W_2 = p(\Omega\Omega_{1x} - \Omega_1\Omega_y) - (\Omega\Omega_{1y} - \Omega_1\Omega_y).$$

The linearity of this expression in p proves that it defines a point transformation; its symbol in the usual Lie notation is

$$(6') \qquad (\Omega\Omega_{1x} - \Omega_1\Omega_x)\frac{\partial}{\partial x} + (\Omega\Omega_{1y} - \Omega_1\Omega_y)\frac{\partial}{\partial y}.$$

4. The general case of two degrees of freedom is equivalent to the motion of a particle constrained to remain on an arbitrary surface, whose first quadratic form we write

* We have borrowed this term from the calculus of variations, although the idea seems quite different. The formula (2) however is precisely the transversality formula connected with the problem $\int W dx =$ minimum. There are several other important analogies between the two theories.

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2. \tag{7}$$

The corresponding contact transformations are

$$W = \Omega\sqrt{E + 2Fp + Gp^2}. \tag{8}$$

The law of transversals becomes

$$E + F(m + p) + Gmp = 0, \tag{9}$$

and expresses orthogonality of directions on the surface (7). Type (8) is also characterized by the fact that each point is converted into a geodesic circle about that point as center. An important special case is dilatation, where the circles are all of equal radii; the characteristic function is then found to be

$$W = \sqrt{\frac{E + 2Fp + Gp^2}{EG - F^2}}. \tag{10}$$

Consider now two transformations

$$W = \Omega\sqrt{E + 2Fp + Gp^2}, \quad W_1 = \Omega_1\sqrt{E + 2Fp + Gp^2}, \tag{11}$$

associated with different potentials on the same surface, and apply formula (4). It is easily verified that W_2 is linear with respect to p. The resulting point transformation is

$$\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y},$$

$$\begin{aligned} \xi &= G(\Omega\Omega_{1x} - \Omega_1\Omega_x) - F(\Omega\Omega_{1y} - \Omega_1\Omega_y), \\ \eta &= E(\Omega\Omega_{1y} - \Omega_1\Omega_y) - F(\Omega\Omega_{1x} - \Omega_1\Omega_x). \end{aligned} \tag{12}$$

Hence if two dynamical systems lead to the same expression for kinetic energy, the alternant of the associated contact transformations is a point transformation.

5. We now inquire whether this can happen when the kinetic energies differ. We may write any two of our transformations in the form

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \sqrt{\alpha_1 + 2\beta_1 p + \gamma_1 p^2}, \tag{13}$$

where the coefficients are arbitrary functions of x and y. The value of the alternant W_2 is then a complicated fractional expression. It suffices to observe that the numerator is a polynomial of the third degree in p, and the denominator is WW_1.

For a point transformation it is *necessary* that the expression shall be rational in p. Omitting the trivial case where both W and W_1 are point transformations (which has no dynamical interest), we see that the quadratics under the radicals in (13) can differ only by a factor. The work of § 4 shows that W_2 is then actually a point transformation.

Hence the alternant of the contact transformations (13) *is a point transformation when and only when* $\alpha_1 : \beta_1 : \gamma_1 = \alpha : \beta : \gamma$.

We have now completed the proof of the theorem stated in § 1.

6. The law of transversals for a transformation of the type

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2} \tag{14}$$

is of the form

$$\alpha + \beta(m + p) + \gamma mp = 0. \tag{15}$$

If this is interpreted on a proper auxiliary surface, namely, one whose length element is proportional to $\sqrt{\alpha dx^2 + 2\beta dx dy + \gamma dy^2}$, it expresses orthogonality. In the x, y plane, however, the directions p and m are conjugate with respect to a central conic. The relation (15) is of linear involutorial character, and depends only on the ratios of α, β, γ. We may therefore state the result of § 4 in geometric terms as follows:

If two contact transformations lead to the same linear involutorial law of transversals, their alternant will be a point transformation.

7. We now show that there are no other transversality laws for which an analogous result holds. In the first place we observe from (2) that if two transformations lead to the same law of transversals, the ratio of their characteristic functions is a function of x and y alone. We therefore form the alternant of W and $\lambda(x, y)W$, finding

$$W_2 = W^2\lambda_y - WW_p(\lambda_x + p\lambda_y) = S\lambda_y - \tfrac{1}{2}S_p(\lambda_x + p\lambda_y), \tag{16}$$

where S represents the square of W. The condition that W_2 shall represent a point transformation is found by placing its second derivative with respect to p equal to zero. This gives

$$(\lambda_x + p\lambda_y)S_{ppp} = 0. \tag{17}$$

The first factor vanishes only when λ is constant, a trivial case, since then the two given transformations coincide. The vanishing of the other factor shows that S must be quadratic in p, that is, W must be of the form (14).

Hence two contact transformations with the same transversality law will have a point transformation for alternant only when they are of the type

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \lambda \sqrt{\alpha + 2\beta p + \gamma p^2}.$$

Transversality is then expressed by a linear involutorial relation (15), *so that for each point the transversal of a given direction is the conjugate direction with respect to a conic with that point as center.*

8. A less important converse result, relating to the type considered in § 3, we state without proof. The only contact transformations which in combination with *every* transformation of type $W = \Omega \sqrt{1 + p^2}$ give a point transformation for alternant are those of the same type. The same is true even if Ω is restricted to the form $a(x^2 + y^2) + bx + cy + d$, a case of interest since then W converts circles into circles. When a vanishes the transformation belongs to the equilong class of Scheffers.

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ON AN INTEGRAL EQUATION WITH AN ADJOINED CONDITION.

BY ANNA J. PELL.

(Read before the Chicago Section of the American Mathematical Society, December 31, 1909.)

IN his doctor dissertation * Professor Cairns develops for infinitely many variables the theory of a quadratic form with an associated linear form, in order to prove the existence of solutions of the following integral equation :

$$(1) \qquad \phi(s) = \lambda \int_a^b K(s, t)\phi(t)dt + \mu p(s),$$

with the adjoined condition

$$(2) \qquad \int_a^b \phi(s)p(s)ds = 0,$$

where $K(s, t)$ is a given continuous symmetric function of s and t, $p(s)$ a given continuous function of s, λ and μ are parameters, and $\phi(s)$ is the function to be determined.

* "Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen," Göttingen, 1907.

A geometrical consideration suggests the possibility of transforming the equations (1) and (2) into an equivalent homogeneous integral equation with a symmetric kernel, the existence of whose solutions has already been shown by Hilbert and others. In this paper such a transformation will be carried out.

If there is a solution $\phi(s)$, the corresponding μ may be expressed

$$\mu = \frac{-\lambda \int_a^b \int_a^b K(s, t) p(s) \phi(t) ds dt}{\int_a^b [p(s)]^2 ds},$$

and the solution $\phi(s)$ satisfies the integral equation

$$(3) \qquad \phi(s) = \lambda \int_a^b L(s, t) \phi(t) dt,$$

where

$$L(s, t) = K(s, t) - \frac{p(s) \int_a^b K(t, t_1) p(t_1) dt_1}{\int_a^b [p(s)]^2 ds}.$$

Conversely any solution of (3) satisfies (1) and (2). The kernel $L(s, t)$, however, is not symmetric and we cannot make any definite conclusions about the existence and character of the characteristic number λ.

Consider now the integral equation

$$(4) \qquad \phi(s) = \lambda \int_a^b M(s, t) \phi(t) dt,$$

where

$$M(s, t) = L(s, t) - p(t) \frac{\int_a^b K(s, t_1) p(t_1) dt_1}{\int_a^b [p(s)]^2 ds}.$$

The kernel $M(s, t)$ is symmetric in s and t, and therefore, unless $M(s, t)$ is identically equal to zero, there exists for at least one real value of λ a solution $\phi(s)$, not identically equal to zero, of the integral equation (4).

Any solution of the equations (1) and (2) satisfies the equa-

tion (4); we must investigate under what conditions a solution of (4) is also a solution of (1) and (2).

Case I. $M(s, t) \not\equiv 0$. There exists a solution $\phi(s)$ of the equation (4); multiply this equation by $p(s)$ and integrate from a to b and we see that $\phi(s)$ satisfies the condition (2), unless the corresponding characteristic number is given by

$$\lambda_0 = \frac{-\int_a^b [p(s)]^2 ds}{\int_a^b \int_a^b K(s, t) p(s) p(t) ds dt}.$$

Suppose first that $\lambda \neq \lambda_0$; then it can easily be verified that $\phi(s)$ satisfies both (1) and (2).

If λ_0 is a characteristic number of the kernel $M(s, t)$, the function $p(s)$ is always a corresponding solution, and $p(s)$ is clearly not a solution of (1) and (2). Let $\psi(s)$ be any other solution of (4) corresponding to λ_0; then the function

$$\psi(s) - \frac{p(s) \int_a^b p(s) \psi(s) ds}{\int_a^b [p(s)]^2 ds}$$

is a solution of (4) and also of (1) and (2). Hence there exists a solution of (1) and (2) unless λ_0 is the only characteristic number of $M(s, t)$ and $p(s)$ the only corresponding solution; in this case $M(s, t)$ has the form

$$M(s, t) = k p(s) p(t), \tag{5}$$

where k is some constant not equal to zero.

Case II. $M(s, t) \equiv 0$. The equations (1) and (2) have no solution not identically equal to zero.

The final result is that the given integral equation always has a solution unless $K(s, t)$ and $p(s)$ satisfy the relation (5) * (where k may take the value zero). Further, the solutions of (1) and (2) form an orthogonal system of functions.

From the expansion theorem for the symmetric kernel $M(s, t)$ we obtain the following: any function $f(s)$ expressible in the form

* The exceptional cases are not indicated in the dissertation referred to.

$$f(s) = \int_a^b K(s,\ t)g(t)dt,$$

where $g(s)$ is any continuous function satisfying

$$\int_a^b p(s)g(s)ds = 0,$$

can be developed into the uniformly convergent series

$$f(s) = \frac{p(s)\displaystyle\int_a^b p(s)f(s)ds}{\displaystyle\int_a^b [p(s)]^2 ds} + \Sigma_i\phi_i(s)\int_a^b \phi_i(s)f(s)ds,$$

where $\phi_i(s)$ are the normalized solutions of (1) and (2).

That this expansion may not hold in case $g(s)$ is any continuous function (as Mr. Cairns states the theorem) is shown by the special example

$$K(s,\ t) = A(s)p(t) + A(t)p(s) + B(s)B(t),$$

$$A(s) \not\equiv cB(s), \quad \int_a^b A(s)p(s)ds = 0, \quad \int_a^b B(s)p(s)ds = 0,$$
$$g(s) = p(s).$$

THE UNIFICATION OF VECTORIAL NOTATIONS.

Elementi di Calcolo vettoriale con numerose Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Bologna, Nicola Zanichelli, 1909. v + 174 pp.

Omografie vettoriali con Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Torino, G. B. Petrini, 1909. xi + 115 pp.

1. IN view of the plan that the fourth international congress of mathematicians held at Rome in 1908 should discuss the notations of vector analysis and perhaps lend the weight of its recommendation to some particular system, Burali-Forti and Marcolongo awhile ago set themselves the laudable but somewhat thankless task of collecting and editing all the historical, critical, and scientific material which might be indispensable to a proper settlement of the question by the congress, and this material they published in a series of five notes beginning in

the twenty-third volume (1907) of the *Rendiconti* of Palermo and running through several succeeding numbers and volumes. It is needless to observe that the work was accomplished with the expected accuracy. It was, however, not done with all the completeness desirable. The attention of the authors was turned almost exclusively to the minimum system most useful in mathematical physics, that is, to the questions of addition of vectors, of scalar and vector products, of differentiation with respect to a scalar, and of differentiation with respect to space (gradient of a scalar function and divergence and curl of a vector function of position). Considerable discussion was given to quaternions but the fact was fully recognized by the authors that many subjects which might rightfully have been treated were omitted — among which the linear vector function (Hamilton) or quotients and Lückenausdrücke (Grassmann) or dyadics (Gibbs) were perhaps the most noteworthy.

The authors not only followed their initial program; they went further and themselves recommended a particular minimum system of notations essentially like any and all of those now employed but differing in the symbols selected. Thus did it appear that these strivers after unification were prone to follow the path of all unifiers and introduce still greater diversity. Seemingly the universal language of vectors, like the universal commercial language, is destined to suffer constantly new amendments at the hands of its zealots. The conception of unification as conceived in the mind of each enthusiastic unifier appears to be that he shall disagree with everybody and that everybody shall then agree with him. For the scalar product of $\mathbf{a}$ and $\mathbf{b}$ the recommendation is $\mathbf{a} \times \mathbf{b}$, for the vector product it is $\mathbf{a} \wedge \mathbf{b}$, for the gradient we have grad, and div and rot for the divergence and curl. The symbol ∇ is abolished. The suggestion $\mathbf{a} \times \mathbf{b}$ for the scalar product seems particularly infelicitous in view of the fact that this notation is in actual use for the vector product. So far as we are aware, this is the first suggestion which violently and confusingly differs from a notation which has become fairly widely established. It is true that Grassmann in some of his papers used the cross as a symbol of scalar multiplication; but the rare and unimportant historical use of a symbol seems hardly a sufficient reason for the present adoption of the symbol in the face of actual modern usage which is tolerably popular. No such objection can be urged against the use of $\wedge$ for the vector product — that notation is altogether new.

With regard to the necessity for new systems of notations we would point out some facts. First there are already available three elaborately and consistently developed systems which represent a considerable portion of the life-work and life-thought of three great minds, namely, the systems of Hamilton, Grassmann, and Gibbs. Any of these is entirely adequate for usage in physics and in addition each represents a distinct attitude toward the science of multiple algebra. Hamilton's theory was developed from the double point of view of the theory of sets and of the exigencies of an analysis for three dimensional space. It represents algebraically the whole domain of linear associative algebra. Grassmann's work gives a general science of extensive magnitudes in any number of dimensions and contributes to algebra the important idea that a product need not be a quantity of the type of either factor. It is the prime geometric algebra. Gibbs's view seems to have been to fuse all these elements and the theory of matrices into a general science of multiple algebra (as distinguished from many individual sciences of different multiple algebras) and to construct from his general point of view a particular system especially fitted for his fellow physicists. In addition to these great scientific analyses of space we have the analysis adopted by the Encyclopedia and many present writers, especially in Germany. This system may apparently be characterized as opportunist in that it seems to have been selected largely in haphazard fashion as a sort of convenient abridged notation. From the sentimental view of scientific fitness and justice it would appear desirable to adopt, if any definitive adoption must be made, one of the systems connected with the name of a great scientist and constructed on scientific principles. From the practical point of view it might be convenient to adopt the opportunist system already so widely used. The hue and cry about the confusion due to the great diversity of notation is largely hysterical. An examination of current literature will show that there is very little diversity and that of the works currently written in vectorial notation not only a plurality but an actual and considerable majority are written in the opportunist system. The chief reason vectors are not more used is not this alleged confusion and diversity so much as it is inertia. The "news items" and "information" and "reasons" which are spread broadcast concerning vectors are about as true, about as fundamental, and about as much founded in fact as those scattered into the air about the markets

for stocks and commodities. With regard to new systems we may mention that Gibbs, with a modesty not universally in evidence, for many years refrained from publishing his system and from allowing it to be published (although he privately printed it for his own convenience and for that of his students) because he felt it did not present sufficiently original differences from existing methods. His attitude was perhaps unfortunate. The world may have been in need of the new systems; it may still be in need of new systems; this, at any rate, is the opinion of the authors whose works are under review.

Further to spread the new system and to offer to Italian scientists and students a treatise on vector analysis in their own language, the authors have written two short volumes of which the titles appear at the head of this review. It is surely a good thing that there is now at least one systematic treatment of vector analysis in Italian. The first volume, Calcolo vettoriale, takes up the matters mentioned as belonging to the minimum system; the second, Omografie vettoriali, discusses what is equivalent to the linear vector function. In their prefaces the authors state that their books differ profoundly both in method and in notation from all previous texts of recent years — differ in method because they intend to operate in an absolute manner on geometric entities, whereas ordinarily vectors and their operations are tachygraphic for coordinates; and differ in notations by the adoption of their own recommendations. It will be impossible for us to discuss these books apart from the contributions of the authors to the *Rendiconti* and to *L'Enseignement Mathématique*, which has so considerately opened its columns to an interchange of views by all interested in vectorial notations. Quotations will be made indifferently from all these sources according as they may be needed.

2. In commencing the detailed review of the two books, mention should be made of the fact that they contain not so much a pure vector analysis as a point and vector analysis with the emphasis on vectors. Thus the vector is introduced as the difference $B - A$ of two points and such equations as

$$P = A + x(B - A), \quad P = O + x\mathrm{i} + y\mathrm{j} + z\mathbf{k},$$

for a line and for the position of a point are found. There is much to be said for this procedure, which introduces the origin explicitly into the analysis in addition to the system i, j, **k**. Its advantages and disadvantages reative to the usual pure

vector analysis will be clear to all readers. On the mathematical and logical side this usage of the authors is highly commendable; it opens the way for the consideration of Grassmann's geometrical algebra and eliminates the origin wherever unessential. Whether physicists, who have a tendency to limit their analysis to an irreducible minimum, will welcome the addition of a little point analysis is doubtful. In other respects the first three chapters give a treatment of addition and multiplication of vectors in much the usual way and with much the usual applications.

With Chapter IV there is an interesting departure from the ordinary texts. The operator i, as a quadrantal versor in a plane, is introduced by the equation* $i\mathbf{x} = \mathbf{u} \times \mathbf{x}$, where $\mathbf{x}$ is normal to $\mathbf{u}$. That this operator i, as regards its repetition, obeys the law $i^2 = -1$ of $i = \sqrt{-1}$ is pointed out. The equations

$$(i\mathbf{a}) \cdot (i\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a} \times (i\mathbf{b}) = \mathbf{b} \times (i\mathbf{a}),$$

however, show the reader at once that here the i is not an ordinary scalar subject to $i^2 = -1$. For if this were the case, $i\mathbf{a} \cdot i\mathbf{b}$ would be $-\mathbf{a} \cdot \mathbf{b}$, and like changes of sign would occur in other formulas. This fact taken with the fact that the operator i really depends on the vector $\mathbf{u}$ which enters into its definition seems to militate greatly against the usefulness of this chapter and the advisability of its general acceptance. It appears unfortunate to use a symbol so familiar as i in a sense which precludes its fuller treatment according to the same formal laws as it ordinarily obeys. This impression is but strengthened by the sight of the equations

$$e^{i\phi}\mathbf{a} \cdot e^{i\phi}\mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \quad e^{i\phi}\mathbf{a} \cdot e^{i\psi}\mathbf{a} = \mathbf{a}^2 \cos(\psi - \phi),$$

and others of that ilk for rotations through various angles. It may well be, however, that the unfavorable impression is due merely to the unfamiliarity of the symbols, or rather, to a familiarity with them under the form of scalars subject to the laws of (complex) scalars.

After a brief mention, in Chapter V, of the differentiation of a vector with respect to a scalar the vitally important subject of differentiation with respect to space, that is, of gradient,

* Here and throughout the review the notations of the authors are translated so far as possible into the notations of Gibbs, which are probably more familiar to readers of the BULLETIN.

divergence, and curl, is treated in Chapter VI. The gradient is defined by the equation *

$$(1) \qquad dV = (\text{grad } V)\cdot dP.$$

Two remarks may be made. In the first place this definition is intrinsic or absolute, that is, devoid of reference to coordinate axes, and therein conforms to the usage we have always indicated as the best for defining ∇V. The second observation is that the authors here and elsewhere use dP, the differential of a point, instead of $d\mathbf{r}$, the differential of a vector. This has the advantage that no origin is implicated in the definition. Whether this advantage is sufficient to compensate for the slight complication of the combined point and vector analysis as against the simple vector analysis is a matter of individual opinion. The authors have a violent dislike for the symbol ∇ and not only give up its use but urge vehemently that it should be universally abandoned. This matter will be discussed later in section 4.

The defining equations for curl $\mathbf{V}$ and div $\mathbf{V}$ are given in the absolute vectorial form

$$(2) \qquad d\mathbf{r} \times \delta\mathbf{r}\cdot \text{curl } \mathbf{V} = \delta\mathbf{r}\cdot d\mathbf{V} - d\mathbf{r}\cdot\delta\mathbf{V}.$$

$$(3) \qquad \text{div } \mathbf{V} = \mathbf{a}\cdot[\text{grad } (\mathbf{a}\cdot\mathbf{V}) + \text{curl } (\mathbf{a}\times\mathbf{V})],$$

where $d\mathbf{r}$, $\delta\mathbf{r}$ are two differential displacements (written dP, δP in the text) and $d\mathbf{V}$, $\delta\mathbf{V}$ are the corresponding differentials of $\mathbf{V}$; and where the equation (3) is supposed to hold for any constant vector $\mathbf{a}$; and where finally we have seen fit for ulterior purposes to rearrange the order of the vectors in the products. These are highly ingenious definitions and have much to commend them. It will be noticed that (2) is suggested at once by the expressions that occur in the proof of Stokes's theorem by the method of variations; the definition of the divergence appears more artificial and is less intimately connected with the fundamental characteristics of div $\mathbf{V}$; both, however, are well adapted to establish some of the important formulas of the differential calculus of vectors and should therefore have the serious attention of students of the presentation of vector analysis.

It should be noticed that as defined by (1) the gradient is immediately interpretable in its physical sense as that vector

* Compare, in Gibbs's notation, $d\mathbf{r}\cdot\nabla V = dV$.

which gives in direction and magnitude the most rapid rise of V. Now the basal significance of curl $\mathbf{V}$ is found in the fact expressed by Stokes's theorem that the induction of curl $\mathbf{V}$ through a surface is equal to the integral of $\mathbf{V}$ around the boundary, and the basal significance of div $\mathbf{V}$ lies in Gauss's theorem that the integral of div $\mathbf{V}$ over a volume is equal to the induction of $\mathbf{V}$ through the bounding surface. Hence for physical reasons definitions like

$$(2') \qquad d\mathbf{S} \cdot \operatorname{curl} \mathbf{V} = \int_{\bigcirc} d\mathbf{r} \cdot \mathbf{V},$$

$$(3') \qquad d\tau \operatorname{div} \mathbf{V} = \int_{\bigcirc} d\mathbf{S} \cdot \mathbf{V},$$

where $d\mathbf{S}$ is an element of surface and $d\tau$ an element of volume, are preferable to the analytic definitions of the authors, and they have the added felicity to be immediately connected with the proofs of Stokes's and Gauss's theorems. They have the disadvantages that they are not so easily available for the proof of formulas of differential calculus and that they mix differential and integral calculus. Whether on the whole the advantages are with (2′), (3′) as against (2), (3) must remain largely a matter of personal preference.

3. It is in this same chapter that the authors reveal their remarkable discovery that the laplacian operator

$$\triangle = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is essentially different according as it is applied to a scalar or to a vector function. Upon this discovery they are especially insistent on every possible occasion.* They even go so far as to introduce different symbols $\triangle$ and $\triangle'$ for the operator according as the operand is scalar or vector. Then they are able to write

$$(4) \qquad \begin{aligned} \triangle V &= \operatorname{div} \operatorname{grad} V, \\ \triangle' \mathbf{V} &= \operatorname{grad} \operatorname{div} \mathbf{V} - \operatorname{curl} \operatorname{curl} \mathbf{V}. \end{aligned}$$

Whereupon they add: You see at once what an immense difference there is between the two operators $\triangle$ and $\triangle'$ for which

* Calcolo vettoriale, p. 72; Omografie vettoriali, p. 61; and *L'Enseignement Mathématique* (1909), p. 462, from which the quotation below is taken.

authors have adopted the single symbol $\triangle_2$ because these operators have the same cartesian expression. That is no reason to use a single sign to designate very different things ; on the contrary it shows once more that the systematic use of coordinates may introduce pseudo-operators which have no longer a geometric and logical character.

Now it is rather curious and decidedly regrettable that two such eminent authors, who work in a country conspicuous for its researches in the logic of mathematics and of whom one is himself illustrious for such investigations, should adduce the equations (4) as a reason for the essential difference of $\triangle$ and $\triangle'$. It merely goes to show how restricted is their point of view in the mathematics of the subject and how deficient in the physical interpretation thereof. If the authors were not so eminent and if we did not have so high a regard for most of their work in mathematics and mathematical physics, we should pass over these remarks of theirs without further comment and in full confidence that no very great number of mathematicians or physicists would fall into their way of thinking. Unfortunately that course is not open to us. The higher the sources from which vicious doctrines are promulgated, the more patient and painstaking must be their refutation. All that equations (4) do is to give two different expressions which are equivalent in the two respective cases to the laplacian operator. The second expression cannot be applied in the first case because the operators div and curl cannot be applied to a scalar; the first expression can be applied in the second case if one knows how to define the gradient of a vector function and the divergence of a linear vector function, but the authors do not give these definitions and there is no need to give them now.*

The laplacian operator is probably best known to mathematicians in connection with Laplace's equation and harmonic functions. It is a fundamental theorem on harmonic functions that the average value of the function upon the surface of a sphere is equal to the value at the center, and it is an immediate corollary that the average value throughout the sphere is equal to the value at the center. Now as the laplacian operator occurs so frequently in fundamental physical problems, it is a reasonable assumption that the operator represents some intrinsic or absolute characteristic of a field and does not depend in any other than an accidental way upon cartesian coordinates.

* See, however, (1), (2), (3) of section 7 below.

And Maxwell pointed out this characteristic which may be called the dispersion. The laplacian operator may in fact be defined by either of the equivalent intrinsic or absolute equations

$$(5) \qquad r^2 \triangle \phi = 6(\bar{\phi} - \phi) = 10(\bar{\bar{\phi}} - \phi),$$

where r is the radius of an infinitesimal sphere and $\bar{\phi}$, $\bar{\bar{\phi}}$ are respectively the average value of ϕ upon the surface and throughout the volume of the sphere. This definition is comparable to (2′), (3′) for divergence and curl in that it depends on integration. But it is incomparably simpler than either of them. For they depend upon scalar products, whereas this depends only on addition as implied in averaging.

In other words the laplacian operator may and should be defined by its intrinsic properties such as are expressed in (5), and when this definition is given it appears that the operator is equally applicable and with the same significance to any quantity ϕ which satisfies the laws of addition, that is, to the elements ϕ of any linear algebra regarded as functions of position and in particular to scalar functions, vector functions, dyadic or linear-vector-function functions, and to planar vector or bivector functions. The laplacian operator is in no wise a pseudo-operator, there is no immense nor even any slight essential difference in its application to various linear fields, and there is no more reason for representing it by different symbols than there is for representing addition by different symbols. With all these facts Maxwell was familiar and so was Gibbs. The really accidental phenomenon is that the laplacian operator can be expanded in either of the forms given by (4).

4. It has been remarked that the authors do not use ∇ for grad, and it is a corollary that they should not use $\nabla \times$ and $\nabla \cdot$ for curl and divergence, which they write as rot and div. In regard to all these operators they remark that: The operators $\nabla \cdot$, $\nabla \times$ are absolute because div and rot may be defined without coordinates as in (2), (3); but ∇, on the contrary, is an essential tachygraph and the sign $\times$ or $\cdot$ which follows it is an operator which without coordinates has no meaning.* It is apparently for such reasons as these that they discard ∇ from their calculus. Now it fails to appear clear to us why grad V or ∇V is not defined by (1) quite as absolutely and just as independently of coordinates as curl $\mathbf{V}$ or $\nabla \times \mathbf{V}$ by (2) and div $\mathbf{V}$ or $\nabla \cdot \mathbf{V}$ by (3). Inasmuch as the definitions ∇V, $\nabla \cdot \mathbf{V}$,

* *L'Enseignement Mathématique* (1909), p. 466.

$\nabla \times \mathbf{V}$ are identical with those for grad V, div $\mathbf{V}$, curl $\mathbf{V}$, or may be taken so, any dependence of one set on coordinates should establish an equal dependence for the others — and we believe that in neither case is there any such dependence.

The question, however, does arise as to the availability of the detached symbols ∇, $\nabla \cdot$, $\nabla \times$, or grad, div, curl. Now in case three totally different symbols like the last set are chosen for the three types of differentiation, there can be little object in separating the operators from the operands and no analytic algorism is suggested. But if the symbols ∇, $\nabla \cdot$, $\nabla \times$ are used and if their use indicates a simple and suggestive algorism by means of which differential formulas may be remembered, then the detaching of the operators from the operand or better the introduction of the idea that the symbols ∇, $\nabla \times$, $\nabla \cdot$ should be interpreted as a combination of the operations ∇, $\times$, $\cdot$ is forcibly recommended.* For those who scorn analytic algorisms and prefer to remember a lot of distinct formulas, this argument is not impressive; but by far the larger number of persons like to have a notation which is algorismic in the sense that in itself it suggests the proper analytic transformations, and there is a very wide belief that there is much of mathematical value in a notation which has the felicity to be suggestive. The question, then, as to the preference of ∇, $\nabla \cdot$, $\nabla \times$ over grad, div, curl, is not one of dependence or independence of coordinates but one of analytic felicity.

To investigate this matter more closely, let it be granted that ∇, $\nabla \cdot$, $\nabla \times$ represent some sort of differentiation so that they may appropriately be called differentiating operators and may be expected to obey the fundamental laws of differentiation

$$(6) \qquad D(u + v) = Du + Dv, \quad D(uv) = D_u(uv) + D_v(uv),$$

where the subscripts u and v denote that u and v respectively are considered as differentiated subject to the constancy of the other. As ∇V is a vector when V is scalar, the operator ∇ may properly be called a vector operator and hence a vector differentiating operator. Next it is but natural to observe that the vector

* These remarks and those which follow, although written in the notation of Gibbs, should not be interpreted as limited to any particular system of notation; the recommendation is merely that, whatever be the notation for the scalar and vector products, the notation should be preserved and combined with ∇ or some equivalent sign of differentiation to express the divergence and curl of a vector function, while ∇ or its equivalent gives the gradient of a scalar function.

characteristic of ∇ is maintained in $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ provided that the $\cdot$ and $\times$ be considered in their usual general significance as operators which respectively combine two vectors into a scalar and into a vector. That there may be less chance for misconceiving our meaning, it may be stated that up to this time none of the operators is supposed to have had any cartesian interpretation and that the $\cdot$ and $\times$ cannot be considered as representing scalar and vector products of ∇ by V in the usual sense. The entire aim is to examine the formulas for differentiation with a view to determining whether or not the symbols obey the laws (6) of differentiation and the laws of scalar and vector products to an extent sufficient to warrant the statement that they are analytically suggestive.*

The formulas of differentiation as given by the authors are

$$\nabla (cV) = c\nabla V, \quad \nabla \cdot \mathbf{C} = 0, \quad \nabla \times \mathbf{C} = 0,$$
$$\nabla (U + V) = \nabla U + \nabla V, \quad \nabla \cdot (\mathbf{U} + \mathbf{V}) = \nabla \cdot \mathbf{U} + \nabla \cdot \mathbf{V},$$
$$\nabla \times (\mathbf{U} + \mathbf{V}) = \nabla \times \mathbf{U} + \nabla \times \mathbf{V}, \quad f(V) \nabla V = \nabla \int f(V) dV,$$
$$\nabla \times (V\mathbf{U}) = V\nabla \times \mathbf{U} + (\nabla V) \times \mathbf{U},$$
$$\nabla \cdot (V\mathbf{U}) = V\nabla \cdot \mathbf{U} + (\nabla V) \cdot \mathbf{U},$$
$$\nabla \cdot (\mathbf{V} \times \mathbf{U}) = \mathbf{U} \cdot \nabla \times \mathbf{V} - \mathbf{V} \cdot \nabla \times \mathbf{U}, \quad \nabla \times (\mathbf{V} \times \mathbf{U}) = ??$$

The results of the first three lines are certainly suggested by the fact that ∇ is a differentiating operator and subject to the first of equations (6). The results of the next line are similarly suggested by the second of equations (6). To show the procedure in greater detail:

$$\begin{aligned} \nabla \times (V\mathbf{U}) &= \nabla_V \times (V\mathbf{U}) + \nabla_{\mathbf{U}} \times (V\mathbf{U}) \\ &= \nabla_V V \times \mathbf{U} + V\nabla_{\mathbf{U}} \times \mathbf{U}. \end{aligned}$$

The scalar V when variable is passed out next to ∇ for differentiation, leaving the $\times$ between two vectors ∇V, $\mathbf{U}$; and the scalar V when constant is passed out past the sign of differentiation. The formula of the last line is treated in a similar way, but the additional laws for the interchange of dot and cross in

* The fact that ∇ may be expanded as $\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$ and that then curl $\mathbf{V}$ and div $\mathbf{V}$ may be regarded as the formal products $\nabla \times \mathbf{V}$ and $\nabla \cdot \mathbf{V}$ obtained according to the laws of multiplication would, from the present point of view, be the last instead of the first argument in justification of the notations.

a triple product or for the interchange of order in a vector product are used.

$$\nabla \cdot (\mathbf{V} \times \mathbf{U}) = \nabla_{\mathrm{V}} \cdot (\mathbf{V} \times \mathbf{U}) + \nabla_{\mathrm{U}} \cdot (\mathbf{V} \times \mathbf{U})$$
$$= \nabla_{\mathrm{V}} \times \mathbf{V} \cdot \mathbf{U} - \nabla_{\mathrm{U}} \cdot (\mathbf{U} \times \mathbf{V})$$
$$= (\nabla \times V) \cdot \mathbf{U} - (\nabla \times \mathbf{U}) \cdot \mathbf{V} = \mathbf{U} \cdot \nabla \times \mathbf{V} - \mathbf{V} \cdot \nabla \times \mathbf{U}.$$

The fact that the notation suggests the result is clear.

It is desirable to go somewhat further. The authors remark * that $\nabla \times (\mathbf{V} \times \mathbf{U})$ cannot be expressed in terms of grad, div, and curl, and they use libellous language toward the operator $\mathbf{a} \cdot \nabla$ — we believe they imply that it is an essential tachygraph! Now we have a fondness for this operator because, if $\mathbf{a}$ is a unit vector, it gives the directional derivative of a vector function.† We propose therefore to inject this operator into their system in a manner quite in sympathy with their definitions of curl and divergence. Consider

$$(7) \quad \mathbf{a} \times (\nabla \times \mathbf{V}) = (\mathbf{a} \cdot \mathbf{V}) \nabla - (\mathbf{a} \cdot \nabla) \mathbf{V} = \nabla (\mathbf{a} \cdot \mathbf{V}) - (\mathbf{a} \cdot \nabla) \mathbf{V},$$

where $\mathbf{a}$ is a constant vector. This formula has been obtained by expanding $\mathbf{a} \times (\nabla \times \mathbf{V})$ just as $\mathbf{a} \times (\mathbf{b} \times \mathbf{V})$ would be expanded and by the transference of ∇ in the first term to a position where it differentiates $\mathbf{V}$ as it should. Whether or not this procedure is regarded as accurate, the definition may be given that

$$(7') \qquad (\mathbf{a} \cdot \nabla) \mathbf{V} = \nabla (\mathbf{a} \cdot \mathbf{V}) - \mathbf{a} \times (\nabla \times \mathbf{V}),$$

and this definition of $(\mathbf{a} \cdot \nabla) \mathbf{V}$ is no more artificial than their definition (3) of div $\mathbf{V}$. Next let $d\mathbf{r}$ be any vector, consider $\mathbf{a}$ as the differential $\delta \mathbf{r}$, and apply the definition (2) of curl.

$$d\mathbf{r} \cdot \mathbf{a} \times (\nabla \times \mathbf{V}) = \mathbf{a} \cdot d\mathbf{V} - d\mathbf{r} \cdot \delta \mathbf{V}, \qquad \text{by (2)},$$
$$d\mathbf{r} \cdot \nabla (\mathbf{a} \cdot \mathbf{V}) = d(\mathbf{a} \cdot \mathbf{V}) = \mathbf{a} \cdot d\mathbf{V}, \qquad \text{by (1)}.$$
$$\therefore d\mathbf{r} \cdot [(\mathbf{a} \cdot \nabla) \mathbf{V}] = \mathbf{a} \cdot d\mathbf{V} - \mathbf{a} \cdot d\mathbf{V} - d\mathbf{r} \cdot \delta \mathbf{V} = d\mathbf{r} \cdot \delta \mathbf{V}.$$
$$(8) \qquad \therefore (\mathbf{a} \cdot \nabla) \mathbf{V} = \delta \mathbf{V} \quad \text{or} \quad (\delta \mathbf{r} \cdot \nabla) \mathbf{V} = \delta \mathbf{V}.$$

The last result is found by canceling the arbitrary vector $d\mathbf{r}$.

* Calcolo vettoriale, p. 68, Omografie vettoriali, p. 51.

† The directional derivative, like the laplacian operator, is a scalar operator and, like it, may be applied to the elements ϕ of any linear field when ϕ is regarded as a function of position. The definition as given in (7′) does not, however, apply in the general case, owing to the vector operations which it contains. In this respect (7′) corresponds to (4) and not to (5).

It is seen from (8) that $(d\mathbf{r}\cdot\nabla)\mathbf{V} = d\mathbf{V}$ and hence that if $\mathbf{a}$ is a unit vector $(\mathbf{a}\cdot\nabla)\mathbf{V}$ is the directional derivative of $\mathbf{V}$ in the direction $\mathbf{a}$. This result may be compared with (1). When V is scalar, ∇V is defined and $d\mathbf{r}\cdot\nabla V = dV$; the operation $(d\mathbf{r}\cdot\nabla)V$ is not defined in this case, but may naturally be defined as equal to $d\mathbf{r}\cdot\nabla V$. When $\mathbf{V}$ is a vector, $\nabla\mathbf{V}$ is not defined but $(d\mathbf{r}\cdot\nabla)\mathbf{V}$ is defined and the probability is suggested that at some future time it may become convenient to define $\nabla\mathbf{V}$ by the relation $d\mathbf{r}\cdot\nabla\mathbf{V} = (d\mathbf{r}\cdot\nabla)\mathbf{V} = d\mathbf{V}$. At any rate with this addition of the directional derivative to the system, expressions for $\nabla\times(\mathbf{V}\times\mathbf{U})$ and $\nabla(\mathbf{V}\cdot\mathbf{U})$* may be found and are identical with those suggested by the formal method.

A word may be added concerning derivatives of the second order. Here there are three important identities.

$$\nabla\cdot\nabla\times\mathbf{V} = 0, \quad \nabla\times\nabla V = 0,$$

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla\nabla\cdot\mathbf{V} - \nabla\cdot\nabla\mathbf{V}.$$

Of these the first two are naturally suggested by the vector characteristics of ∇ and the third, which is equivalent to (4), tallies with the formal expansion analogous to that used in (7). Now it is by no means our intention to regard these formal derivations of the fundamental formulas of differentiation as proofs of those formulas, but merely as proofs of the statement that the notations involving ∇ are analytically suggestive and that for this reason these notations are far preferable to those like grad, div, curl. It may perfectly well be that they have disadvantages which should cause their abandonment; but these disadvantages should be clearly stated and the statements should be true and not mistaken. The only statement which we have seen and which we regard as true relative to the disadvantages of the notations ∇, $\nabla\cdot$, $\nabla\times$ is that they do not suggest the physical significance of the operations as well as grad, div, curl do. Whether this nominal infelicity outweighs the analytic suggestiveness must be left to individual opinion.†

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* The result $\nabla(\mathbf{U}\cdot\mathbf{V}) = (\nabla\mathbf{U})\cdot\mathbf{V} + (\nabla\mathbf{V})\cdot\mathbf{U}$ which the formal method indicates is correct; but, as it is meaningless without definitions of $\nabla\mathbf{U}$ and $\nabla\mathbf{V}$, another form involving the directional derivative and the curl is usually given when treating only the minimum system.

† The reader will recall that at times there have been serious objections urged against replacing the S and V in the notations $S\mathbf{ab}$ and $V\mathbf{ab}$ for the scalar and vector products by other symbols, for the reason that no other notation so forcibly suggests which product is scalar and which is vector; nevertheless the S and V are not used much now.

5. The first half of the Calcolo vettoriale has now been covered. The second half, which is entitled applications, starts with a treatment of geometric applications, especially the tangent, normal, and binormal to curves. Here the use of vectors has decided advantages over the usual methods.

The second chapter on applications deals with fundamental theorems of integral calculus more than with applications. The formulas

$$\int \nabla \times \mathbf{V} d\tau = \int d\mathbf{S} \times \mathbf{V}, \quad \int \nabla V d\tau = \int V d\mathbf{S}, \quad \int \nabla \cdot \mathbf{V} d\tau = \int \mathbf{V} \cdot d\mathbf{S},$$

which are equivalent to an integration are given and proved. These formulas and many similar ones are all consequences of the operational equation (where any sign or no sign of multiplication may be inserted after ∇ and $d\mathbf{S}$)

$$\iiint d\tau \nabla (\ ) = \iint d\mathbf{S} (\ )$$

of which the proof or for which a justification may easily be given. Green's theorem, Stokes's theorem, and the formula for the rate of change of the flux of a fluid through a surface are the other topics of the chapter. The applications to mechanics come next and are tolerably numerous — velocity and acceleration of a point and their resolutions along various directions, central motion, kinematics of a rigid body, motion of a rigid body in its plane, equilibrium of strings, motion of a rigid body in space. The selection of topics and the method of discussion are both admirable. It is here that the authors find considerable use for their operator $e^{i\phi}$ which establishes a rotation through the angle ϕ.

The applications of a more physical character follow. A short chapter on hydromechanics sets forth the derivation of the equations of motion, vortical motion, and velocity potential. It is interesting to remark how short, direct, and elegant is the treatment of these fundamental questions by vectorial methods. The elements of the theory of the equilibrium of an isotropic elastic body is given. The volume closes with applications to electromagnetism including retarded potentials, Maxwell's equations, the Poynting vector, integrals of the equations, and finally Lorentz's equations. It may therefore be seen that this volume affords a very good introduction to the elementary general theories of mathematical physics in addition to its presentation of the most important parts of vector analysis. The

work cannot fail to be of interest to physicists and to students of vectorial methods. In it may be found numerous suggestions that merit wide-spread adoption. The large amount of material which has been put into a small space without any apparent crowding or obscurity is especially noteworthy and has been accomplished largely by adherence to the program of using purely vectorial methods. Somewhat greater reference to coordinates might have made the work easier to read for those who previously were unacquainted with vectors, but certain compensating disadvantages would undoubtedly have arisen.

6. The second volume, Omografie vettoriali, is divided into three chapters which treat respectively homographies or linear transformations with a fixed origin, differentiation with respect to a point, and applications to mathematical physics. In addition there is an introduction which presents a few generalities on linear operators or transformations and an appendix which contains some added developments in the analytic theory of homographies and some applications to the differential geometry of surfaces. The linear transformation or homography is designated by a small Greek letter and the vector which results from the application of the homography α to a vector $\mathbf{u}$ is denoted by $\alpha\mathbf{u}$. The invariants of α are written as $I_3\alpha$, $I_2\alpha$, $I_1\alpha$ and are defined by the equations

$$I_3\alpha = \frac{\alpha\mathbf{u} \times \alpha\mathbf{v} \cdot \alpha\mathbf{w}}{\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}}, \quad I_2\alpha = \left[\frac{d}{dx} I_3(x + \alpha)\right]_{x=0}$$

with an additional equation involving the second derivative to define $I_1\alpha$. It is shown that $I_3\alpha$ is independent of the three independent vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ which enter into its definition and that consequently the other two invariants are also really invariants. The reader will observe that the authors have no hesitation about adding together a number x and a homography α. They regard a number, whenever convenient, as a linear transformation. The meaning of the expression $x + \alpha$ would probably be taken from the equation

$$(x + \alpha)\mathbf{u} = x\mathbf{u} + \alpha\mathbf{u}, \quad \mathbf{u} \text{ arbitrary.}$$

Now although this equation may be regarded as a justification of the usage of the authors, the question does arise as to whether or not their usage is really good. From the algebraic or matricular points of view, which they almost wholly ignore, the quantity α is an element of a quadrate algebra containing

nine units and having a modulus (often called the idemfactor), but the unit 1 is not one of the units as it is in the case of the ordinary complex numbers or in the case of quaternions, and the idemfactor, which in those cases is 1, is not 1 in the algebra of homographies α because 1 is not an element of the system. For reasons that perhaps are merely puristic we believe that it is better to regard 1 and the idemfactor as distinct and to refrain from identifying multiplication by a number with multiplication in the system. Statements like: If α, β are homographies and m is a real number, then

$$I_1(\alpha+\beta)=I_1\alpha+I_1\beta, \quad I_1(m\alpha)=mI_1\alpha, \quad I_1m=3m,$$

seem particularly confused and infelicitous.

After a brief mention of singular homographies, of the fixed directions of a homography, and of the identical equation which a homography satisfies, the authors pass to the consideration of various types of homographies. According as one of the equations *

$$\mathbf{x}\cdot\alpha\mathbf{y}=\mathbf{y}\cdot\alpha\mathbf{x} \quad \text{or} \quad \mathbf{x}\cdot\alpha\mathbf{y}+\mathbf{y}\cdot\alpha\mathbf{x}=0$$

may be satisfied, the homography is called a dilatation or an axial homography. These correspond to what are often called self-conjugate and anti-self-conjugate or symmetric and skew-symmetric matrices or linear vector functions. The symbols D and V are introduced so that $D\alpha$ shall represent the dilatation of α or self-conjugate part of α and $V\alpha$ shall represent the vector of the axial remainder $\alpha - D\alpha$ of α. The result is that α may be written as

$$\alpha=D\alpha+V\alpha\times, \quad \text{and} \quad K\alpha=D\alpha-V\alpha\times$$

is thereupon taken as the definition of the conjugate of α. Both these equations must be regarded as operational equations and not as equations in multiple algebra. In this respect they are like $x+\alpha$. To obtain the transformation of plane areas regarded as vectors, the symbol R is introduced by the definition

$$R\alpha(\mathbf{x}\times\mathbf{y})=\alpha\mathbf{x}\times\alpha\mathbf{y}, \quad \text{and} \quad C\alpha=I_1\alpha-\alpha$$

is another definition which is reserved, however, for the appendix. There follow a large number of formulas connecting

* Readers familiar with current notations in linear associative algebra or with the notations of Hamilton, Gibbs, Cayley, or Clebsch-Aronhold for such equations as these will notice the lack of symmetry in the authors' use of the sign of multiplication and probably will regret that the idea of a post-operator was not introduced on a par with that of a pre-operator; this, however, would have involved the authors in serious notational difficulties.

the symbols I_r, D, V, K, R as applied to homographies (including real numbers). An additional symbol H is introduced so that $H(\mathbf{u}, \mathbf{v})$ may represent what in Gibbs's system is the dyad $\mathbf{vu}$. More relations between the symbols are given. Finally the chapter ends with a discussion of singular homographies and of versors and perversors.

From what has been said it will appear that the point of view of the authors is operational and not algebraic. Whether by taking this method and introducing these symbols they have materially added to the unity or unification of vectorial notations may be debated. Had they not been so insistent on the necessity of unification, we should say that these new methods and notations added an interesting and instructive diversity to the subject, and that the more points of view we had the better off we were both in a scientific and in a pedagogical way. Undoubtedly the best and most thorough way for anybody to learn a subject that is new and unfamiliar and unsatisfactory to him is to rewrite the subject according to his own desires; this replaces mere receptivity by original activity and lends a zest to the study. We should be happy to see everyone who is interested in vectors and who believes in their necessity adopt the authors' method and make the analysis suit himself. The adherents of unification would probably regard this proffered liberty as an invitation to license in any case other than their own. In fact although they are perfectly willing to use the operators i and $e^{i\phi}$ in a somewhat unusual way and to add numbers and homographies in a manner not in accord with the most careful practice, they are unwilling to let others use a single symbol for the laplacian operator or the notations ∇ and $\mathbf{a}\cdot\nabla$ which are of long standing and like the laplacian operator are almost universally believed to be essential operators independent of the axes of reference.

The question of the choice between the operational and algebraic treatment of strains deserves the most careful consideration if the proposition to abandon one and confine the attention to the other is seriously maintained. Linear operators are important. Owing to the researches of Volterra, Hadamard, Fréchet, and others, their importance is becoming rapidly extended to the domain of higher analysis. Hence it seems as though so much of the theory of linear operators as is applicable in general should be presented in the treatment of strains. The real question is whether the method should be applied to

strains in a detail which is not available in general and to the exclusion of the algebraic treatment. Without going into the matter of multiple algebra in general, mention may be made of the very important subject of matrices, which may now be studied so readily in the excellent presentation given by Bôcher in his Introduction to higher algebra. That there is a sort of isomorphism between matrices of the third order and strains can hardly be denied, and the isomorphism extends to matrices of higher order and strains in spaces of higher dimensions. Apart from the introduction of an arbitrary factor of proportionality, collineations in $n-1$ dimensions may replace strains in n dimensions. The subject of matrices is naturally so presented that one who is familiar with it cannot be said to be unfamiliar with the theory of strains, especially as that theory is presented by Gibbs. And it seems only fair toward the student who for any reason learns the theory of strains to present that theory in a way which makes it conversely true that one who is familiar with the theory of strains cannot be said to be really unfamiliar with the theory of matrices. It is doubtful if the operational method is fair in this sense. Moreover although the algebraic method may not always appear quite so direct as the operational, it has the advantage of possessing an important algorism — the algorism of multiplication — which adds considerably to the ease of acquisition and offers some insight into the general domain of multiple algebra. In fine, the algebraic method seems to us to afford so much of the operational point of view as is useful for the general theory of linear operators and in addition to offer intimate points of contact with the theory of matrices and the theory of multiple algebra.*

7. The chapter on derivatives is shorter but none the less important. If u is any entity which is a function of position, the authors define the derivative

$$\frac{du}{dP} \quad \text{by} \quad \frac{du}{dP}dP = du, \tag{9}$$

and they point out that the derivative of a vector is a homo-

* We believe that a large number of writers fail to bring out clearly and perhaps even fail to realize the setting of the simple vector algebra in the wide domain of general multiple algebra. To this is attributable many a comment founded exclusively on a too restricted point of view. Whatever may be advisable pedagogically, the only scientific aspect of vector analysis is in its place in multiple algebra.

graphy and the derivative of a homography is a linear operator which converts a vector into a homography. If **u** is a vector, the derivative of **u** is not the linear vector function $\nabla\mathbf{u}$ of Gibbs; it is the conjugate $(\nabla\mathbf{u})_c$ of that function, because the authors apply the differential vector dP after the operator where Gibbs writes $d\mathbf{r}\cdot\nabla\mathbf{u} = d\mathbf{u}$ and applies it before. In like manner the derivative of a homography is not $\nabla\alpha$, but one of its five conjugates. It may be noted that, as the derivative $d\mathbf{u}/dP$ of the present authors and the derivative $\nabla\mathbf{u}$ of Gibbs are both linear vector functions, they differ from Hamilton's $\nabla\mathbf{u}$ which is a quaternion. Numerous formulas for differentiation are given. It should be remarked that Gibbs was familiar with the authors' notation for derivatives but abandoned it in favor of the notation ∇u, for reasons known probably to no one. Victor Fischer in his Vektordifferentiation und Vektorintegration resumes this notation and extends it by using a dot and cross in a manner suggested by $\nabla\cdot\mathbf{u}$ and $\nabla\times\mathbf{u}$.

The authors next proceed to define a vector grad α by

$$\text{(10)} \qquad \text{grad}\,\alpha = \left(\frac{d\alpha}{dP}\mathbf{i}\right)\mathbf{i} + \left(\frac{d\alpha}{dP}\mathbf{j}\right)\mathbf{j} + \left(\frac{d\alpha}{dP}\mathbf{k}\right)\mathbf{k}.$$

The reader must not be so unwary as to be led to believe that the components of the vector grad α are the parentheses. The parentheses are linear vector functions which arise from applying the derivative of a homography to a vector, and the vector grad α is that which results from operating with these homographies upon the vectors after the parentheses. Although the definition of grad α is thus given in terms of the system **i**, **j**, **k**, the vector is really independent of any set of axes. It is unfortunate that an absolute definition like those of (1), (2), (3) was not given. It looks as if the authors had temporarily fallen back to some extent into the fatal slough of tachygraphy against which they are so careful to warn us. They go on to remark that in case the homography α reduces to a number m, the noteworthy result

$$(\text{grad}\,m)\cdot dP = dm, \quad \text{Cf. (1)},$$

arises; and with delightful ingenuousness they add that this relation does not appear to have a correlative for other homographies than numbers. Of course not! If the authors had seen fit to call the vector defined by (10) curl α or $\sqrt{\alpha}$ or anything else selected at random from the vast realm of mathe-

matical notations, they might have observed a similar lack of analogy. There is only one exception — if they had called their vector div $K\alpha$, they would have been surrounded on every side with the most persistent analogies.

Now after seeing the authors complain so much more bitterly than anybody else about the chaos of present vectorial notations and protest so vigorously against the use of a single symbol for the laplacian operator, it is difficult to say whether it is amusing or depressing to see these same authors introducing the old familiar symbol grad in a sense which they naively admit has small analogy with its former significance. Do they imagine that their present selection of notation will alleviate the chaotic condition? And do they find that the use of the same symbol for unrelated things is justifiable when it is not for things essentially identical?* If they had but listened to themselves half so attentively as they would have old hands at vector analysis listen to them, they would at least for safety's sake have used some other symbol than grad for (10) when they discovered that (10) had practically no connection with the well known grad. For this reason their choice of grad cannot be attributed to carelessness. It must be attributed to deliberateness. And this in face of the fact that various persons have used the term div in essentially the same sense as the authors use grad. Especial mention may be made of the extremely incisive and suggestive remarks of Prandtl† who treats the very subject of elasticity for which the authors are most in need of this new symbol. Even if an astute logician found no objection to the use of grad for div, it would seem as though a profound student of elasticity must feel intuitively that the forces in an elastic body arise from the divergence rather than from the slope of the fundamental homography connecting the normals to plane areas with the pressures upon them.‡

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* It is interesting to quote from the authors, *L'Enseignement*, p. 466: "En conclusion, peut-on admettre, dans les mathématiques, un même nom, un même signe, pour indiquer deux choses différentes? Nous ne le croyons pas; par conséquent *nous n'avons pas suivi et nous ne suivrons jamais cette voie, qui conduit inévitablement à faire des confusions.*" The italics are ours.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 13, pp. 436-449. This keen article should have the careful attention of all who are interested in vectors.

‡ The true inwardness of the incidental relation $dr.\nabla m = dm$ which arises when m is a numerical homography is seen by writing m in its proper form as mI where I is the idemfactor. Then $\nabla \cdot (mI) = \nabla m.\, I + m\nabla \cdot I$. But as I is constant, $\nabla \cdot I = 0$ and $\nabla \cdot (mI) = \nabla m$. Perhaps if the authors had not confused m and mI they would not have been led to mistake their relation grad $m \cdot dP = dm$ for an analogy with the gradient formerly defined.

Although this review is becoming lengthy we may perhaps be allowed the space to indicate what is probably the logical method of procedure in treating variable strains. With slight modifications we might use the authors' own excellent definitions (1), (2), (3) and write

$$(1) \qquad d\mathrm{r}\cdot\nabla u = d\mathrm{r}\cdot\mathrm{grad}\ u = du,$$

$$(2) \qquad d\mathrm{r}\times\delta\mathrm{r}\cdot\mathrm{curl}\ u = \delta\mathrm{r}\cdot du - d\mathrm{r}\cdot\delta u,$$

$$(3) \qquad \mathrm{div}\ u = \mathbf{a}\cdot[\mathrm{grad}\ (\mathbf{a}\cdot u) + \mathrm{curl}\ (\mathbf{a}\times u)],$$

where in (1) the entity u might be a scalar, vector, linear vector function, etc., and where in (2) and (3) the entity u may be a vector, linear vector function, etc., but not a scalar. With these definitions the reader may readily show the relation between div α and what the authors call grad α. If the methods of section 4 are adopted and extended, the definitions of curl and div become dependent on that of ∇ and the last two definitions may be suppressed. In fact these last two are introduced in the first instance so that the curl and divergence of a vector may be defined without introducing the gradient of a vector which would be a linear vector function. In like manner these definitions may be retained and the curl and divergence of a linear vector function may be defined without using the gradient of a linear vector function, which is an operator that converts vectors into linear vector functions. It is interesting to note that from the point of view of double multiplication (which must be considered in any general theory of Lückenausdrücke) the definition of the curl may be written

$$d\mathrm{r}\times\delta\mathrm{r}\cdot\nabla\times\Phi = (d\mathrm{r}\delta\mathrm{r} - \delta\mathrm{r}d\mathrm{r}):\nabla\Phi$$

and becomes formally identical with the equation

$$\alpha\times\beta\cdot\gamma\times\Phi = (\alpha\beta - \beta\alpha):\gamma\Phi, \quad (\alpha,\ \beta,\ \gamma \text{ vectors}),$$

which is a special case of a very general relation.

8. It will not be feasible to give any account of the last chapter of the Omografie, which contains applications to elasticity and to electrodynamics. There remains no space for such comments and there is very little to say. In bringing this review to a close it should be stated that the preponderating length of our adverse criticisms must not be interpreted as a wholesale condemnation of the two volumes. It has doubt-

less been noticed that the criticisms have been directed against those particular points at which, we feel confident, the authors have made incorrect statements or have unwisely abandoned fruitful algorisms and have thereby left the reader with a wrong or an unfortunately restricted point of view. There is no need to emphasize the excellent features of the work. A large number of these features are common to a considerable number of previous texts on vector analysis; many of them are new. The fact that there are so many points in which the volumes do not meet our approval is in itself evidence of the value of the books to all students of vectorial methods. In order to acquire a thorough appreciation of a subject it is necessary to examine various methods and points of view, and the restricted or even the wrong ones furnish an amount of instruction which is comparable with that furnished by those that are general and right. The present advanced state of the theory of functions of a real variable is due in no small measure to the inaccuracies or the narrow vision of earlier writers, and a considerable amount of present day instruction in this subject goes to showing that which is not true and contrasting it with that which is true.

For the benefit of vector analysis and cognate fields of mathematics we sincerely urge the general study of this work of Burali-Forti and Marcolongo and we especially recommend that each student follow their example and construct the system that pleases him most. That will be the best possible monument to the movement for unification. It will accomplish a real unity of knowledge and out of the resulting incidental diversity there will come a general and perchance not very slow elimination of the less fit and selection of the more fit. What the resulting residual system may be we will not venture to predict, but that there will be such a system fifty years hence we fully believe. And whatever that system may be it should and probably will conform to two requirements:

1° *Correct ideas* relative to vector fields,

2° *Analytic suggestiveness* of notation.

The first requirement may be fulfilled by proper teaching regardless of notation, whether vectorial, quaternionic, or cartesian; for the physicist the second is perhaps now best exemplified by the system of Gibbs, but the future may develop something preferable.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *January*, 1910.

SHORTER NOTICE.

Allgemeine Formen- und Invariantentheorie; Band I. Binäre Formen. By W. Fr. MEYER. Leipzig, Göschen (Sammlung Schubert, volume XXXIII), 1909. viii + 376 pp.

IN accordance with the general plan of the Schubert series, the treatise on invariants commences with concrete examples to lead up to general theorems, presupposes no previous knowledge of the subject, yet presents a systematic discussion which includes all the essentials of this important discipline.

The entire theory of the quadratic equation, as developed in elementary algebra, is reproduced in all detail, and the same ideas are applied to systems of equations. Anharmonic forms, involutions, Jacobians, are all explained and illustrated.

Now come linear substitutions, first translations, then inflations, and finally reciprocations. The general substitution is shown to be made up of these three, and those functions which are unchanged by the three elementary operations are therefore invariant under the general substitution. Conversely, the linear substitutions generate groups which may be classified as one, two, or three parameter groups, according to Lie.

Properties of the self corresponding elements of a general substitution and of the double elements of an involution are treated in a manner that makes this chapter a valuable appendix to a course in projective geometry.

The preceding elementary and very concrete discussion occupies 118 pages; it is followed by a chapter of 40 pages on bilinear and quadratic forms, with an introduction of the concept of the differential operator. Here again every transformation is built up from the elementary ones, the effect of each operation upon functions of the coefficients being minutely examined. As an appendix some twenty pages are now added on symbolic representation, no use of which is made in any part of the book except in the appendix at the end; in the latter the fundamental theorem is proved that every invariant can be expressed symbolically in terms of the elementary determinant forms. While this theorem is perhaps desirable for the sake of completeness, it is presented from such a different point of view that the discussion is out of harmony with the rest of the book. The treatment is so concise that the proof of the theorem will hardly be convincing to the reader, let alone any possibility of using the new method in his own work.

The second part is concerned with the differential equations connected with invariants of binary forms. Given a function $J(a_1, a_2, \cdots)$ which is invariant under the transformation $A_1(h) \equiv x_1 = \xi_1 + h_1\xi_2,\ x_2 = \xi_2$ by which

$$f(x) \equiv \sum_{i=0}^{n} n_i a_i x_1^{n-i} x_2^i$$

goes into

$$\phi(\xi) \equiv \sum_{i=1}^{n} n_i \alpha_i \xi^{n-i} \xi_2^i,$$

n_i being the binomial coefficient $n!/(n-i)!\, i!$; if $J(\alpha) = J(a)$ under $A_1(h)$ and $A_2(h)$, it is called an invariant of translation. Since α_i is a function of h as well as the a, we have

$$J(\alpha) = J(a) + h \left.\frac{dJ(\alpha)}{dh}\right]_{h=0} + \cdots,$$

from which $J'(\alpha)_{(n=0)} = 0$.

From this equation the first invariant operator

$$\nabla_1 = \sum_{i=1}^{n} i a_{i-1} \frac{\partial}{\partial a_i}$$

follows, and similarly the second,

$$\nabla_2 = \sum_{i=0}^{n} (n-i) a_{i+1} \frac{\partial}{\partial a_i}.$$

There follows a good discussion of the laws of combination of these operators and their application to a series of simultaneous forms.

The concepts of order and weight, commutators, and infinitesimal transformation are now taken up and everything reduced to the fundamental operators. There are numerous examples given for the student to work out; the algebraic work in each illustrates the idea nicely, but many could have their value increased by showing their geometric meaning, as was done in the preceding chapter.

The second chapter in the second part is concerned with relative invariants. The theorem that all the coefficients of a binary form lacking the second term are relative invariants is given an elegant proof. It is shown how to construct forms of

a given order and weight and that every invariant form for f_n is the source of a covariant for f_{n+r}. All these results are generalized to apply to a system of simultaneous forms.

The concept of an invariant is extended to apply to certain transcendental forms, including the logarithm and the elliptic integrals. By means of the former it is shown that every symmetric function of the roots can be rationally expressed in terms of the sum of the powers of the roots and a number of related theorems are derived (Waring's formulas). It is now easy to derive the expressions for the discriminant of an equation, the resultant of two such equations, and the expressions for the elementary relative invariants in terms of the roots. A second volume is in preparation which is to extend the preceding theory to ternary and quaternary forms.

To students of analytic geometry and of algebraic functions Professor Meyer's treatise will be of real assistance.

VIRGIL SNYDER.

NOTES.

THE April number (volume 11, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "The theorem of Thomson and Tait and natural families of trajectories," by EDWARD KASNER; "The introduction of ideal elements and a new definition of projective n-space," by F. W. OWENS; "The groups of classes of congruent quadratic integers with respect to a composite ideal modulus," by ARTHUR RANUM; "A simplified treatment of the regular singular point," by G. D. BIRKHOFF; "The strain of a gravitating, compressible elastic sphere," by L. M. HOSKINS.

AT the meeting of the London mathematical society held on March 10 the following papers were read: By W. F. SHEPPARD, "Forms of the remainder in the Euler-Maclaurin sum formula"; by J. W. NICHOLSON, "The scattering of light by a large conducting sphere"; by Miss H. P. HUDSON, "The 3–3 birational space transformation."

THE following papers have been read at recent meetings of the Edinburgh mathematical society. January 19: by R. SANGANA,

"Series for calculating Euler's constant and the constant in Sterling's theorem"; by D. Mc. SOMMERVILLE, "A problem in voting" and "Classification of geometries with projective matrix." February 11: by L. NAVANIENGAR, "The locus of points at which two sides of a given triangle subtend equal or supplementary angles"; by J. I. CRAIG, "Orthogonal trajectories in vectorial coordinates." March 11: by G. P. CARSLAW, "The Bolyai-Lobachevsky non-euclidean geometry; an elementary interpretation and some results which follow from it"; by R. J. T. BELL, "The locus of the lines that intersect three given lines"; by R. C. ARCHIBALD, "Note on Wallace's line."

THE Royal academy of sciences, letters and arts of Belgium announces the following prize problems for 1911:

"New investigations in the development of real or analytic functions in series of polynomials"; prize 800 francs.

"A summary of the various memoirs on systems of conics in space and a new contribution to the theory of these systems"; prize 600 francs.

Competing essays must be written in French or Flemish and sent to the secretary under the usual conditions before August 1, 1911.

THE publishing house of Gauthier-Villars in Paris announces the following additions to the series of monographs on the theory of functions edited by Professor E. BOREL: Principes de la théorie des fonctions entières de genre infini, by O. BLUMENTHAL; Leçons sur la théorie de la croissance, by A. DENJOY; Leçons sur les séries de polynomes à une variable complexe, by P. MONTEL; Leçons sur le prolongement analytique, by L. ZORETTI; L'inversion des intégrales définies, by V. VOLTERRA; Quelques principes fondamentaux de la théorie des fonctions de plusieurs variables complexes, by P. COUSIN; Leçons sur les correspondances entre variables réelles, by J. DRACH; Leçons sur la fonction $\zeta(s)$ de Riemann et son application à la théorie des nombres premiers, by H. v. KOCH.

THE following courses in advanced mathematics are offered by the various American universities during the academic year 1910–1911.

CORNELL UNIVERSITY. — By Professor J. MCMAHON: Theory of probabilities, two hours; Vector analysis, two hours.

— By Professor J. H. TANNER : Theory of equations, three hours. — By Professor J. I. HUTCHINSON : Theory of functions of a complex variable, three hours. — By Professor V. SNYDER : Descriptive geometry, three hours ; Birational transformations, two hours, first term. By Professor F. R. SHARPE : Mechanics, two hours. — By Professor W. B. CARVER : Advanced calculus, three hours. — By Professor A. RANUM : Theory of groups, two hours. — By Dr. D. C. GILLESPIE : Differential geometry, two hours. — By Dr. C. F. CRAIG : Applications to mechanics and physics, two hours. — By Dr. F. W. OWENS : Differential equations, two hours. — By Dr. J. V. MCKELVEY : Advanced analytic geometry, three hours. — By Dr. L. L. SILVERMAN : Algebra of logic, two hours.

PRINCETON UNIVERSITY. — By Professor H. B. FINE : Theory of algebraic numbers, three hours, first term. — By Professor H. D. THOMPSON : Coordinate geometry, three hours. — By Professor L. P. EISENHART : Mechanics, three hours ; Differential geometry, three hours. — By Professor O. VEBLEN : Linear groups and invariants, three hours, second term ; Projective geometry, II, three hours, first term ; Projective geometry, I, three hours. — By Professor G. D. BIRKHOFF : Differential equations, three hours ; Differential equations of physics, three hours. — By Professor E. SWIFT : Theory of functions of a complex variable, I, three hours.— By Professor J. H. McL. WEDDERBURN: Theory of functions of a complex variable, II, three hours, second term.

YALE UNIVERSITY. — By Professor J. PIERPONT : Abelian functions, two hours ; Thermodynamics, two hours ; Theory of functions of a complex variable, two hours ; Modern analytic geometry, two hours. — By Professor P. F. SMITH: Geometrical analysis, one hour ; Differential geometry, two hours ; Elementary differential geometry, two hours. — By Professor E. W. BROWN : Elementary mechanics, two hours ; Advanced mechanics, two hours ; Advanced calculus, three hours. — By Professor W. R. LONGLEY : Calculus of variations, two hours ; Potential theory and harmonic analysis, one hour. — By Dr. A. W. GRANVILLE : Elementary differential equations, one hour. — By Dr. G. M. CONWELL : Finite groups, two hours ; Partial differential equations of physics, one hour. — By Dr. G. F. GUNDELFINGER : Advanced analytic geometry, two hours. — By Dr. D. D. LEIB : Transformations of space, two hours.

The following courses in mathematics are announced for the summer semester, 1910.

University of Strassburg. — By Professor H. Weber: Definite integrals and introduction to the theory of functions, four hours; Algebra, two hours; Seminar, two hours. — By Professor F. Schur: Projective geometry, four hours; Theory of ordinary differential equations, two hours; Seminar, two hours. — By Professor J. Wellstein: Forms and matrices, two hours; Vector analysis, three hours; Seminar, two hours. — By Professor L. v. Mises: Graphical statics, two hours; Aërial mechanics, two hours. — By Professor P. Epstein: Introduction to higher mathematics, two hours. — By Professor M. Simon: Algebraic analysis in connection with the methods of elementary arithmetic, two hours.

For the year 1908–1909 the list of doctorates with mathematics as the major subject conferred by German universities is as follows (the list for 1906–1908 appeared in the February Bulletin, pages 268–274):

Berlin.

Lichtenstein, L. "Zur Theorie der gewöhnlichen Differentialgleichungen und der partiellen Differentialgleichungen zweiter Ordnung. Die Lösungen als Funktionen der Randwerte und der Parameter."

Breslau.

Freund, E. "Entwickelung willkürlicher Funktionen vermittelst meromorpher."

Goldman, F. "Poncelet'sche Polygone bei Kreisen."

Jopke, A. "Synthetische Untersuchungen über lineare Kegelschnittsysteme erster, zweiter, und dritter Stufe."

Kliem, F. "Ueber Oerter von Treffgeraden entsprechender Strahlen in eindeutig und linear verwandten Strahlengebilden erster bis vierter Stufe."

Giessen.

Lepper, H. "Ueber die invarianten Bildungen von Formen mit digredienten Schichten von Variabeln."

Schmidt, K. "Untersuchungen über Kurven dritter Ordnung im Anschluss an eine Grassmann'sche Erzeugungsweise."

WAGNER, R. "Ueber binäre bilineare und quaternäre quadratische Formen."

Göttingen.

BOLTZE, E. "Grenzschichten an Rotationskörpern in Flüssigkeiten mit kleiner Reibung."

HARR, A. "Zur Theorie der orthogonalen Funktionensysteme."

IHLENBURG, W. "Ueber die geometrischen Eigenschaften der Kreisbogenvierecke."

KOCH, H. "Ueber die praktische Anwendung der Runge-Kuttaschen Methode zur numerischen Integration von Differentialgleichungen."

SCHIMMACK, R. "Axiomatische Untersuchungen über die Vektoraddition."

SPEISER, A. "Die Theorie der binären quadratischen Formen mit Koeffizienten und Unbekannten in einem beliebigen Zahlkörper."

Greifswald.

FINKE, P. "Ueber Schaaren von ∞^5 Kurven im gewöhnlichen Raume."

HAUSSLEITER, H. "Zur Theorie der Pfaffschen Systeme."

LIER, O. "Ueber Flächenschaaren, die durch Berührungstransformation in Kurvenschaaren überführbar sind."

WERNER, A. "Ueber Systeme von drei Pfaffschen Gleichungen im Raume von fünf Dimensionen."

ZIEMKE, E. "Ueber partielle Differentialgleichungen erster Ordnung mit Integralvereinen, die als Punktmannigfaltigkeiten zweifach ausgedehnt sind."

Halle.

BOLDMANN, O. "Zur Theorie der übergeschlossenen Gelenkmechanismen."

JONAS, H. J. "Ueber W-Strahlensysteme, Flächendeformation und äquidistante Kurvenschaaren."

Jena.

GÜNTZEL, F. "Ueber Gruppierungen und Realitätsverhältnisse gewisser Punkte bei Raumkurven vierter Ordnung erster Spezies."

ROEGNER, M. "Die Steiner'sche Hypocykloide."

Kiel.

JANSEN, H. "Lückenlose Ausfüllung des R_n mit gitterförmig angeordneten n-dimensionalen Quadern."

NEUENDORFF, R. "Ueber Kreispunktpolarkurven."

Königsberg.

NEUMANN, A. "Ueber quadratische Verwandtschaften in Ebene und Raum, insbesondere Kreis- und Kugelverwandtschaften."

Leipzig.

FÖRSTER, R. "Beiträge zur specielleren Theorie der Riemannschen P-Funktionen 3ter Ordnung."

MEYER, C. "Zur Theorie des logarithmischen Potentials."

Munich.

BERWALD, L. "Krümmungseigenschaften der Brennflächen eines geradlinigen Strahlensystems und der in ihm enthaltenen Regelflächen."

BÖHM, F. "Parabolische Metrik im hyperbolischen Raum."

BURMESTER, H. "Untersuchung der wahren Hellegleichen auf der Kugel nach dem Lommel-Seeligerschen Satz."

DEBYE, P. "Der Lichtdruck auf Kugeln von beliebigem Material."

DEGENHART, H. "Ueber einige zu zwei ternären quadratischen Formen in Beziehung stehende Konnexe."

HOWLAND, L. A. "Anwendung binärer Invarianten zur Bestimmung der Wendetangenten einer Kurve dritter Ordnung."

NOETHER, F. "Ueber rollende Bewegung einer Kugel auf Rotationsflächen."

SCHMID, A. "Anwendung der Cauchy-Lipschitz'schen Methode auf lineare partielle Differentialgleichungen."

ZAPP, E. "Untersuchung eines speziellen Falles des Drei- und Vierkörperproblems."

Rostock.

Jecke, R. H. "Beiträge zur Geometrie der Bewegung."

Lange, M. "Vereinfachte Formeln für die trigonometrische Durchrechnung optischer Systeme."

Strassburg.

Malessa, G. "Fokale Eigenschaften korrelativer Grundgebilde."

Tübingen.

Casper, M. "Ueber die Darstellbarkeit der homomorphen Formenschaaren durch Poincaré'sche Z-Reihen."

Fritz, H. "Die Darstellung willkürlicher Funktionen in Anwendung auf die Statistik."

Oehler, H. "Ueber die Gleichungssysteme, welche man aus einer Matrix variabler Elemente durch Nullsetzen der Determinanten gegebener Ordnung erhält.

Würzburg.

Widder, W. "Untersuchungen über die allgemeinste lineare Substitution mit vorgegebener $p$ter Potenz."

Zilling, J. "Ueber die infinitesimale Deformation der Minimalflächen."

During the academic year 1908–1909 the following doctorates with mathematics as the major subject were conferred by the University of Paris:

Diennes, P. "Essai sur les singularités des fonctions analytiques."

Gambier, B. "Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes."

Verone, T. "Contribution à la théorie des ondes liquides."

Mr. L. N. G. Filon, professor of pure mathematics in University College, London, and Mr. G. H. Hardy, fellow and mathematical lecturer in Trinity College, Cambridge, have been elected to membership in the Royal Society of London.

Mr. G. I. Taylor, of Trinity College, Cambridge, was awarded a first Smith's prize for his essay on "Discontinuous motion in gases."

Dr. E. Meissner, of the technical school of Zürich, has been promoted to a professorship of rational mechanics.

Professor H. Heegaard, of the military academy at Waedbeck, has been appointed professor of mathematics at the University of Copenhagen, as successor to Professor H. G. Zeuthen, who will retire at the close of the present academic year.

Professor A. S. Chessin has recently delivered lectures on the modern theory of the gyrostat at the U. S. Naval Academy and at Lafayette College.

At Columbia University, Professor Edward Kasner has been promoted to a full professorship of mathematics. Mr. H. B. Curtis has been appointed instructor in mathematics in Barnard College.

At Cornell University, Dr. F. R. Sharpe, Dr. W. B. Carver and Dr. A. Ranum have been promoted to assistant professorships of mathematics. Mr. W. A. Hurwitz and Mr. E. J. Miles have been appointed instructors in mathematies.

Mr. A. S. Hawkesworth has been appointed professor of higher mathematics at the University of Pittsburg.

At Haverford College, Professor A. H. Wilson, of the Alabama Polytechnic Institute, has been appointed associate professor of mathematics, as successor to Professor W. H. Jackson, who returns to England.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

BACHMANN (P.). Niedere Zahlentheorie. 2ter (Schluss-) Teil. Additive Zahlentheorie. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, X, 2.) Leipzig, Teubner, 1910. 8vo. 10 + 480 pp. Cloth. M. 17.00

DINGELDEY (F.). Sammlungen von Aufgaben zur Anwendung der Differential- und Integralrechnung. In 2 Teilen. 1ter Teil: Aufgaben zur Anwendung der Differentialrechnung. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, XXXII, 1.) Leipzig, Teubner, 1910. 8vo. 6 + 202 pp. Cloth. M. 6.00

LEBON (E.). Gaston Darboux. Biographie, bibliographie analytique des écrits. Paris, Gauthier-Villars, 1910. 8vo. 8 + 72 pp. Fr. 7.00

LORIA (G.). Spezielle algebraische und transzendente ebene Kurven-Theorie und Geschichte. Autorisierte, nach dem italienischen Manuskript bearbeitete deutsche Ausgabe von F. Schütte. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, V, 1.) In 2 Teilen. 1ter Band. 2te verbesserte und vermehrte Auflage. Leipzig, Teubner, 1910. 8vo. 18 + 488 pp. Cloth. M. 18.00

MASON (M.). See MOORE (E. H.).

MICKIEWICZ (B.). Das sogennante Fermatsche Problem. Krakau, 1909. 8vo. 28 pp. M. 2.00

MINKOWSKI (H.). Geometrie der Zahlen. Leipzig, Teubner, 1910. 2te Lieferung. 8vo. Pp. 8 + 241–256. M. 1.00

MOORE (E. H.), WILCZYNSKI (E. J.) and MASON (M.). The New Haven Mathematical Colloquium. Lectures delivered before members of the American Mathematical Society in connection with the summer meeting, held September 5th to 8th, 1906, under the auspices of Yale University. New Haven, Yale University Press, 1910. 8vo. 10 + 222 pp. Cloth. $3.00

NEUMANN (C.). Ueber das logarithmische Potential einer gewissen Ovalfläche. Leipzig, Teubner, 1909. M. 3.00

OLIVO (A.). Sulla soluzione dell'equazione cubica di Tartaglia: studio storico-critico. Milano, Frigerio, 1909. 16mo. 36 pp.

PENKMAYER (R.). Beweis des Satzes von Fermat. (Neubearbeitung.) München, Lindauer, 1910. 8vo. 9 pp. M. 0.50

PILGRIM (L.). Vereinfachte Behandlung der schiefwinkligen Koordinaten im Raum. Stuttgart, Metzler, 1909.

ROESER (E.). Die Verfolgungskurve auf der Kugel. (Diss.) Halle, 1909.

ROTHENBERG (S.). Geschichtliche Darstellung der Entwicklung der Theorie der singulären Lösungen totaler Differentialgleichungen von der ersten Ordnung mit zwei variablen Grössen. Leipzig, Teubner, 1908. M. 3.60

SCHÜTTE (F.). See LORIA (G.).

STAUDE (O.). Analytische Geometrie des Punktpaares, des Kegelschnittes und der Fläche zweiter Ordnung. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, XXX, 1.) 1ter Teilband. Leipzig, Teubner, 1910. 8vo. 10 + 548 pp. Cloth. M. 22.00

VOLKMANN (P.). Erkenntnistheoretische Grundzüge der Naturwissenschaften und ihre Beziehungen zum Geistesleben der Gegenwart. Allgemein wissenschaftliche Vorträge. 2te, vollständig umgearbeitete und erweiterte Auflage. (Wissenschaft und Hypothese, IX.) Leipzig, Teubner, 1910. M. 6.00

WILCZYNSKI (E. J.). See MOORE (E. H.).

II. ELEMENTARY MATHEMATICS.

AUBERT (P.) et PAPELIER (G.). Exercices d'algèbre, d'analyse et de trigonométrie. Tome II, à l'usage des élèves de mathématiques spéciales (2e année.) Paris, Vuibert, 1910. 8vo. 363 pp.

BAUDOIN (P.). See BOURLET (C.).

BOURLET (C.) et BAUDOIN (P.). Cours abrégé de géométrie, publié avec de nombreux exercices théoriques et pratiques et des applications au dessin géométrique. I. Géométrie plane, 6e, 5e, 4e, B. 4e édition, revue. Paris, Hachette, 1909. 16mo. 14 + 408 pp. Fr. 2.50

CANIN (O.). Tafel der Logarithmen des Sinusversus. Berlin, 1909. M. 0.90

CIAMBERLINI (C.). Aritmetica e geometria, per la terza classe complementare. Torino, Paravia, 1910. 8vo. 82 pp. L. 1.20

DURELL (C. V.). A course of plane geometry for advanced students. Part 2. London, Macmillan, 1910. 8vo. 372 pp. Cloth. 7s. 6d.

EIESLAND (J. A.). Advanced algebra for technical schools and colleges. Morgantown, W. Va., Eiesland, 1910. 8vo. 156 pp. $2.00

HARTDEGEN (F.). Kurzer Abriss der Geometrie. Leipzig, Bange, 1910. 200 pp. M. 1.20

HÖFLER (A.). Didaktik des mathematischen Unterrichts. (Didaktische Handbücher für den realistischen Unterricht an höheren Schulen. Herausgegeben von A. Höfler und F. Poske. 1ter Band.) Leipzig, Teubner, 1910. 8vo. 18 + 510 pp. Cloth. M. 12.00

KNOPS (K.) und MEYER (E.). Lehr- und Uebungsbuch für den Unterricht in der Mathematik an den höheren Mädchenschulen, Lyceen und Studienanstalten. Essen, Baedeker, 1910. 8vo.

Heft 6 für Klasse II der Lyceen. 107 pp. Cloth. M. 1.20

Heft 6a für die Klassen II und I der realen Studienanstalten. 224 pp. Cloth. M. 2.00

Heft 7 für die Klasse I der Lyceen. 120 pp. Cloth. M. 1.20

KUNDT (F.). Arithmetische Aufgaben mit einem Anhange von Aufgaben aus der Stereometrie für höhere Mädchenschulen und die unteren Klassen der Studienanstalten. Auf Grund der Ausführungsbestimungen zu dem Erlasse vom 18. August 1908 über die Neuordnung des höheren Mädchenschulwesens bearbeitet. Leipzig, Teubner, 1910. 8vo. 6 + 172 pp. Cloth. M. 2.00

LENNES (N. J.). See Slaught (H. E.).

MANNOWRY (G.). Methodologisches und Philosophisches zur Elementar-Mathematik. Haarlem, Visser, 1909. 8vo. 8 + 279 pp. M. 9.50

MARTINI ZUCCAGNI (A.). Elementi di algebra, ad uso delle scuole tecniche e normali. 3a edizione, corretta e migliorata. Livorno, Belforte, 1910. 16mo. 8 + 126 pp. L. 1.00

—— e NARDI (P.). Trattato di geometria elementare, ad uso delle scuole secondarii inferiori e specialmente delle scuole tecniche e normali. 2a edizione, corretta e migliorata. Livorno, Belforte, 1910. 16mo. 12 + 322 pp. L. 1.75

MEYER (E.). See KNOPS (K.).

NARDI (P.). See MARTINI ZUCCAGNI (A.).

OTTO (F.) und SIEMON (P.). Lehr- und Uebungsbuch der Arithmetik und Algebra für höhere Mädchenschulen. Nach den ministeriellen Bestimmungen vom 18. VIII und 12. XII. 1908 bearbeitet. Pensum für Klasse IV–I. 4te Auflage. Leipzig, Hirt, 1910. 8vo. 200 pp. M. 2.25

PAPELIER (G.). See AUBERT (P.).

QUAIO (E.). Raccolta di calcoli fatti, con 90 tabelle ed istruzioni pratiche sul modo di usarle. 2a edizione, ampliata del manuale "Conti e calcoli fatti," Milano, Hoepli, 1910. 16mo. 11 + 341 pp.

SCHUSTER (A.). Einführung in die elementare Mathematik. Kempten, Kösel, 1909. 8vo. 4 + 169 pp. Cloth.

SELLENTHIN (B.). Mathematischer Leitfaden mit besonderer Berücksichtigung der Navigation. Auf Veranlassung der Kaiserlichen Inspektion des Bildungswesens der Marine bearbeitet. 2te umgearbeitete Auflage. Leipzig, Teubner, 1910. 8vo. 10 + 452 pp. Cloth. M. 8.40.

SIEMON (P.). See OTTO (F.).

SLAUGHT (H. E.) and LENNES (N. J.). Plane geometry, with problems and applications. Boston, Allyn and Bacon, 1910. 12mo. 286 pp. Cloth. $1.00.

SUPPANTSCHITSCH (R.). Lehrbuch der Geometrie für Gymnasien und Realgymnasien. Mittelstufe, Planimetrie und Stereometrie. Wien, Tempsky, 1910. K. 4.50.

WARREN (I.). Elements of plane trigonometry for the use of schools. 8th edition. London, Hodges, 1910. 8vo. 194 pp.

III. APPLIED MATHEMATICS.

ANTOMARI (X.). Cours de géométrie descriptive, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, aux Ecoles centrales des arts et manufactures, des ponts et chaussées et des mines de Paris et de Saint-Etienne. 4e édition. Paris, Vuibert, 1910. 8vo. 636 pp.

BIANCHI MALDOTTI (E). Manuale di idraulica, 2a edizione, riveduta e notevolmente ampliata. Torino, Fiandesio, 1910. 16mo. 20 + 256 pp.

HALL (W. L.) and MAXWELL (H.). Surface conditions and stream flow. Washington (Office of the superintendent of documents), 1910. 8vo. 16 pp.

HAMMER (E.). Mess- und Rechen-Uebungen zur praktischen Geometrie. 4te Auflage. A. Ausgabe für Bau-Ingenieure. Stuttgart, Metzler, 1910. 198 pp. Cloth. M. 3.30

—— Dasselbe. B. Ausgabe für Maschinen-Ingenieure und Architekten. Stuttgart, Metzler, 1910. 78 pp. Cloth. M. 1.40

HORT (W.). Technische Schwingungslehre. Einführung in die Untersuchung der für den Ingenieur wichtigsten periodischen Vorgänge aus der Mechanik starrer, elastischer, flüssiger und gasförmiger Körper, sowie aus der Elektrizitätslehre. Berlin, Springer, 1910. 8vo. 7 + 227 pp. Cloth. M. 6.40

KEMPE (H. R.). The engineer's yearbook of formulae, rules, tables, data and memoranda, 1910. London, Lockwood, 1910. 8vo. 8s.

LAMB (H.). The dynamical theory of sound. New York, Longmans, 1910. 8vo. 8 + 303 pp. Cloth. $3.50

MACDONALD (H. M.). The diffraction of electric waves round a perfectly reflecting obstacle. London, Dulau, 1910. 4to 2s.

MAXWELL (H.). See HALL (W. L.).

OETTINGEN (A. VON). Elemente der projektiven Dioptrik. Leipzig, Teubner, 1910.

SALMOIRAGHI (A.). Istrumenti e metodi moderni di geometria applicata. Parte I: Teoria degli istrumenti misuratori, descrizione e norme pratiche per l'uso. Vol. II. Milano, 1909. 8vo. Pp. 497–912.

TESAR (L.). Die Mechanik. Eine Einführung mit einem metaphysischen Nachwort. Leipzig, Teubner, 1909. 8vo. 14 + 220 pp. M. 4.00

TIMERDING (H. E.). Die Theorie der Kräftepläne. Eine Einführung in die graphische Statik. (Mathematisch-physikalische Schriften für Ingenieure und Studierende herausgegeben von E. Jahnke, No. 7.) Leipzig, Teubner, 1910. 8vo. 699 pp. Cloth. M. 3.00

WALKER (G. W.). The initial accelerated motion of electrified systems of finite extent, and the reaction produced by the resulting radiation. London, Dulau, 1910. 4to. 2s. 6d.

WHITTALL (W. J. H.). An elementary lecture on the theory of life assurance. Delivered at a meeting of the Birmingham Insurance Institute, January, 1889. 2nd edition, revised. London, Layton, 1910. 8vo. 43 pp. 2s.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and forty-eighth regular meeting of the Society was held in New York City on Saturday, April 30, 1910, extending through the usual morning and afternoon sessions. The following forty-one members were present:

Mr. E. S. Allen, Mr. F. W. Beal, Professor W. J. Berry, Dr. E. G. Bill, Professor G. D. Birkhoff, Professor E. W. Brown, Mr. R. D. Carmichael, Miss Emily Coddington, Professor F. N. Cole, Professor J. L. Coolidge, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor C. C. Grove, Dr. G. F. Gundelfinger, Dr. L. C. Karpinski, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Dr. H. F. MacNeish, Dr. Emilie N. Martin, Mr. A. R. Maxson, Mr. H. H. Mitchell, Professor C. L. E. Moore, Professor Frank Morley, Professor G. D. Olds, Professor W. F. Osgood, Dr. H. B. Phillips, Mr. H. W. Reddick, Professor R. G. D. Richardson, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Dr. W. M. Strong, Professor J. H. Tanner, Professor C. B. Upton, Professor Oswald Veblen, Mr. H. E. Webb, Miss M. E. Wells, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Ex-President T. S. Fiske and Professor Frank Morley at the afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. F. W. Beal, Princeton University; Professor W. J. Berry, Brooklyn Polytechnic Institute; Mr. J. K. Lamond, Yale University; Mr. R. M. Mathews, University High School, Chicago, Ill.; Professor F. E. Miller, Otterbein University; Mr. J. E. Rowe, Johns Hopkins University; Mr. W. H. Terrell, Clyde, N. C.; Mr. George Wentworth, Exeter, N. H.; Mr. W. A. Wilson, Yale University. Eight applications for membership in the Society were received.

Professor Bôcher was elected by the Council to succeed Professor Osgood as a member of the Editorial Board of the *Transactions*. Professor Dickson was appointed to fill the unexpired term of Professor E. B. Van Vleck, who retires from the Editorial Board in July.

It was decided to hold the April, 1911, meeting of the Society at Chicago, Ill. The publication of the lectures delivered at the Princeton Colloquium by Professors G. A. Bliss and Edward Kasner was placed in charge of the Committee of Publication. The New Haven Colloquium Lectures have recently been issued by the Yale University Press.

The following papers were read at this meeting:

(1) Dr. H. B. Phillips: "Application of Gibbs's indeterminate product to the algebra of linear systems."

(2) Dr. H. B. Phillips: "Concerning a class of surfaces associated with polygons on a quadric surface."

(3) Professor Virgil Snyder: "Conjugate line congruences contained in a bundle of quadric surfaces."

(4) Professor W. B. Carver: "Ideals of a quadratic number field in canonic form."

(5) Professor G. A. Miller: "On a method due to Galois."

(6) Dr. E. H. Taylor: "On the transformation of the boundary in conformal mapping."

(7) Professor W. B. Fite: "Concerning the invariant points of commutative collineations."

(8) Professor R. G. D. Richardson: "On the saddle point in the theory of maxima and minima and in the calculus of variations."

(9) Mr. H. H. Mitchell: "Note concerning the subgroups of the linear fractional group $LF(2, p^n)$."

(10) Mr. H. H. Mitchell: "The subgroups of the linear group $LF(3, p^n)$."

(11) Professor C. L. E. Moore: "Some infinitesimal properties of five-parameter families of lines in space of four dimensions."

(12) Professor Edward Kasner: "Forces depending on the time, and a related transformation group."

(13) Professor F. H. Safford: "Sturm's method of integrating $dx/\sqrt{X} + dy/\sqrt{Y} = 0$."

(14) Dr. G. F. Gundelfinger: "On the geometry of line elements in the plane with reference to osculating circles."

In the absence of the authors Dr. Taylor's paper was presented by Professor Osgood, and the papers of Professors Snyder, Carver, Miller, Fite, and Safford were read by title. Abstracts of the papers follow below.

1. In his vice-presidential address before the American Asso-

ciation for the Advancement of Science (1886) Gibbs suggested a method for obtaining the results of Grassmann's linear algebra by the use of what he called an indeterminate product. Following this method Dr. Phillips gives proofs of a number of the fundamental theorems of Grassmann.

2. If the lines of a polygon of $2n$ sides lie on a quadric surface, it follows from a theorem of Poncelet that the lines joining the odd vertices to the non-adjacent even ones lie on a surface of class $n - 2$. Dr. Phillips discusses these surfaces and the configurations of lines for the cases $n = 5$ and $n = 6$.

3. In Professor Snyder's paper a method was developed for determining all the congruences formed by the generators of a bundle of quadric surfaces, and for distinguishing the necessary and sufficient condition that the two systems generate congruences that are rationally separable. The lines l, l' through a basic point B_i of a bundle of quadrics $\Sigma\lambda_i H_i$ are arranged in pairs, each of which uniquely determines the other, defining an involution I. A cone K with vertex at B_i fixes a congruence σ in Σ and its image K' in I fixes another congruence τ; the lines of σ, τ are formed by conjugate systems of generators of the same family of quadrics. The singular cones at B_i, B_k are birationally equivalent. The congruences σ, τ define an infinite discontinuous birational group of point transformations which leave their common focal surface invariant. Finally, both congruences define a relation $f(\lambda) = 0$ which is the equation of a contact curve of the discriminant $\Delta(\lambda)$ in the λ plane. All the contact curves of $\Delta(\lambda)$ having an odd characteristic can be obtained in this way.

4. In his Vorlesungen über Zahlentheorie, Sommer shows that any ideal of the quadratic number field $k(\sqrt{m})$ can be reduced to the canonic form $(i, i_1 + i_2\omega)$, where i, i_1, and i_2 are rational integers, i_2 a factor of i and i_1, and ω is $\sqrt{m}$ or $\frac{1}{2}(1 + \sqrt{m})$ according as $m \not\equiv 1 \pmod 4$ or $m \equiv 1 \pmod 4$. In the present paper Dr. Carver uses a slightly modified canonic form, viz., $r(s, t + \omega)$, r, s and t being rational integers and ω having the meaning given above, with the conditions $s > t \geqq 0$ and

$$t^2 \equiv m \pmod s, \qquad \text{if } m \not\equiv 1 \pmod 4;$$
$$(2t + 1)^2 \equiv m \pmod{4s}, \qquad \text{if } m \equiv 1 \pmod 4.$$

A method is developed for factoring, multiplying, and dividing (when division is possible) ideals in this canonic form.

5. If H represents any subgroup of a group G, all the operators of G may be represented, without repetition, in either of the following two ways:

$$G = H + HS_2 + HS_3 + \cdots + HS_\gamma$$
$$= H + T_2H + T_3H + \cdots + T_\gamma H.$$

If for each HS_α $(\alpha = 2, 3, \cdots, \gamma)$ it is possible to find some $T_\beta H$ so that all the operators of HS_α coincide with those of $T_\beta H$, then H is an invariant subgroup of G, and vice versa. Galois called attention to this important case and named it a proper decomposition of G. Professor Miller considers the general case where HS_α has exactly ρ operators in common with $T_\beta H$ and proves that, if this condition is satisfied, the operators of both HS_α and $T_\beta H$ transform H into a group which has exactly ρ operators in common with H. In particular, if all the operators of HS_α coincide with those of $T_\beta H$ then H is invariant under each of the operators of HS_α. He also proves that in any group whatsoever it is possible to select the operators $S_2, S_3, \cdots, S_\gamma$ in such a way that all the operators of G may be represented, without repetition, in either of the following ways, H being any arbitrary subgroup of G:

$$G = H + HS_2 + HS_3 + \cdots + HS_\gamma$$
$$= H + S_2H + S_3H + \cdots + S_\gamma H.$$

In Weber's Lehrbuch der Algebra, volume 2, 1896, page 8, it is observed that the second of these decompositions is also equal to

$$H + S_2^{-1}H + S_3^{-1}H + \cdots + S_\gamma^{-1}H.$$

Hence the S's may also be replaced by their inverses without affecting either decomposition as a whole.

6. The solution of the problem of mapping the interior of a simply connected region S on the interior of a circle was completed by Professor Osgood in his proof of the existence of the Green's function of the most general simply connected plane region. The question as to whether the boundary of S, when this boundary is of the most general nature, will be transformed

continuously into the circumference of the circle, was answered by Professor Osgood in a set of theorems published in volume 9 of the BULLETIN. The major part of Dr. Taylor's paper is given to the proofs of these theorems.

7. In view of the erroneous statement by Reye in his Geometrie der Lage that a space collineation A with just four invariant points can be commutative only with collineations that have the same invariant points, Professor Fite determines in detail what must be the invariant points of collineations commutative with A. He extends the results for three dimensional space to collineations in space of $n - 1$ dimensions, and determines the nature and number of the collineations that are commutative with both A and B, B being any collineation commutative with A.

8. Professor Richardson shows that the minimum of the function $f(x, y, z)$ for those values of the variables that satisfy the relation $g(x, y, z) + \pi h(x, y, z) = 0$ is a function $M(\pi)$ of the parameter π, which when maximized gives the same constant as the minimum of the function $f(x, y, z)$ for those values of the variables which satisfy the relations $g(x, y, z) = 0$, $h(x, y, z) = 0$.

In the calculus of variations the two problems corresponding are also found to be equivalent:

1. To find a function $y(x)$ which satisfies the boundary conditions

$$(A) \qquad y(0) = y(1) = 0$$

and the integral relations

$$\int_0^1 g(x, y, dy/dx)dx = 0, \quad \int_0^1 h(x, y, dy/dx) = 0,$$

and minimizes the integral

$$(B) \qquad \int_0^1 f(x, y, dy/dx).$$

2. The minimum of the integral (B) for those functions $y(x)$ which satisfy the boundary conditions (A) and the integral condition

$$\int_0^1 \{g(x, y, dy/dx) + \pi h(x, y, dy/dx)\}dx = 0$$

is a function of the parameter π. To determine π in such a way that the minimum is maximized.

In both theories the results admit of immediate generalization.

9. Moore and Wiman have determined the subgroups of the linear fractional group $LF(2, p^n)$. Mr. Mitchell gives another method for the determination of those subgroups which contain additive groups. It is based on the solution of the Diophantine equation

$$\Omega = 1 + (p^m - 1)\frac{\Omega}{d_1 p^m} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} \quad (f_i = 1, 2).$$

10. In this paper Mr. Mitchell extends the results obtained by him concerning the subgroups of the linear group $LF(3, p)$, to the more general group $LF(3, p^n)$. The group is represented as a collineation group in the finite plane $PG(2, p^n)$.

The maximal subgroups are: the $LF(3, p^k)$, where k is a divisor of n; the hyperorthogonal groups $HO(3, p^k)$, which appear for n even and k a factor of $\frac{1}{2}n$; groups leaving invariant a point, line, triangle, and conic; a G_{720} (for $p = 5$), $G_{\frac{1}{2}7!}$ $(p = 5)$, the G_{360}, G_{216}, and G_{168}.

11. Professor Moore makes use of the ten coordinates of a line in S_4 to discuss some properties of five-parameter families which involve second and higher differentials. The interpretation in the geometry of circles in ordinary space is then given.

12. The forces considered by Professor Kasner lead to differential equations of the form $\ddot{x} = \phi(x, y, t)$, $\ddot{y} = \psi(x, y, t)$. In connection with the trajectories (xy curves), it is of interest to introduce the xt curves and the yt curves, t being represented as a space coordinate. The transformations of x, y, t converting a pair of equations of the above type into one of the same type form an infinite group involving three arbitrary functions.

13. In Professor Safford's paper Sturm's method of integrating $dx/\sqrt{X} + dy/\sqrt{Y} = 0$ is discussed for a somewhat more general form of X than usual. The integrating factor is found as a function of xy and the coefficients of X. The determination of this factor has previously been in most cases dependent upon a knowledge of the integral to be obtained instead of upon the given differential equation.

14. By interpreting the projective geometry within a linear line complex as a geometry of line elements in the plane by

means of a transformation of Lie, Dr. Gundelfinger effects a classification of ordinary differential equations of the first order with respect to the arrangement of the ∞^2 osculating circles to their integral curves and develops a theory of reciprocal line element loci.

F. N. Cole,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

The twenty-seventh regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 8–9, 1910. Professor L. E. Dickson, Vice-President of the Society and Chairman of the Section, presided at the three sessions held on Friday morning and afternoon and Saturday morning. The attendance at the various sessions included twenty visitors and the following forty-four members of the Society:

Mr. R. P. Baker, Mr. W. H. Bates, Professor G. A. Bliss, Professor Oskar Bolza, Dr. R. L. Börger, Dr. H. E. Buchanan, Dr. Thomas Buck, Dr. H. T. Burgess, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor W. B. Ford, Professor A. G. Hall, Professor E. R. Hedrick, Mr. T. H. Hildebrandt, Professor O. D. Kellogg, Professor Kurt Laves, Dr. A. C. Lunn, Dr. W. D. MacMillan, Mr. E. J. Miles, Professor G. A. Miller, Dr. R. L. Moore, Professor E. H. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Dr. Anna J. Pell, Professor Alexander Pell, Professor H. L. Rietz, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. R. R. Shumway, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor B. F. Yanney, Professor J. W. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening nearly all of the members present at the meeting dined together at the Quadrangle Club, at which time the question of holding the next meeting of the Section at Minneapolis was discussed. A letter was read from the President of the University of Minnesota urging the Section to meet

there in connection with the annual convocation of the American Association for the Advancement of Science. Both the president and secretary of Section A of the Association, Professors E. H. Moore and G. A. Miller, urged the desirability of closer affiliation between the mathematicians and other scientists, especially the astronomers and physicists. It was finally voted to hold the December meeting of the Section at Minneapolis, unless the Council should decide that the annual meeting of the Society, with the presidential address, will be held at Chicago at that time. It had been previously announced that the Council had considered favorably the question of holding a meeting of the entire Society at Chicago either in December, 1910, or in April, 1911, depending upon the convenience of the President, Professor Bôcher.

The following papers were read at this meeting:

(1) Professor D. R. CURTISS: "An extension of Descartes's rule of signs."

(2) Professor C. H. SISAM: "On a class of r-spreads in space of n dimensions."

(3) Dr. ANNA J. PELL: "On a functional equation."

(4) Dr. H. E. BUCHANAN: "The equations of variation for the straight line elliptical orbits of three finite masses."

(5) Professor J. B. SHAW: "On quaternions."

(6) Professor W. B. FORD: "On the determination of the asymptotic developments of a given function (second paper)."

(7) Professor OSKAR BOLZA: "An application of 'general analysis' to a problem in the calculus of variations."

(8) Dr. H. T. BURGESS: "Differential equations of the projective curves on a quadric whose tangents belong to a line complex."

(9) Professor E. J. WILCZYNSKI: "On the general theory of congruences of straight lines."

(10) Mr. R. P. BAKER: "On a class of equations of state representing normal and abnormal three-state bodies."

(11) Dr. L. I. NEIKIRK: "A theorem on (m, n) correspondence."

(12) Mr. W. H. BATES: "Note on the generalization of the formulas of Gauss and Codazzi."

(13) Mr. W. H. BATES: "On the medium curvature of a hypersurface (second paper)."

(14) Professor G. A. MILLER: "Groups of transformations of Sylow subgroups."

(15) Professor G. A. MILLER: "Extensions of theorems due to Cauchy."

(16) Dr. R. L. MOORE: "On Duhamel's theorems."

(17) Miss IDA M. SCHOTTENFELS: "The Fano geometry."

(18) Professor F. R. MOULTON: "On the problem of the spherical pendulum."

(19) Professor L. E. DICKSON: "Equivalence of families of quadratic forms."

Professor Dickson's paper was read by title. Professor Bolza's paper appeared in full in the May number of the BULLETIN. Abstracts of the other papers follow below, the numbers corresponding to those in the above list.

1. Descartes's rule of signs gives merely an upper limit for the number of positive roots of an algebraic equation $f(x) = 0$. To determine the exact number of such roots we must usually have recourse to other methods. In Professor Curtiss's paper it is, however, shown that there always exists a polynomial $\phi(x)$ such that the number of variations of sign in the coefficients of $F(x) = f(x)\phi(x)$ is exactly equal to the number of positive roots of $f(x) = 0$. Further, $\phi(x)$ can be so chosen that $F(x)$ has only $m + 1$ coefficients, where m is the degree of $f(x)$, and these coefficients are determinants of a matrix formed from the coefficients of $f(x)$. From this is derived a criterion for the number of real roots in any interval (a, b). The paper concludes with a graphical method for applying these tests.

2. The principal theorem in Professor Sisam's paper is the following: If V_r is an analytic r-spread belonging to a space of n dimensions $(n \geqq r + 1)$; if through a generic point of V_r there does not pass a continuum of spaces of t dimensions lying on V_r; and if the section of V_r by a generic hyperplane through the tangent r-dimensional space at a generic point P of V_r is an $(r - 1)$-spread the tangent cone to which at P has a double t-dimensional space; then V_r is generated by t-dimensional spaces in such a way that the tangent r-dimensional space to V_r is invariant along each t-dimensional space.

3. The functional equation $\lambda\phi(s) = T[\phi(s)]$, where T is a single-valued fractional transformation with the two properties

$$(f) \qquad \int_a^b f(s)T[f(s)]\,ds \geqq 0,$$

$$(f_1, f_2) \qquad \int_a^b f_1(s)T[f_2(s)]\,ds = \int_a^b f_2(s)T[f_1(s)]\,ds,$$

is reduced by means of a biorthogonal system of functions to an infinite system of linear equations whose coefficients form a limited symmetric matrix. Dr. Pell considers the character of solutions corresponding to a point, or continuous spectrum of λ, and finds conditions for the solvability of the non-homogeneous functional equation, together with some expansion theorems.

4. In his dissertation Dr. Buchanan discussed some periodic orbits of three bodies near the straight line circular solutions. The present paper is a report on the extension to the straight line elliptical solutions. For $e = 0$, the solutions obtained here reduce to those obtained in the circular case. The equations of variation, formed in the usual way, are a system of homogeneous linear equations of the twelfth order, but two of them are independent of the other four. The coefficients of the equations are expanded as power series in the eccentricity, the coefficients of these power series being simply periodic. The solutions are found to exist as power series in e. The characteristic exponents are found to be uniquely determined by the periodicity conditions and to have the same character as in the circular case.

5. In this paper Professor Shaw develops formulas for certain functions of two, three, four, or more quaternions, by means of which the various differentiating operators of quaternions may be readily expressed. Thus formulas are found for the quaternion operator

$$\frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

which are as manageable as those for ∇. Theorems of integration are given. These functions furnish an application to quaternions of general formulas previously given in the author's papers on general associative algebra.

6. In Professor Ford's paper a method is derived for determining the asymptotic developments of a given function, and application is made to a number of important functional types. The results supplement those of a former paper (see April number of the *Annals of Mathematics*); also those of Barnes, Lindelöf, Wiman, Mattson, and others concerning the asymptotic develments of integral functions defined by canonical (Weierstrass)

products. A new type of development is obtained for these latter functions, thus enabling one to study their behavior for large values of the variable in regions which were precluded in the previous developments.

8. Dr. Burgess shows that the differential equation of the curves, upon the quadric $\phi(x) \equiv x_1x_4 + x_2x_3 = 0$, whose tangents belong to the line complex $F(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}) = 0$, is homogeneous in six arguments and of the form

$$F(u^2dv,\ v^2du,\ -udv - vdu,\ -dv,\ du,\ vdu - udv) = 0,$$

where the parametric equations of the quadric ϕ are

$$\rho x_1 = uv, \quad \rho x_2 = u, \quad \rho x_3 = v, \quad \rho x_4 = -1.$$

The case for a *linear* line complex is then studied in detail and the theory of elementary divisors is applied to get the canonical forms of the differential equations, together with their integrals. Next, the case where $F(p) = 0$ is quadratic is attacked in the particular form in which $F(p) = 0$ is the line equation of a quadric surface. Here the theory of elementary divisors is again applied and the canonical forms of the differential equations are found.

9. The new theory of congruences presented by Professor Wilczynski is based upon the consideration of the completely integrable system of partial differential equations

$$(D) \qquad \begin{aligned} y_v &= mz, \quad z_u = ny, \\ y_{uu} &= ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v. \end{aligned}$$

The integral surfaces of (D) are the two sheets of the focal surface of the congruence, the curves $u =$ const. and $v =$ const. upon these surfaces being their intersections with the developables of the congruence. The lines of the congruence are obtained by joining corresponding points (u, v) of the two surfaces. All of the known results in the general theory of congruences, so far as they are of a projective nature, present themselves in a very simple way from this point of view, and a large number of new theorems make their appearance. The so-called Laplace transformation which was first associated with the theory of congruences by Darboux appears in a new light, enabling one

to formulate and solve far more general problems than has hitherto been possible. The general theory is applied to two particular problems: first, to the determination of those congruences which belong to a linear complex and for which the first Laplace transformation gives rise to a congruence that likewise belongs to a linear complex; second, to the study of those congruences whose focal surfaces are quadrics. Both of these problems are closely connected with the differential equation

$$\frac{\partial^2\theta}{\partial u \partial v} = e^{2\theta} - e^{-2\theta}.$$

10. The equations considered in Mr. Baker's paper are of the form $p = tF - TM$, in which F represents the gas law and is explicitly discussed for Van der Waal's form $1/(v-1)$ and for Planck's form $-\log(1-1/v)$; $M = 1/v^2 - c/v^3 + d/v^4$ and is an extension of Van der Waal's term which may ultimately be a power series; and $T = mt/(t^2 + kt + l^2)$ is a special form of Amagat's multiplier. The conditions for the existence of a triple point when $T = 1$ are worked out for both the gas laws mentioned and give in each case an area in the (c, d) plane within which triple points are possible. If $k > 0$, the surface is of the same type as for a normal three state body, the LS line ending at a critical point or not, according to the dependence of c, d on the temperature, quite small movements of the (c, d) point deciding. If $k < 0$ and the (c, d) point passes out of the triple point area before the critical temperature is reached, the equation represents the type of an abnormal body. If this does not happen, a form approximating the "Tamann idea" is reached, though the complete form requires seven values of v for given p, t. The particular form of the gas law is indifferent as far as type is concerned; and the reversal of sign of the molecular pressure is neither necessary nor sufficient for abnormality, but may occur with a very low temperature for the triple point. By a proper choice of the three scales and the constants, the equations may be compared with the observations.

11. Emil Weyr in a paper * published in 1870 defines an n-fold involution of points on a straight line as the points whose abscissas are roots of the equation $f(x) - \lambda\phi(x) = 0$, where the base functions $f(x)$ and $\phi(x)$ are rational integral functions of

* *Math. Annalen*, vol. 3 (1870), pp. 34–44.

degree n, and λ is an arbitrary parameter. He finds its double elements and other properties. In a second paper † he introduces the two involutions $f(x) - \lambda\phi(x) = 0$ and $F(y) - \mu\psi(y) = 0$, of degrees m and n, and calls these involutions projective when $A\lambda\mu + B\lambda + C\mu + D = 0$. He considers the plane curves defined by the projective involutions and develops their properties. Dr. Neikirk considers an (m, n) correspondence given by a rational integral equation $f(x, y) = 0$ of degree m in x and n in y which has at least a single set of correspondences of the type $(x_1, x_2, x_3 \cdots, x_m; y_1, y_2, y_3 \cdots, y_n)$ and shows that $f(x, y) = 0$ may be reduced to $\theta(x) + \lambda\phi(x) = 0$, $\chi(y) + \lambda\psi(y) = 0$, where θ, ϕ, χ, and ψ are rational integral functions, the first two of degree m, the last two of degree n, and λ is a variable parameter.

12. In his first paper Mr. Bates uses Maschke's symbolic notation in generalizing the ordinary method of deriving the Gauss and Codazzi formulas.

13. In a paper read before the Society in December, 1909, Mr. Bates expressed those medium curvatures of R_n that have even subscripts, in terms of the first fundamental quantities of R_n. In the present paper, he makes use of a theorem of Maschke to express those medium curvatures of R_{n-1} that have odd subscripts, in terms of the first fundamental quantities of R_n and the function which defines R_{n-1} in R_n. The formula for the first medium curvature K_1 contains as special cases the curvature of ordinary plane and twisted curves and the medium curvature of the surface $f(x, y, z) = 0$.

14. Professor Miller proves the following theorems: The necessary and sufficient condition that a transitive group of degree n is a group of transformations of Sylow subgroups is that its subgroup composed of all its substitutions which omit a given letter involves one and only one Sylow subgroup of degree $n - 1$. Hence there are only two alternating and also only two symmetric groups which are groups of transformations of Sylow subgroups. If the Sylow subgroups of order p^m in a group are transformed under this group according to an imprimitive group, the number of these Sylow subgroups is of the form $(1 + k_1 p^\alpha)(1 + k_2 p^\alpha)$ where none of the integers k_1, k_2, α is zero. Moreover, the systems of imprimitivity must be transformed

† *Prager Sitzungsberichte*, 1870, pp. 14–19.

according to some non-regular group. If a group involves only 3 Sylow subgroups of order 2^m these subgroups generate a group of order $3 \cdot 2^m$; if it involves exactly 4 Sylow subgroups of order 3^m they must generate a group whose order is either $4 \cdot 3^m$ or $8 \cdot 3^m$; if it involves exactly 6 Sylow subgroups of order 5^m they generate a group whose order is either $12 \cdot 5^m$ or $24 \cdot 5^m$. The group generated by these Sylow subgroups is a characteristic subgroup of the entire group if it does not coincide with it.

15. The object of Professor Miller's second paper was to extend a theorem due to Cauchy and to indicate how Sylow's theorem is an almost immediate result of this extended theorem, which was stated as follows: The number of those operators of a group G, involving H_1, H_2 as subgroups, which transforms H_1 into a group having exactly ρ operators in common with H_2 is divisible by h_1h_2/ρ, h_1 and h_2 being the orders of H_1 and H_2 respectively. This theorem was proved by Cauchy for the special case where $\rho = 1$, and this special case was used by him to prove that every group whose order is divisible by a given prime number p must involve operators of order p. The extended theorem may be used, in a similar way, to prove that every group whose order is divisible by p^a must involve a subgroup of order p^a.

16. Dr. Moore discusses certain conditions under which the limit of the sum of one set of infinitesimals is the same as that of another set.

17. In the *Annals of Mathematics*, second series, volume 11, number 2, page 60, Mr. George M. Conwell discusses the Fano configuration and its group. Miss Schottenfels in the present paper discusses the postulates under which the Fano geometry was developed and its extension to n-space. See *Giornali di Matematiche*, volume 30 (1892).

18. The problem of the spherical pendulum gives rise to the differential equations

$$\frac{d^2x}{dt^2} + (a_0 + a_1z)x = 0, \quad \frac{d^2y}{dt^2} + (a_0 + a_1z)y = 0,$$

$$\frac{d^2z}{dt^2} + (b_0 + b_2z^2)z - g = 0,$$

where x, y and z are subject to the conditions

$$x^2 + y^2 + z^2 = l^2.$$

Professor Moulton treats the third equation as one of a class possessing periodic solutions. The existence of the solutions is proved and practical methods of constructing them as power series in a parameter are given. Then the first two equations become homogeneous linear differential equations having simply periodic coefficients of the form first treated by Hill in his memoir on the motion of the lunar perigee. A distinct treatment of these equations is given, both as to the existence of the solutions and as to practical methods of finding them.

19. Professor Dickson discusses the equivalence of two pencils of quadratic forms in n variables. If in the pencil $A = \lambda_1 A_1 + \lambda_2 A_2$, we replace the generators A_1 and A_2 by $\bar{A}_1 = gA_1 + hA_2$ and $\bar{A}_2 = g'A_1 + h'A_2$, where $gh' - g'h \neq 0$, then $\bar{A} = \lambda_1 \bar{A}_1 + \lambda_2 \bar{A}_2$ becomes $\bar{\lambda}_1 A_1 + \bar{\lambda}_2 A_2$, where $\bar{\lambda}_1 = \lambda_1 g + \lambda_2 g'$, $\bar{\lambda}_2 = \lambda_1 h + \lambda_2 h'$. An elementary divisor $(a_1\lambda_1 + a_2\lambda_2)^a$ of $|A|$ corresponds to the elementary divisor $(\bar{a}_1\lambda_1 + \bar{a}_2\lambda_2)^a$ of $|\bar{A}|$, where $\bar{a}_1 = a_1 g + a_2 h$, $\bar{a}_2 = a_1 g' + a_2 h'$. It follows that two non-singular pencils of quadratic forms are equivalent if and only if the elementary divisors of one pencil can be derived from those of the second by a linear transformation on the two parameters.

To discuss the problem of equivalence in a given field F, use is made of the results in *Transactions*, volume 10 (1909), pages 347–360. For the above replacement of generators, the elementary divisor $(\lambda - c)^a$ of $|\lambda A_1 - A_2|$ corresponds to the elementary divisor $(\lambda - \gamma)^a$ of $|\lambda \bar{A}_1 - \bar{A}_2|$, where $\gamma = (g' + ch')/(g + ch)$. Hence from the ultimate canonical types (obtained in the paper cited) of non-singular pairs of quadratic forms we may derive canonical types of pencils of forms by restricting the roots $c_1, c_2, \cdots, c_m$ to sets not equivalent under linear fractional transformation in the initial field F.

H. E. SLAUGHT,
Secretary of the Section.

GROUPS GENERATED BY TWO OPERATORS EACH OF WHICH IS TRANSFORMED INTO A POWER OF ITSELF BY THE SQUARE OF THE OTHER.

BY PROFESSOR G. A. MILLER.

(Read before the Chicago Section of the American Mathematical Society, January 1, 1910.)

§ 1. *Introduction.*

Two special cases of the category of groups defined by the heading of this paper have been considered; viz., when the square of each of the two generators transforms the other generator either into itself* or into its inverse.† It was observed that in the former of these two cases the orders of the two generators are not restricted, while in the latter each of these orders must divide 8. Each of these special cases led to a very elementary category of solvable groups. It will be proved that the more general category defined by the heading of this paper is also composed entirely of solvable groups of simple structure.

As an instance of how such generalizations may lead to very complex categories of groups we may give the theorem that *every symmetric group can be generated by two operators whose squares are commutative.* In fact, the symmetric group of degree n is evidently generated by the following two cyclic substitutions whose squares are commutative:

$$t_1 = (x_1 x_2 x_3 \cdots x_{n-1}), \quad t_2 = (x_1 x_n).$$

From the theorem that every symmetric group whose degree exceeds 8 can be generated by two substitutions of orders 2 and 3 respectively ‡ it results directly that all such groups are included in the category of groups defined by the condition that each of them can be generated by two operators which are transformed into themselves by the square and cube respectively of the other.

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*BULLETIN, vol. 16 (1910), p. 173.

‡ *Annals of Mathematics*, vol. 9 (1907), p. 48.

‡ BULLETIN, vol. 7 (1901), p. 426.

§ 2. *General Considerations.*

The conditions imposed upon the two generators s_1, s_2 of the group G, as expressed by the heading of this paper, give rise to the following equations:

$$s_1^{-2}s_2s_1^2 = s_2^{\alpha}, \quad s_2^{-2}s_1s_2^2 = s_1^{\beta}.$$

If at least one of the two numbers α, β is even, the corresponding operator is of odd order and hence it must be generated by its square. In this case G is generated by a cyclic group and an operator transforming this cyclic group into itself. As many properties of these groups are well known, we shall confine our attention, in what follows, to the consideration of cases in which both α and β are odd. The group H generated by s_1^2, s_2^2 clearly belongs to the elementary category of groups which may be generated by two operators each of which transforms the other into a power of itself.* Hence the commutator subgroup of H is cyclic and the order of H divides the quotient obtained by dividing the product of the orders of s_1^2, s_2^2 by the order of the commutator subgroup.

It is easy to see that H is invariant under G, since α and β are odd and the following equations are satisfied:

$$s_1^{-1}s_2^2s_1 = s_2^2s_2^{-2}s_1^{-1}s_2^2s_1 = s_2^2s_1^{1-\beta}, \quad s_2^{-1}s_1^2s_2 = s_1^2s_2^{1-\alpha}.$$

From the fact that

$$s_2^{-2}s_1^{-2}s_2^2 = s_2^{2\alpha-2}s_1^{-2} = s_1^{-2\beta}$$

it results that

$$s_1^{2(\beta-1)}s_2^{2(\alpha-1)} = 1.$$

Hence the two operators $s_1^{2(\beta-1)}$, $s_2^{2(\alpha-1)}$ are invariant under G. To find multiples of the orders of s_1, s_2 we may transform these invariant operators by $s_2^2s_1^2$ respectively, as follows:

$$s_2^{-2}s_1^{2(\beta-1)}s_2^2 = s_1^{2\beta(\beta-1)} = s_1^{2(\beta-1)}, \quad \text{or} \quad s_1^{2(\beta-1)^2} = 1;$$

$$s_1^{-2}s_2^{2(\alpha-1)}s_1^2 = s_2^{2\alpha(\alpha-1)} = s_2^{2(\alpha-1)}, \quad \text{or} \quad s_2^{2(\alpha-1)^2} = 1.$$

Hence the theorem: *If two operators s_1, s_2 satisfy the equations $s_1^{-2}s_2s_1^2 = s_2^{\alpha}$, $s_2^{-2}s_1s_2^2 = s_1^{\beta}$, their orders divide $2(\beta-1)^2$ and $2(\alpha-1)^2$ respectively, their squares generate a group which involves a cyclic commutator subgroup and is invariant under the*

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* *Quar. Jour. of Math.*, vol. 37 (1906), p. 286.

group generated by s_1 *and* s_2, *and they satisfy the equation* $s_1^{2(\beta-1)} = s_2^{2(1-\alpha)}$.

The quotient group of G with respect to H is dihedral, since it can be generated by two operators of order 2. As H is solvable and this quotient group is solvable, G is always solvable. That is, *if two operators are such that each is transformed into a power of itself by the square of the other, they generate a solvable group whose fourth derived is identity.* As an instance of such a group, we may cite the symmetric group of order 24. This is generated by any two of its operators of order 4 which do not have a common square, and each of these two operators is transformed into its inverse by the square of the other. In this case the third derived is already identity. To obtain a group whose third derived is not identity we may consider the group of order 48 generated by two operators of order 8 each of which is transformed into its inverse by the square of the other.* This group of order 48 illustrates also that the orders of s_1, s_2 may be actually $2(\beta-1)^2$, $2(\alpha-1)^2$ respectively.

The theorem stated above implies that the numbers α, β always fix an upper limit for the orders of s_1, s_2 except when at least one of these numbers is unity. When both of them are unity there results the special case noted in the introduction. If only one of them (α) is unity the defining relations assume the form

$$s_1^{-2}s_2s_1^2 = s_2, \quad s_2^{-2}s_1s_2^2 = s_1^{\beta} \ (\beta \neq 1).$$

Hence $2 = 2\beta + kn$, n being the order of s_1 and $k \neq 0$. That is, if we assume $\alpha = 1$, $\beta \neq 1$, it results that the order of s_1 is a divisor of $2(\beta-1)$. These conditions do not fix an upper limit for the order of s_2, as may be seen from the following special case. The two operators s_1, s_2 may evidently be the generators of the dicyclic group of order 16, s_2 generating the cyclic group of order 8 and β being 3. If s_2 is multiplied by an operator of arbitrary order which is commutative with each of the operators s_1, s_2, the product thus obtained and s_1 will again satisfy the given conditions. This proves that the order of s_2 may be an arbitrary multiple of 8 when $\alpha = 1$ and $\beta = 3$, but the order of s_1 must be 4.

It was observed above that the orders of s_1, s_2 divide $2(\beta-1)^2$, $2(\alpha-1)^2$ respectively and that the $2(\beta-1)$th power of the former of these operators is equal to the $2(1-\alpha)$th power of the

* *Annals of Mathematics*, vol. 9 (1907), p. 51.

A negative circular form in the plane pencil transforms a sheaf into a congruent sheaf, the rays of the two sheaves being arranged in opposite senses; i. e., it is a reflection across one of the double rays of the projectivity. A section by a line not passing through the common center gives rise to a negative circular form upon that line.

A circular form is expressible exponentially in terms of a certain involution. For example, the positive circular form on a line is

$$C_{(a,\,b,\,w)} = e^{\mathfrak{e}\,w},$$

where a, b are the base points of the first point row, w is a real parameter, e the Naperian base, and $\mathfrak{e}$ is the elliptic involution

$$\mathfrak{e} = \frac{b,\,-a}{a,\,b}.$$

For a negative form, the involution in the exponent is hyperbolic

$$h = \frac{b,\,a}{a,\,b}.$$

The double points of these involutions coincide with the double points of the corresponding circular form.

All positive circular forms with the same point pair a, b form a continuous one-parameter group. Negative circular forms do not form a group. The totality of all positive and negative circular forms with the same point pair a, b forms a discontinuous group which contains the continuous group of positive circular forms as a subgroup.

The circular forms are not the only projectivities expressible exponentially in terms of an involution. The consideration of linear systems of projectivities, in particular a sheaf

$$lp + mq,$$

where l and m are numerical quantities, leads to the fact that, unless all the projectivities of the sheaf are involutions, there is but one involution contained in the sheaf. If the projectivity q is fixed, say identity, the double points of the involution contained in the sheaf coincide with the double points of p, and p is expressible in terms of the identity and this involution double points.

A direct (gleichläufig) projectivity with real and distinct double points is then

$$p = e^{hw},$$

The combination square or power of a projectivity

$$[p^2] = \frac{[e_1e_2 \cdot pp]}{[e_1e_2]} = \frac{[a_1a_2]}{[e_1e_2]}$$

is a numerical quantity, since it is the ratio of the two Stäbe connecting the base points of each point row.

A projectivity p is directly or oppositely projective; i. e., the points of the two point rows are arranged in the same or in opposite senses according as $[p^2]$ is positive or negative. If $[p^2] = 0$, then $[a_1a_2] = 0$ and the projectivity is degenerate.

A double point d of a projectivity p satisfies the equation

$$dp = rd,$$

where r is a numerical quantity, hence it is altered only as to mass by the projectivity. This equation easily transforms into a quadratic in r whose coefficients depend only upon the base points of the two point rows. The roots of this equation are termed the principal numbers of the projectivity and the equation itself its principal equation.

If a projectivity leaves every point of the line unaltered as to mass and position, it is identity. If it alters the mass only, it is a coincidence (Deckung). If the sum of the principal numbers is zero, the projectivity is an involution. The properties of the involution follow together with an application to the vector equations of the ellipse and the hyperbola. The equation of the ellipse, for example, comes out to be

$$x = a(\cos w + \mathsf{e} \sin w) = ae^{\mathsf{e}w},$$

where x is any point of the curve, a the semi-major axis, w the eccentric angle, e the Naperian base, and e the elliptic involution which changes one set of diameters into the conjugate set.

The consideration of projectivities with conjugate imaginary principal numbers is introduced by a study of what are called positive and negative circular forms (Abbildungen), after a designation due to the elder Grassmann. A positive circular form in the plane pencil is a projectivity transforming a sheaf of lines into a congruent sheaf, i. e., a rotation. A positive circular form upon a line is a section of the congruent sheaves by a line not passing through their common center. Every projectivity with conjugate imaginary principal numbers is identical geometrically to a positive circular form and can differ from it analytically only by a numerical factor.

to the product of a point by a numerical quantity. Similar considerations hold for a projectivity established between two sheaves of lines belonging to the same plane pencil of rays.

The commutative and associative laws hold for the sum of any number of symbols representing projectivities on the same line or in the same pencil. The product of two such symbols p, q is called the resultant product (Folgeprodukt) and is defined as a third projectivity on the same line obeying the associative law

$$x(pq) = (xp)q.$$

The resultant product is in general non-commutative. The method for calculating this resultant product is given, and we are led at once to the theorem that all projectivities on a line form a group.

A projectivity $p = a_1, \ a_2/e_1, \ e_2$ possesses an inverse $p^{-1} = e_1, e_2/a_1, a_2$, provided the exterior product $[a_1a_2] \neq 0$; *i. e.*, provided the base points of the second point row are not coincident. In the opposite case, p is degenerate and does not possess an inverse. If p possess an inverse, the equation

$$xp = x'$$

is solvable for x.

The combination product $[yz \cdot pq]$, where y and z are points on the line, is defined by the equation

$$[yz \cdot pq] = \tfrac{1}{2}\{[yp \cdot zq] - [zp \cdot yq]\}.$$

A number multiplying any one of the letters y, z, p, q may be written before the entire expression, and the symbol itself is distributive over a sum replacing any one of its constituents. The expression

$$\frac{[yz \cdot pq]}{[yz]}$$

is independent of the particular points y, z used in forming it. Hence

$$\frac{[yz \cdot pq]}{[yz]} = \frac{[e_1e_2 \cdot pq]}{[e_1e_2]} \equiv [pq].$$

This new symbol $[pq]$ is called the combination product of the projectivities p and q, and is always commutative in contradistinction to the resultant product.

If the point calculus were an end in itself, or if it found its chief application in developing this elementary geometry, it would be rather more curious than useful. But the advantage comes in studying linear transformations in the n-ary field and is perhaps increasingly manifest the greater the value of n. Hence the reader's interest will be aroused and will increase as he reads on in the third and last chapter. This chapter is devoted to projectivities on a line and in a plane pencil and contains rather more than two thirds the entire volume.

For the analytic representation of the points of a point row, two non-coincident base points e_1 and e_2 are chosen whose masses are determined so that a third point e, not coinciding with either of the others, but otherwise arbitrary as to mass and position, shall be the sum of the other two. This third point is the unit point. Any point x of the point row is then given by the formula

$$x = x_1e_1 + x_2e_2,$$

where x_1 and x_2 are numerical quantities. A second point row on the same line whose base points and unit point are respectively a_1, a_2, and a is projectively related to the first when the base points and unit point of the one are made to correspond to the base points and unit point of the other, each to each.

So far this is not unlike the usual introduction to the study of linear transformations in the binary field. The divergence comes in the next step. A factor p* (Abbildungsfaktor) is defined so that

$$e_1p = a_1,\ e_2p = a_2,\ ep = a\,;\ i.\ e.,\ (e_1 + e_2)\,p = e_1p + e_2p,$$

and also

$$xp = (x_1e_1 + x_2e_2)p = x_1a_1 + x_2a_2.$$

The operation p so defined may be represented formally by what the author calls an extensive fraction

$$p = \frac{a_1,\ a_2}{e_1,\ e_2}.$$

This brings into evidence the base points of the two point rows. The operation indicated by p is distributive over the sum of any number of points on the line and associative when applied

* Many of the symbols in the text are expressed by German letters.

The present volume is devoted to the binary field and is filled with detail much of which must be passed over without notice. It is furnished with a good table of contents and a register of material and names, which aid substantially in following the main lines of thought. What follows may serve to illustrate these. There are three Hauptteile or chapters. In the first of these the reader is introduced to the Punktrechnung by means of which points and lines are added, subtracted, and multiplied and the laws governing these operations formulated. Thus a point is represented by two factors, a scalar m called its mass and a vector f depending only upon its position. The sum of two points $m_1 f_1$ and $m_2 f_2$ is a point ms whose mass m is $m_1 + m_2$. and which has the position of the center of gravity of the two given points. Points of zero mass are ideal or at infinity. Points of unit mass are said to be simple points, others are called multiple points.

The process of multiplying one point a by another b is called exterior (äussere) multiplication and is indicated by the symbol $[ab]$. It is non-commutative, $[ab] = -[ba]$. The product is conceived of as a force acting along the segment from a to b and is called a Stab — a designation used by the author in 1894 (Punktrechnung und Projektive Geometrie, Halle) and since quite generally adopted by German writers. If a and b are finite points, the product $[ab]$ vanishes only when $a = b$.

The difference between two simple points is the segment (Strecke) connecting them. The point calculus will naturally be more familiar to the student of vector analysis than to one accustomed to think in terms of ordinary coordinate systems.

In the second chapter one finds an application of the point calculus to elementary projective geometry. A group of four points on a line in the order a, c, b, d, is called, after Von Staudt, a Punktwurf and its anharmonic ratio is defined as the double ratio

$$\frac{[ac]}{[cb]} \div \frac{[ad]}{[db]}.$$

This, being the double ratio of four Stäbe, on the same line, is a numeral quantity. The definition of projectively related ranges and pencils follows, together with the generation of curves of second class and of second order. The harmonic properties of the complete quadrilateral and the Pascal and Brianchon theorems are derived easily and thus we have the elementary part of projective geometry.

The auxiliary function θ is the temperature and the obvious physical mode of·solution is Liouville's method of successive substitutions.*

A case of special interest physically is that in which k is defined by

$$k\kappa = 1.$$

Is there any method of numerical computation better than approximate integration?

HAVERFORD COLLEGE,
March, 1910.

GRASSMANN'S PROJECTIVE GEOMETRY.

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung dargestellt. Von HERMANN GRASSMANN. Erster Band: *Binäres.* B. G. Teubner, 1909. 8vo. xii + 360 pp.

MODERN projective geometry is two-sided. Either use is made of algebraic analysis in its development or it is developed from the fundamental concepts of point, line, plane by means of certain axioms and postulates. In the one case it is analytic, in the other synthetic. Usually the two methods of presentation are more or less combined, with the emphasis laid upon the one or the other. If the analytic method is adopted, operations are usually carried out in cartesian space with the aid of a system of coordinates. The synthetic method makes no use o coordinate systems.

Professor Grassmann's work is analytic in character in that use is made of algebraic analysis. It is unique in discarding the usual coordinate systems and adopting ideas due to Möbius and to the elder Grassmann. These ideas found expression in the Baryzentrische Calcul and in the Ausdehnungslehre.

In the last quarter century a number of writers have made use of these ideas; notably, Stéphanos, H. Wiener, Segre, Peano, Aschieri, Study, Burali-Forti. It is the author's purpose to bring the results of these writers and of others together into a connected course covering the fields of binary and ternary linear transformations. This is certainly a most worthy purpose and mathematicians will be grateful to the author for the evident care and devotion with which he has set about the performance of his task.

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* Maxime Bôcher, An introduction to the study of integral equations. Cambridge, Eng., 1909.

parallel planes and the radiation is homogeneous and everywhere normal to the isothermal surfaces.

If either of these two last restrictions is discarded, an equation is obtained of the form

$$E - \frac{1}{[k(\kappa)]^2}\frac{\partial^2 E}{\partial m^2} = \int_0^1 E d\kappa.$$

By a series of substitutions, the solution of this equation can be led back to the solution of an ordinary integral equation of the second kind with symmetric kernel. Each step is directly suggested by physical conceptions. Write

$$A = E - \frac{1}{k}\frac{\partial E}{\partial m}, \quad B = E + \frac{1}{k}\frac{\partial E}{\partial m},$$

whence

$$2E = A + B.$$

Put further

$$\theta(m) = \int_0^1 E d\kappa,$$

so that

$$A + \frac{1}{k}\frac{\partial A}{\partial m} = B - \frac{1}{k}\frac{\partial B}{\partial m} = \theta(m).$$

By the ordinary device of an integrating factor $e^{\pm km}$ these equations lead to

$$A = A_0(\kappa)e^{-km} + k\int_0^m e^{-k(m-\xi)}\theta(\xi)d\xi,$$

$$B = B_0(\kappa)e^{-k(m'-m)} + k\int_m^{m'} e^{-k(\xi-m)}\theta(\xi)d\xi,$$

where $0 < m < m'$ and A_0, B_0 are arbitrary functions of κ.

By addition, putting

$$2E_0 = A_0(\kappa)e^{-km} + B_0(\kappa)e^{-k(m'-m)},$$

we have

$$2E = 2E_0 + k\int_0^{m'} e^{-k|\xi-m|}\theta(\xi)d\xi.$$

Finally, integrating from 0 to 1 with respect to κ and writing

$$2K(x) = \int_0^1 ke^{-k|x|}d\kappa, \quad \theta_0(m) = \int_0^1 E_0 d\kappa,$$

we have

$$\theta(m) = \theta_0(m) + \int_0^{m'} K(\xi - m)\theta(\xi)d\xi.$$

$$s_1 = abcdefgh \cdot ijlmnop, \quad s_2 = aiem \cdot cogk.$$

When s_1 is of order 8, s_2 must be of order 4 since s_2^2 is not commutative with s_1. When s_1 is of order 4, the order of s_2 is either 4, 2, or 1, as may be readily seen from the following substitutions:

$$s_1 = aecg, \quad s_2 = abcd \cdot efgh\,; \quad s_1 = abcd, \quad s_2 = ac.$$

Finally, when s_1 is of order 2, the order of s_2 is evidently 2 or 1. Hence the theorem: *If two non-commutative operators satisfy the relations* $s_1^{-2}s_2s_1^2 = s_2^3$, $s_2^{-2}s_1s_2^2 = s_1^5$, *their orders are one of the following pairs of numbers:* 8, 4; 4, 4; 4, 2; 2, 2.

When s_1 is of order 8, H is abelian and of order 8. From the following equations it results that s_1^2 is transformed into its inverse by $(s_1s_2)^2$:

$$(s_1s_2)^{-1}s_1^2s_1s_2 = s_2^{-1}s_1^2s_2 = s_1^2 \cdot s_1^{-2}s_2^{-1}s_1^2 \cdot s_2 = s_1^2s^2,$$

$$(s_1s_2)^{-2}s_1^2(s_1s_2)^2 = s_2^{-1}s_1s_2^2s_1s_2 = s_2 \cdot s_2^{-2}s_1s_2^2 \cdot s_1s_2 = s_2s_1^{-2}s_2 = s_1^{-2}.$$

Hence the order of G is a multiple of $8 \cdot 4 \cdot 2 = 64$ whenever s_1 is of order 8. That the order of G may be exactly 64 results directly from the given substitutions, as they generate an imprimitive group of degree 16 and order 64. From the properties of the dihedral group it results that s_1, s_2 may be so selected that the order of G is an arbitrary multiple of 64 and that the third derived of each one of these groups is identity. The categories of groups which result when the orders of s_1, s_2 have the other possible sets of values are still more elementary and their fundamental properties are easily derived from the general theorems of the preceding section.

THE SOLUTION OF AN INTEGRAL EQUATION OCCURRING IN THE THEORY OF RADIATION.

BY PROFESSOR W. H. JACKSON.

(Read before the American Mathematical Society, December 30, 1909.)

PROFESSOR Arthur Schuster * has discussed the propagation of heat by radiation when the isothermal surfaces are

* "The influence of radiation on the transmission of heat." *Phil. Magazine*, Feb., 1903.

As $(s_1s_2)^2$ is transformed into its inverse by s_1, it is also transformed into its inverse by s_2 and hence it is transformed into itself by half the operators of G. To prove that the order of H is exactly 32 when s_1 is of order 16 it is only necessary to observe that s_1^4 is not invariant under G, and this results from the continued equation

$$(s_1s\ )^{-1}s_1^4s_1s_2 = s_2^{-1}s_1^4s_2 = s_1^4 \cdot s_1^{-4}s_2^{-1}s_1^4 \cdot s_2 = s_1^4s_2^8 = s_1^{-4}.$$

These results give rise to the theorem : *If two operators of order* 16 *are such that each of them is transformed into its fifth power by the square of the other, these squares generate an invariant subgroup of order* 32 *involving a commutator subgroup of order* 2.

It has been proved that the group generated by $(s_1s_2)^2$ is invariant under G and that the order of the corresponding quotient group is a divisor of 128. To prove that this invariant subgroup has at most two operators in common with the invariant subgroup H it is only necessary to observe that a subgroup of order 4 in the former of these invariant subgroups can not be contained in the latter. This results directly from the facts that s_1s_2 is not commutative with s_1^4 or s_2^4 and that the operators of order 4 in H which are not generated by s_1^4 or s_2^4 are non-commutative with s_1^2, while $(s_1s_2)^2$ is commutative with this operator. Since the commutators of G are contained in the invariant subgroup generated by s_1^2, s_2^2, $(s_1s_2)^2$, and this invariant subgroup has a commutator subgroup of order 2, it results that the second derived of G is identity. It is also evident that the quotient group of G with respect to this invariant subgroup is the four-group. Hence the theorem : *If each of two operators is transformed into its fifth power by the square of the other, the orders of these operators divide* 16 *and the second derived of the group generated by them is identity.* Hence these operators may have a common order only when this order is 1, 2, 4, 8, or 16, and if they are non-commutative they can have different orders only when these orders are one of the two pairs 2, 4 ; 4, 8.

As a third and final special case we consider the relations

$$s_1^{-2}s_2s_1^2 = s_2^3, \quad s_2^{-2}s_1s_2^2 = s_1^5,$$

where α and β are unequal. According to the theorems of the preceding section, the orders of s_1, s_2 divide 8 and 4 respectively. It is easy to verify by means of substitutions that the orders of s_1, s_2 may actually be 8 and 4 respectively.

group, while they are transformed into their inverses by the remaining operators. The fourth derived of each of these groups is identity while the third derived is of order 2. The quotient group of G with respect to the subgroup generated by $(s_1s_2)^3$, s_1^2, s_2^2, is the symmetric group of order 6 and this subgroup has the group of order 2 for its commutator subgroup. The quotient group of G with respect to the invariant group generated by $(s_1s_2)^3$ is either the symmetric group of order 24 or the group of 48 which may be generated by the two substitutions given in the preceding paragraph. Hence the theorem : *If two non-commutative operators are such that each is transformed into its third power by the square of the other, they are of the same order and this common order is* 2, 4, *or* 8. *If this order is* 2 *they generate a dihedral group ; if it is* 4 *they generate a group whose quotient group with respect to a cyclic invariant subgroup is the symmetric group of order* 24 ; *and if it is* 8 *the corresponding quotient group is either this symmetric group or a group of order* 48 *which can be represented as a transitive substitution group of degree* 8.

If s_1, s_2 satisfy the two conditions

$$s_1^{2}s_2s_1^{-2} = s_2^5, \quad s_2^{-2}s_1s_2^2 = s_1^5,$$

their orders divide 16, since $5^4 - 1$ is not divisible by 32. It also follows from the theorems of the preceding section that $s_1^8s_2^8 = 1$, and hence $s_1^8 = s_2^8$. To verify that the common order of s_1, s_2 may actually be 16 we may employ the following substitutions :

$$s_1 = abcdefghijklmnop, \quad s_2 = ahcfmlojipknedgb.$$

When s_1, s_2 represent these substitutions, H is of order 32 and involves a commutator subgroup of order 2. Since

$$s_1s_2 = bf \cdot cg \cdot em \cdot hp \cdot jn \cdot ko$$

the order of G is 128. In the next paragraph we shall prove that the order of H is 32 and that the order of G is a multiple of 128 whenever s_1 is of order 16.

The operator $(s_1s_2)^2$ is invariant under half the operators of G while it is transformed into its inverse by the remainder of these operators since

$$(s_1s_2)^{-2} = s_2^{-1}s_1^{-1}s_2^{-1}s_1^{-1} = s_2 \cdot s_2^{-2}s_1^{-1}s_2^2 \cdot s_2^{-3}s_1^{-1}$$
$$= s_2s_1^{-5}s_2^{-3}s_1^{-1} = s_2s_1 \cdot s_1^{-6}s_2^{-3}s_1^6 \cdot s_1^{-7} = (s_2s_1)^2.$$

ever neither of these operators is identity. When s_1, s_2 are commutative they generate the four-group or a subgroup of this group. This trivial case will be excluded in what follows. That is, we shall assume that s_1, s_2 are non-commutative. When each of these operators is of order 2 they generate a dihedral group and every dihedral group may evidently be generated by two operators satisfying the given conditions. The only two possible cases which remain to be considered are when the common order of s_1, s_2 is either 4 or 8. In each of these cases the order of s_1s_2 is a multiple of 3 since

$$s_1^{-1}s_2^{-1}\cdot s_2^2\cdot s_2s_1 = s_1^{-1}s_2^2s_1 = s_2^2\cdot s_2^{-2}s_1^{-1}s_2^2\cdot s_1 = s_2^2s_1^{-2},$$

$$s_1^{-1}s_2^{-1}\cdot s_2^2s_1^{-2}\cdot s_2s_1 = s_1^{-1}s_2s_1^{-2}s_2s_1 = s_1^{-3}\cdot s_1^2s_2s_1^{-2}\cdot s_2s_1 = s_1^{-2}s_2^4 = s_1^2,$$

$$s_1^{-1}s_2^{-1}\cdot s_1^2\cdot s_2s_1 = s_1\cdot s_1^{-2}s_2^{-1}s_1^2\cdot s_2s_1 = s_1s_2^{-2}s_1 = s_2^{-2}\cdot s_2^2s_1s_2^{-2}s_1 = s_2^2.$$

When the common order of s_1, s_2 is 4 the invariant subgroup H generated by s_1^2, s_2^2 is the four-group. If the order of s_1s_2 is 3, G must be the symmetric group of order 24. In fact, in this case G is generated by two operators which are such that each is transformed into its inverse by the square of the other and hence its properties are known, as was observed in the introduction. The only case which remains to be considered is the one where the common order of s_1, s_2 is 8. In this case H is the quaternion group and the order of G is a multiple of 48. When the order of s_1, s_2 is 3, G is one of the four groups of order 48 which involve the non-twelve group of order 24. Since $(s_1s_2^2)^2 = s_1s_2^2s_1s_2^2 = s_1s_2^4\cdot s_2^{-2}s_1s_2^2 = 1$, this group can be represented as a transitive substitution group on 8 letters. It is easy to verify that the following generators of this transitive substitution group satisfy the conditions imposed upon s_1, s_2:

$$s_1 = acfhbdeg, \quad s_2 = agdfbhce.$$

These generators are directly obtained from those given by Cole.*

From what precedes it is not difficult to deduce some fundamental properties of the infinite system of groups generated by two operators of order 8, each of which is transformed into its third power by the square of the other. In each of these groups $(s_1s_2)^3$ generates an invariant subgroup whose operators are separately invariant under half the operators of the entire

*Bulletin, vol. 2 (1893), p. 188.

latter. Hence it results that the order of each of these operators is a divisor of $2(\alpha - 1)(\beta - 1)$. That is, *if two operators are such that each is transformed into a power of itself by the square of the other, the order of each of them is a divisor of twice the product of the indices of these powers diminished by unity.* When α and β are different this evidently furnishes a lower limit for the order of one of the operators than the one given before. Since $s_1^{2(\beta-1)}$ and $s_2^{2(\alpha-1)}$ are invariant under G, it results that

$$s_1^{\beta^{\alpha-1}} = s_1, \quad s_2^{\alpha^{\beta-1}} = s_2.$$

Hence the order of s_1 is a divisor of $\beta^{\alpha-1} - 1$, and that of s_2 is a divisor $\alpha^{\beta-1} - 1$. These conditions will sometimes give lower limits for these orders than those given above, as may be seen from the special cases in the following section.

The preceding considerations prove that two operators which satisfy the two conditions

$$s_1^{-2} s_2 s_1^2 = s_2^{\alpha}, \quad s_2^{-2} s_1 s_2^2 = s_1^{\beta}$$

have their orders limited by these conditions when, and only when, each of the numbers α, β is different from unity. When α, β satisfy this condition, s_1^2, s_2^2 generate an invariant subgroup whose order is a divisor of the product of their orders. For every pair of odd values for α, β there is an infinite system of groups whose fourth derived must be unity and each of these systems must include the dihedral groups. The properties of the invariant subgroup generated by s_1^2, s_2^2 are known, since these operators transform each other into powers, and every G may be constructed by adjoining to this invariant subgroup the operator $s_1 s_2$ and then adjoining to the invariant subgroup thus obtained the operator s_1. While these fundamental properties apply to every possible G, a number of interesting special properties apply to given values of α and β, as may be seen from the examples of the following section.

§ 3. *Special Cases.*

The following special cases may serve to illustrate some of the preceding theorems. Suppose that s_1, s_2 are any two operators which satisfy the two equations

$$s_1^{-2} s_2 s_1^2 = s_2^3, \quad s_2^{-2} s_1 s_2^2 = s_1^3.$$

From the general results of the preceding section it follows that $s_1^4 = s_2^4$, $s_1^8 = s_2^8 = 1$, and that s_1, s_2 have the same order when-

where h is the double point involution and w a real parameter. On the other hand an opposite (gegenläufig) projectivity expressed in terms of its double points involution is

$$q = he^{hw}.$$

Group properties follow as in the case of the circular forms.

Two projectivities for which the combination product $[pq]$ vanishes are called harmonic. The name is due to Segre as well as many of their properties (*Crelle,* volume 100).

The domain of all the projectivities on a line is defined by means of four unit fractions (Ausdehnungslehre)

$$\mathrm{e}_{11} = \frac{e_1,\, 0}{e_1,\, e_2}, \quad \mathrm{e}_{12} = \frac{e_2,\, 0}{e_1,\, e_2}, \quad \mathrm{e}_{21} = \frac{0,\, e_1}{e_1,\, e_2}, \quad \mathrm{e}_{22} = \frac{0,\, e_2}{e_1,\, e_2}.$$

These are all degenerate projectivities, the second and third being parabolic involutions, and hence the power of each vanishes. Any projectivity

$$p = \frac{a_1,\, a_2}{e_1,\, e_2}$$

can be expressed as a linear function of these four units. For if we put

$$a_1 = a_{11}e_1 + a_{12}\, e_2, \quad a_2 = a_{21}e_1 + a_{22}e_2,$$

then

$$p = a_{11}\mathrm{e}_{11} + a_{12}\mathrm{e}_{12} + a_{21}\mathrm{e}_{21} + a_{22}\mathrm{e}_{22}.$$

The proof consists in applying p, thus expressed, to each of the base points e_1, e_2 and showing that they transform into a_1, a_2 respectively.

The four units are linearly independent. For the assumption of an equation of the form

$$c_{11}\mathrm{e}_{11} + c_{12}\mathrm{e}_{12} + c_{21}\mathrm{e}_{21} + c_{22}\mathrm{e}_{22} = 0$$

leads to the vanishing of all the c's since e_1 and e_2 are distinct points.

The above representation of p is unique. For if p could be expressed by a set of a's and also by a set of b's, the difference between the two expressions would vanish and thus the a's equal the b's, each to each.

Of course any four linearly independent projectivities may be taken for a base. It is convenient to choose four which are mutually harmonic; these are, when expressed in terms of the unit fractions,

$$1 = \mathsf{e}_{11} + \mathsf{e}_{22}, \quad h_1 = \mathsf{e}_{11} - \mathsf{e}_{22}, \quad h_2 = \mathsf{e}_{12} + \mathsf{e}_{21}, \quad \mathsf{e} = \mathsf{e}_{12} - \mathsf{e}_{21},$$

where 1 is identity, h_1 and h_2 are hyberbolic involutions, and e is an elliptic involution. It is easily shown that they are mutually harmonic by means of the laws governing the multiplication of unit fractions and the condition which two projectivities must satisfy in order to be harmonic. Any projectivity on the line is then

$$p = a + a_1 h_1 + a_2 h_2 + a_3 \mathsf{e}.$$

This expression of p leads at once to the Stéphanos (*Mathematische Annalen*, volume 22) representation of the projectivities on a line by the points of ordinary space; viz., the images of the four fundamental projectivities 1, h_1, h_2, e are taken to be the vertices of a fundamental tetrahedron whose unit point is arbitrary. The image of any projectivity p is then the point whose homogeneous coordinates referred to this tetrahedron are a, a_1, a_2, a_3.

The images of all degenerate projectivities fill the quadric

$$a^2 - a_1^2 - a_2^2 + a_3^2 = 0$$

with respect to which the fundamental tetrahedron is self-conjugate. Two harmonic projectivities are imaged upon the points which are conjugate with respect to this quadric. The involutions on the line are imaged upon the points of the plane determined by the images of the three fundamental involutions h_1, h_2, and e.

A number of details of this representation are given together with a discussion of the representation of the involutions on the line by the points of a plane.

The projectivities upon a line form a system of higher complex quantities since they satisfy the four conditions laid down by Study (*Göttinger Nachrichten*, 1889). On comparing the multiplication table of the four fundamental projectivities with the multiplication table of the four units 1, i, j, k of the Hamilton quaternion system, it follows that any projectivity may be expressed as a complex quaternion, viz.,

$$p = a + a_1\sqrt{-1}i + a_2\sqrt{-1}j - a_3 k.$$

A theorem called Study's theorem (Cf. Encyclopädie, I. 1) brings these results together:

The group of all the projectivities on a line forms a system of

higher complex quantities of quaternion type. It possesses another real form which is like the system of ordinary quaternions and these two real forms are the only ones in which systems of higher complex quantities of quaternion type can appear.

The work is manifestly a labor of love. An interesting circumstance in connection is the fact that this first volume was brought out on the hundredth anniversary of the birthday of the author's father.

L. WAYLAND DOWLING.

LINEAR DIFFERENTIAL EQUATIONS.

Vorlesungen über lineare Differentialgleichungen. Von LUDWIG SCHLESINGER. Leipzig and Berlin, Teubner, 1908.

IT is over ten years ago that the author of the present "Vorlesungen" completed the publication of his well-known Handbuch der Theorie der linearen Differentialgleichungen. As every one familiar with the older book well knows, it was intended to be, as its name implied, a handbook containing a complete treatment of all that was at that time known about the subject. It seemed natural therefore, to expect under the title of "Vorlesungen" a briefer version of the same subject, adapted to the needs of the younger student and rendered more palatable for him by a proper selection of topics and by a more elementary treatment. And in a certain sense the "Vorlesungen" may indeed be considered as an introduction into the theory of linear differential equations, in so far at least as all of the most important results of the theory built up by Fuchs and his successors are discussed. But the method of treatment is so novel and the artistic unity of the book is preserved to such an extraordinary extent that we must look upon it as an important addition to analysis rather than as a treatise of more or less pedagogical merit.

It is well known that Riemann's discussion of the hypergeometric function furnished Fuchs with the fundamental ideas which led to the modern theory of linear differential equations, which theory may be said to date from Fuchs's paper of 1865. But we now know that Riemann himself had intended to construct a general theory of linear differential equations upon the same general principles which had led to such brilliant results in

the theory of abelian functions. In the two fragments, published in 1876, after his death, he formulates a general problem which may briefly be stated as follows: Given m points $a_1, \cdots, a_m$ in the plane of the complex variable with each of which is associated one of the linear n-ary substitutions with constant coefficients $A_1, \cdots, A_m$; to determine a system of n functions of the complex variable x which shall have the given points $a_1, \cdots, a_m$, and no others, as branch points in such a way that when x describes a closed path in the positive direction around a_i, the functions y_κ shall undergo the linear substitution A_i. Riemann speaks of two different systems of functions which are nowhere "infinite of infinite order" which have the same branch points, the same fundamental substitutions and the same poles, as belonging to the same *class*.

It is easy enough to see that the solutions of a linear differential equation of the nth order are functions of this general character. But if the points a_i and the substitutions A_i and the poles are chosen arbitrarily, it is a problem of great difficulty to demonstrate the existence of a system of functions of the class defined by them. This is what is known as Riemann's problem and in its solution Schlesinger's Vorlesungen culminate. The method adopted consists essentially in studying the relations between the parameters which occur in the coefficients of the differential equation, the quantities which determine the linear substitutions A_i and the branch points a_i. These (transcendental) relations are shown to be of such a character that, the latter quantities being arbitrarily assigned, the former may be chosen in such a way that the corresponding differential equations have as their solutions a system of functions with the required properties.

As we have already indicated, the solution of Riemann's problem is the culminating feature of the book and appears only at the end of a long series of investigations in which all of the most essential properties of linear differential equations are studied. But all of these are presented in a novel way. Instead of considering the problems of integration in connection with a single linear differential equation of the nth order, Schlesinger follows Koenigsberger's example by considering a system of n differential equations of the first order. Such a system is characterized by its n^2 coefficients as well as by a system of n^2 solutions. It is owing to this fact that the entire theory appears as an application of the calculus of matrices of

n^2 functions which was originally developed by Volterra, and which certainly manifests itself here as a most important analytical instrument.

Let us consider the system of differential equations

$$(B) \qquad \frac{dy_\kappa}{dx} = \sum_{\lambda=1}^{n} y_\lambda a_{\lambda\kappa}(x) \qquad (\kappa = 1, 2, \cdots, n),$$

where the coefficients $a_{\lambda\kappa}$ are supposed to be real, finite and continuous functions of x in the interval $(p \cdots q)$. Let the values

$$(I) \qquad y_1 = y_1^{(0)}, \quad \cdots, \quad y_n = y_n^{(0)}$$

be arbitrarily prescribed for $x = x_0 = p$. In order to demonstrate the existence of a system of real continuous solutions of (B) which satisfy these initial conditions, the author makes use of the method of interpolation, which proceeds as follows, in close analogy with Riemann's definition of a definite integral: Let r be any value of x for which

$$p < r \leqq q.$$

Divide the interval $(p \cdots r)$ into m parts by interpolating $m - 1$ points $x_1, \cdots, x_{m-1}$, and let us put $x_0 = p$, $x_m = r$. In each of the m subintervals $(x_{\nu-1} \cdots x_\nu)$ obtained in this way choose an arbitrary value $\xi_{\nu-1}$,

$$x_{\nu-1} \leqq \xi_{\nu-1} < x_\nu,$$

and consider in place of the differential equations (B), the difference equations

$$y_\kappa^{(1)} - y_\kappa^{(0)} = (x_1 - x_0) \sum_{\lambda=1}^{n} y_\lambda^{(0)} a_{\lambda\kappa}(\xi_0),$$

$$(C) \qquad y_\kappa^{(2)} - y_\kappa^{(1)} = (x_2 - x_1) \sum_{\lambda=1}^{n} y_\lambda^{(1)} a_{\lambda\kappa}(\xi_1), \quad (\kappa = 1, 2, \cdots, n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_\kappa^{(m)} - y_\kappa^{(m-1)} = (x_m - x_{m-1}) \sum_{\lambda=1}^{n} y_\lambda^{(m-1)} a_{\lambda\kappa}(\xi_{m-1}).$$

The author shows in the first lecture that, under the assumptions made in regard to the continuity of the functions $a_{\lambda\kappa}$, the functions $y_\kappa^{(m)}$ defined by these difference equations approach definite limits y_κ as m becomes infinite, while each of the subintervals approaches the limit zero, these limits being indepen-

dent of the particular method of subdivision employed, as well as of the choice made of the points $\xi_{\nu-1}$ in the various subintervals. Moreover the n functions obtained in this way are shown to be continuous functions of x which satisfy the system of differential equations (B) and the initial conditions (I).

In the second lecture we are told to consider not a single system of solutions (B), but n such systems corresponding to the n systems of initial conditions

$$\begin{array}{l} y_1 = y_{11}^{(0)},\ y_{21}^{(0)},\ \cdots,\ y_{n1}^{(0)}, \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ y_n = y_{1n}^{(0)},\ y_{2n}^{(0)},\ \cdots,\ y_{nn}^{(0)}, \end{array} \qquad \text{for } x = x_0 = p,$$

where the determinant

$$|y_{i\kappa}^{(0)}|$$

is supposed to be different from zero. The n systems of solutions $(y_{11}, \cdots, y_{1n}), \cdots, (y_{n1}, \cdots, y_{nn})$ will be obtained by the limit process just indicated from the difference equations

$$(C') \qquad y_{i\kappa}^{(\nu)} - y_{i\kappa}^{(\nu-1)} = (x_\nu - x_{\nu-1}) \sum_{\lambda=1}^{n} y_{i\lambda}^{(\nu-1)} a_{\lambda\kappa}(\xi_{\nu-1}),$$
$$(i, \kappa = 1, 2, \cdots, n;\ \nu = 1, 2, \cdots, m)$$

which are merely the equations (C) written down for each separate set of initial conditions.

The equations (C') may be written in the form

$$(1) \qquad y_{i\kappa}^{(\nu)} = \sum_{\lambda=1}^{n} y_{i\kappa}^{(\nu-1)} \left[a_{\lambda\kappa}(\xi_{\nu-1})(x_\nu - x_{\nu-1}) + \delta_{\lambda\kappa}\right],$$

where $\delta_{ii} = 1, (i = 1, 2, \ldots n)$, and $\delta_{i\kappa} = 0$ for $i \neq \kappa$. But these may be looked upon as the equations for the multiplication of two square matrices, since the single relation between matrices

$$(a_{i\kappa})(b_{i\kappa}) = (c_{i\kappa})$$

is equivalent to the n^2 relations

$$c_{i\kappa} = \sum_{\lambda=1}^{n} a_{i\lambda}\, b_{\lambda\kappa}, \qquad (i, \kappa = 1, 2, \ldots n)$$

and since equations (1) are obviously of this form.

Let us follow the author in using the simple parenthesis as a symbol for a matrix as well as in its elementary significance, although that practice occasionally gives rise to a formidable

collection of parentheses in a single equation. We suggest incidentally that this might easily be avoided, in the interests of clearness, by using some other symbol for a matrix, say a square bracket. Equations (1) may then be written as follows:

(2) $(y^{(\nu)}_{i\kappa}) = (y^{(\nu-1)}_{i\kappa})(a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa})$ $(\nu = 1, \ldots, m)$,

which gives rise to the symbolic formula

$$(3) \qquad (y^{(m)}_{i\kappa}) = (y^{(0)}_{i\kappa})\prod_{\nu=1}^{m}(a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}),$$

where the product upon the right member is, of course, a product of matrices taken in the proper order.

Let the matrix $(y^{(0)}_{i\kappa})$ of initial values be the unit matrix $(\delta_{i\kappa})$. The author now introduces the following symbol:

$$(\alpha) \qquad \overset{\frown}{\int_p^r} [a_{i\kappa}(x)dx + \delta_{i\kappa}]$$

to denote the matrix of the limits

$$\lim_{m=\infty}\prod_{\nu=1}^{m}[a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}] \quad (i, \kappa = 1, 2, \cdots, n).$$

The symbol (α) is to be read "integral matrix from p to r," and is used with great success throughout the book. The author tells us that he has chosen this symbol so as to emphasize its analogy with the ordinary integral, and at the same time to remind us of the initial letter of the word product, just as the integral sign reminds us of the first letter of the word sum.

Let us put $r = x$ and think of x as variable in the interval from p to q. The integral matrix

$$[\eta_{i\kappa}(x)] = \overset{\frown}{\int_p^r} [(a_{i\kappa}(x)dx + \delta_{i\kappa}]$$

will represent a matrix of n^2 functions such that

$$\eta_{i1}(x), \cdots, \eta_{in}(x) \qquad (i = 1, 2, \cdots, n)$$

constitute n systems of simultaneous solutions of system (B) which satisfy the initial conditions

$$\eta_{i\kappa}(p) = \delta_{i\kappa},$$

and the most general matrix of simultaneous solutions is obtained from $(\eta_{i\kappa})$ in the form

$$(y_{i\kappa}) = (c_{i\kappa})(\eta_{i\kappa}),$$

where the elements of the matrix $(c_{i\kappa})$ are arbitrary constants.

The equations (B) may be written

$$\left(\frac{dy_{i\kappa}}{dx}\right) = (y_{i\kappa})(a_{i\kappa}),$$

whence

$$(a_{i\kappa}) = (y_{i\kappa})^{-1}\left(\frac{dy_{i\kappa}}{dx}\right),$$

the determinant of the matrix $(y_{i\kappa})$ being different from zero in the whole interval $(p \cdots q)$. Thus the coefficients of (B) may be expressed in terms of n systems of simultaneous solutions. Schlesinger denotes this operation by the symbol $D_x(y_{i\kappa})$, so that

$$D_x(y_{i\kappa}) = (y_{i\kappa})^{-1}\left(\frac{dy_{i\kappa}}{dy}\right),$$

and speaks of this as the derivative matrix of $(y_{i\kappa})$ with respect to x. If the solutions of (B) are not subjected to the initial conditions

$$\eta_{i\kappa}(p) = \delta_{i\kappa}$$

nor any other specific conditions, he writes

$$(y_{i\kappa}) = \overset{\frown}{\int}(a_{i\kappa}dx + \delta_{i\kappa}),$$

a notation which corresponds to the indefinite integral of ordinary analysis.

At the end of the second lecture we find the laws of combination of these new symbols of derivation and integration, which correspond closely to the familiar ones of the infinitesimal calculus, the principal difference being that symbolic multiplication of matrices takes the place of addition.

The third lecture introduces the integrating factors of Lagrange and Jacobi, and the idea of adjoint system, which is used in the familiar way for the purpose of showing how to integrate non-homogeneous systems by quadratures if the corresponding homogeneous systems have already been solved. This theory again finds an important application in enabling us to deduce, by successive approximation, a series for the solu-

tions of the system (B), which is convergent in the whole domain of continuity of the coefficients.

The transition from functions of a real to functions of a complex variable is made by following the same general ideas that dominate the corresponding step in the theory of ordinary integrals. By separating all of the variables involved into their real and imaginary parts, i. e., by putting

$$y_\kappa = u_\kappa + \sqrt{-1}v_\kappa, \quad a_{i\kappa} = \alpha_{i\kappa} + \sqrt{-1}\beta_{i\kappa}, \quad x = \xi + \sqrt{-1}\eta,$$

a system of total linear differential equations is obtained from (B). Its integral matrix is shown to be independent of the path, if this path be restricted to a simply connected portion of the plane at every point of which the coefficients $a_{i\kappa}$ are analytic. It then follows easily that all of the values of an integral matrix for a given value of x, taken over a path not so restricted, may be obtained from one of them by multiplication with constant matrices. This leads to the notion of fundamental equation, the representation of the elements of a canonical fundamental system in the well-known form and all of the rest of the classical theory.

Schlesinger's solution of Riemann's problem has had to undergo some severe criticisms by Plemelj.* The discussion initiated by this attack is certainly worthy of careful attention on the part of all who might be tempted to apply these methods to similar problems. The reviewer has not been able to find the time to form a final opinion upon the main points involved in this controversy. It certainly does not invalidate the value of the work as a whole.

The following unimportant misprints may be noted: page 10, line 23, read $\mathfrak{x}_{\lambda-2}$ in the exponent in place of $\mathfrak{x}_{\lambda-1}$; page 40, line 13, read $u_{i\kappa}^{(\nu)}$ instead of $(u_{i\kappa}^{(\nu)})$; page 76, equation (12), read $a_{i\kappa}$ instead of $\alpha_{i\kappa}$.

E. J. WILCZYNSKI.

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*See *Jahresbericht der Deutschen Mathematiker-Vereinigung*, January 31, 1909, and June 11, 1909.

SHORTER NOTICES.

La Geometria non-euclidea, esposizione storico-critica del suo Sviluppo. Da ROBERTO BONOLA. Bologna, Zanichelli, 1906. vi + 213 pp.

Wissenschaft und Hypothese IV: *Die nichteuklidische Geometrie, historisch-kritische Darstellung ihrer Entwicklung.* Von ROBERTO BONOLA, Professor an der Scuola Normale zu Pavia. Autorisierte deutsche Ausgabe besorgt von Prof. Dr. HEINRICH LIEBMANN. Leipzig, Teubner, 1908. viii +244 pp.

IN the development of the sober science of mathematics a certain dramatic and even sensational element has been furnished by non-euclidean geometry, the history of which is therefore unusually interesting. By its very nature this subject lends itself easily to a historical and critical treatment, like that of the admirable book under review. Unfortunately, the very fascination of the subject has apparently retarded its growth along the substantial lines of actual detailed knowledge: the tendency has been to regard it as a curious and elegant plaything, rather than as the valuable adjunct to euclidean geometry, which it undoubtedly is. The slowness of its growth is illustrated by the fact that although more than eighty years have elapsed since Lobachevsky published his first epoch-making researches, it is only recently that quadric surfaces in non-euclidean space have been carefully studied and classified.

This and most other recent investigations are not mentioned by Bonola. Indeed, as he himself states, the character of his book is distinctly elementary. He begins by taking the reader back to the early period of questioning and doubt as to Euclid's fifth postulate, then carries him through the storm and stress period of creation by Gauss, Lobachevsky, and Bolyai, and is finally content to land him safely in the harbor of modern thought, where projective geometry, differential geometry, and the theory of continuous groups all afford cumulative evidence of the validity of the new doctrine.

In Chapter 5 the author gives several well-known methods of representing or imaging a non-euclidean space in a euclidean space, but omits to mention one introduced by Klein and oincaré and used successfully by Weber and Wellstein in their Encyklopädie der Elementar-Geometrie and by Liebmann (the

translator of Bonola's work into German) in his own Nichteuklidische Geometrie, namely, one in which non-euclidean lines and planes are represented by euclidean circles and spheres, respectively.

Two supplementary chapters, one on non-euclidean statics and one on Clifford parallels, Clifford surfaces, and the Clifford-Klein problem, and in the German edition another supplementary chapter on the construction of Lobachevsky parallels, add considerably to the value of the book. There is an index of authors cited, but no general index.

It would be fortunate if we could have an English translation of so valuable and interesting a work; for in English there is nothing covering even approximately the same ground except possibly the scattered papers of G. B. Halsted.

ARTHUR RANUM.

Analytic Geometry. Revised Edition. By E. W. NICHOLS. D. C. Heath and Company, 1908. xi + 282 pp.

THE general scope of this book is the same as that of the first edition which appeared fifteen years earlier. In the first edition the last chapter — a discussion of surfaces — was written by Professor A. L. Nelson and in the new edition this chapter has been entirely rewritten. Otherwise comparatively few changes in the subject matter have been made. The revised edition is very neatly bound in flexible covers — the style so largely used by D. C. Heath and Company lately. The printing, too, is distinctly better than in the former edition.

"The aim of the author has been to prepare a work for beginners, and at the same time to make it sufficiently comprehensive for the requirements of the usual undergraduate course." The first part of this aim has been more successfully carried out than the second. The book is written clearly and contains numerous, well-chosen problems. The conventional order of topics is followed — the conic sections being discussed separately with little emphasis upon their relation to each other. Probably the book is more elementary than would be acceptable in the best engineering schools.

G. H. SCOTT.

Complete Arithmetic. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Boston, Ginn and Company, 1909. v + 474 pp.

THIS book preserves and combines most of the strong features of two well-known series of arithmetics — the Wentworth and

the more modern Smith texts. Briefly, it embodies the spirit of the newer series in the forms of the older. It is "thoroughly modern in spirit and in material" but is free from all traces of "fad-ism" found in so many texts of recent years. The book is strictly topical, although the authors frankly admit, in the preface, that under certain conditions the recurrent treatment of topics may be preferable. The problem material is carefully chosen and is given in great abundance. And, at frequent intervals, under the heading "Problems without Numbers" are given sets of questions that combine review and generalization very effectively. It might have been well to present the metric system earlier and then give practical problems in it through a longer interval.

G. H. SCOTT.

Theorie des Potentials und der Kugelfunktionen. Von DR. A. WANGERIN. I. Band., Leipzig, Göschen (Sammlung Schubert. Band LVIII.), 1909. 8 + 255 pp. M. 6.60.

THIS is the first of two volumes dealing in an elementary way with the subject indicated by the title. The second volume will treat of spherical harmonics and their applications to the potential of the sphere. The present volume is confined to the derivation of the characteristic properties of the potential. The treatment follows Gauss for the potential due to solids, Weingarten for that due to surfaces. The potential function for other laws than the Newtonian is briefly considered. The last section gives in some detail the problems of potential and attraction of a homogeneous ellipsoid.

The development is very skilfully handled. The text begins with very elementary data, and builds up the integrals for the attractions of solids and surfaces, with applications to circular arcs, straight segments, and surface of circle and sphere. It is thus made to connect easily with an ordinary course in integral calculus. The potential function noticed by Lagrange is then introduced as a point function whose three partial derivatives are the three components of the attraction. The conceptions of equipotential surface and lines of force follow. The next chapter derives the usual characteristic properties of the potential function, as a function of a position in space, for points outside the attracting mass. The holomorphism of the function and its derivative as to x, y, or z, its order at infinity, and the vanishing of its concentration are shown. Next the characteristics for

points belonging to the attracting mass are developed, and the discontinuity of the second derivatives as the point goes through a bounding surface is shown and the values of the saltus determined. Following this, like problems are taken up for surface distributions. The closing chapter of the section is devoted to proving that the properties enumerated for the potential as necessary are also sufficient, hence characteristic.

The second section considers the function for other laws than that of the inverse square of the distance. It is shown in particular that the Newtonian is the only law which gives a constant potential inside a spherical shell whose density is a function of the distance from the center. It is not however the only law for which the attraction on an external point due to the shell is equal to that of an equal mass concentrated at the center of the shell. The shape of the "solid of greatest attraction" is considered. The logarithmic potential and the potential due to a double distribution, as a Leyden jar, are each given a chapter.

The book would seem to be quite teachable. Gauss's, Green's, and Stokes's theorems are not dragged in, but show up naturally when needed to further the development. The student sees clearly all the time the drift of the development and why it proceeds as it is does. He learns how to attack such problems, but he also becomes acquainted with a class of point functions particularly useful in mathematical physics. Difficult questions of higher analysis are passed over, yet the treatment is careful and tends to inspire to further research.

JAMES BYRNIE SHAW.

Geometrie der Kräfte. By H. E. TIMERDING. Leipzig and Berlin, Teubner, 1908. 8vo. xi + 381 pages.

In this book the author has developed the geometry of forces as an independent discipline, a branch of pure mathematics. While the word force (Kraft) has been retained in preference to stroke or vector, great pains have been taken to free it from the "physiological, physical, and metaphysical" attributes which belong to it originally. A force is a matter of definition, being defined as a vector with which is associated a numerical factor. The resulting theory is then applicable to any quantity which satisfies the definition, for example to momentum quite as well as to force in the ordinary physical sense. The subject matter is not new. In different forms and

from various points of view it has appeared in several places. For example the part which is applicable to the kinetics of rigid bodies has been worked out in Ball's Theory of Screws, and a parallel development for the statics of deformable bodies appears in the writings of Sir William Thomson. The authors' purpose in rewriting to a large extent the Theory of Screws is explained by the following quotation from the preface. "As a consequence of the English genius it (Ball's work) shows, even in those parts which do not have immediate practical applications, a fine appreciation for the essence and claims of science. On the other hand it seeks in no way to attain to that unity and purity which we Germans have been accustomed to regard as the goal of all scientific work. It appears more as a geometric illustration of mechanics than as a special subject independent of mechanics. The latter appears to us as the only purpose which a geometry of forces has to fulfill." It is intended that this development shall provide a bridge from geometry to mechanics without assuming a hybrid character; that it shall be a development such as the geometry of motion has had under the name of kinematics.

The first five chapters stand apart from the rest of the book and are devoted to an exposition of Grassmann's Ausdehnungslehre and Hamilton's Quaternions. Any mention of vector products carries with it the question as to whose notation is employed. After giving in a footnote the notations of Grassmann, Hamilton, Heaviside, Gibbs, and Lorentz, Professor Timerding adopts one which is not in agreement with any of the others. He denotes the inner or scalar product by $\mathbf{a} \times \mathbf{b}$ and the outer or vector product by $[\mathbf{ab}]$. In the following chapters almost no use is made of the terminology and results of vector analysis, and much of the first five chapters could have been omitted. It is remarked in the preface that the introduction of the methods of Grassmann's analysis into the first part of the book without making use of them in the sequel may be open to criticism. The justification is to be found in the purpose of a strictly systematic development of the subject without regard to subsequent application of the early steps.

For the principal part of the book the choice of material has been determined by two main considerations. On the one hand a development of the geometry of forces must be broad enough to include the two manifestations of mechanical energy, the kinetic energy of motion and the potential energy of deforma-

tion. On the other hand the author has demanded a minimum prerequisite knowledge of analysis. This requirement is limited to elementary analytic geometry and calculus, to which may be added a few of the formulas for vector products given in the introductory chapters.

The exposition of the geometry of forces begins in Chapter 6 with a consideration of instantaneous rotations. This is followed by chapters on forces and force systems, foundations of line geometry, and equilibrium. The next six chapters are devoted to the theory of screws and are followed by two chapters on deformations, the point of view throughout this portion being purely geometric. The concepts of mechanics are taken up in the last six chapters, of which two are given to deformable bodies. The four chapters on kinetics of rigid bodies deal with the equations of motion in general, free motion under no applied forces, motion with two degrees of freedom, and with three degrees of freedom. The special case of a system of forces in a plane is excluded throughout, and in the free motion of a rigid body it is assumed that the axis of rotation does not have a fixed direction in space.

The necessary complications of notation in this subject have been reduced by the systematic use of different styles of type, thus avoiding an excessive number of accents and subscripts. For the convenience of the reader the scheme of notation is exhibited in a table at the end of the book.

There are many misprints, but fortunately most of them are self-evident and will not cause confusion.

W. R. Longley.

NOTES.

The seventeenth summer meeting of the American Mathematical Society will be held at Columbia University on Tuesday and Wednesday, September 6–7. Abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary not later than August 20.

The April number (volume 32, number 2) of the *American Journal of Mathematics* contains the following papers: "The reduction of families of bilinear forms," by H. E. Hawkes; "Basic systems of rational norm-curves," by J. R. Conner; "Surfaces invariant under infinite discontinuous birational

groups defined by line congruences," by V. SNYDER; "The apparent size of a closed curve," by C. A. LUNN; "On linear transformations which leave an Hermitian form invariant," by J. I. HUTCHINSON.

THE April number (volume 11, number 3) of the *Annals of Mathematics* contains: "A generalization of the game called nim," by E. H. MOORE; "A simple method for graphically obtaining the complex roots of a cubic equation," by R. E. GLEASON; "The topography of certain curves defined by a differential equation," by F. R. SHARPE; "Abel's theorem and some addition formulæ for elliptic integrals," by H. H. BARNUM; "On the determination of the asymptotic developments of a given function," by W. B. FORD; "The integral roots of certain inequalities," by W. H. JACKSON.

AT the meeting of the London mathematical society held on April 28 the following papers were read: By W. F. SHEPPARD, "On the accuracy of interpolation by finite differences;" by G. H. HARDY, "Note on Maclaurin's test for the convergence of series;" by A. J. C. CUNNINGHAM, "The factorization of $2^{77} + 1$ and the divisibility of $2^p - 2$ by p^2, p being prime."

THE proceedings of the Fourth international congress of mathematicians, which was held at Rome in April, 1908, have been distributed to the members, and can now be obtained from the publisher. Especially should public and university libraries avail themselves of the opportunity to procure this carefully prepared digest of the progress of pure and applied mathematics. The complete report appears in three volumes, and can be obtained from the firm of E. Loescher, in Rome, for 35 francs. The volumes are not sold separately. The titles are as follows: Atti del IV Congresso internazionale dei Matematici, publicati per cura del Segretario Generale G. CASTELNUOVO, Roma; Tipografia della R. Accademia dei Lincei. Volume I (Relazione sul congresso, Discorsi e conferenze), 8vo, iv+216 pp., 1909; Volume II (Comunicazioni delle Sezioni I e II, matematica pura), 8vo, 315 pp., 1909; Volume III (Comunicazioni delle Sezioni III A, III B e IV, matematica applicata), 8vo, 583 pp., 1909.

AT a meeting of the Cambridge mathematical club on March 8 Sir ROBERT BALL delivered a lecture on Halley's comet.

The Club was founded four or five years ago with the hope of promoting the exchange and discussion of mathematical ideas among teachers of mathematics resident in Cambridge and the more proficient of their pupils. The meetings are held two or three times a term, and are usually comparatively small and quite informal, though occasionally a lecture or paper of a more formal character is presented. The subjects for discussion are sometimes branches of mathematical or physical theory, sometimes questions regarding mathematical teaching or examining. There is no publication in any way connected with the Club. The present President is Sir GEORGE DARWIN, and the Secretaries are Dr. T. J. I'A. BROMWICH and Mr. G. H. HARDY.

THE committee on the teaching of mathematics to engineering students, appointed at the Chicago joint meeting of 1907, will present its report at the meeting of the Society for the promotion of engineering education at Madison, Wis., June 23–25. Mathematicians who may be interested are invited to attend this important meeting.

THE mathematics section of the central association of science and mathematics teachers has issued a pamphlet entitled "Applied problems in algebra and geometry," compiled by a committee appointed for the purpose, containing 103 exercises taken from practical applications of these sciences. Copies may be secured, at 5 cents each, from the secretary of the section, Miss Mabel Sykes, Bowen High School, Chicago, Illinois.

TWO new chairs in mathematics have been established at the University of Paris, and one at Lyons and one at Toulouse; the occupants have not yet been announced.

THE following courses in mathematics are announced for the summer semester:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Surfaces and space curves, four hours; Elliptic functions, four hours; Selected chapters in the theory of analytic functions, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Elementary theory of functions, four hours; Automorphic functions, four hours; Seminar, two hours. — By Professor R. LEHMANN-FILHÉS: Absolute perturbations according to Hansen, four hours; Analytic mechanics, four hours. — By Professor G.

Hettner: Introduction to the theory of ordinary differential equations, two hours. — By Professor J. Knoblauch: Applications of elliptic functions, four hours; Integral calculus, four hours; Selected chapters in differential calculus, four hours. — By Professor I. Schur: Differential calculus, with exercises, five hours; Theory of algebraic equations, two hours.

The following courses in mathematics are announced for the academic year 1910–1911.

University of Chicago. — By Professor E. H. Moore: Introduction to general analysis: Theory of functions of infinitely many variables; Integral equations in general analysis; Seminar on the foundations of pure mathematics; each two hours throughout the year. — By Professor L. E. Dickson: Finite groups, four hours, first term; General algebra, four hours, second term; Quadratic forms, four hours, third term. — By Professor F. R. Moulton: Modern theories of analytic differential equations with applications to celestial mechanics, four hours, all three terms. — By Professor E. J. Wilczynski: Theory of plane curves, four hours, first term; Projective differential geometry of ruled surfaces and space curves, four hours, second term; Projective differential geometry of non-ruled surfaces and congruences, four hours, third term. — By Professor K. Laves: Analytic mechanics, four hours, first and second terms. — By Professor H. E. Slaught: Differential equations, four hours, first term. — By Professor G. A. Bliss: Elliptic integrals, four hours, second term; Theory of definite integrals, four hours, third term; Fundamental existence theorems, two hours, second and third terms. — By Dr. A. C. Lunn: Hydrodynamics, four hours, first term; Differential equations of mathematical physics, the conduction of heat, four hours, third term.

Summer Quarter, June 20 to September 2. — By Professor E. H. Moore: General analysis, four hours; Seminar on the foundations of mathematics, four hours; Graphical methods in algebra, four hours, all second term. — By Professor L. E. Dickson: Theory of substitutions, four hours; Differential calculus, five hours. — By Professor J. W. A. Young: Critical review of secondary mathematics, four hours; Advanced algebra, five hours. — By Professor G. A. Bliss: Functions of a complex variable, four hours; Modern analytic geometry, four hours. — By Professor E. J. Wilczynski: Projective differ-

ential geometry, four hours; Integral calculus, five hours; Synoptic course in mathematics, five hours. — By Professor A. L. UNDERHILL: Differential equations, five hours; Plane analytic geometry, five hours; College algebra, five hours.

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. COLE: Theory of functions of a complex variable, three hours; Theory of plane curves, three hours. — By Professor JAMES MACLAY: Differential equations, three hours, second half-year; Differential geometry, three hours, second half-year.— By Professor D. E. SMITH: History of mathematics, two hours; Seminar in the history and teaching of mathematics.— By Professor C. J. KEYSER: Modern theories in geometry, three hours; Principles of mathematics, three hours. — By Professor EDWARD KASNER: Vector analysis, two hours, first half-year; Geometry of differential equations, two hours.

Summer session (July 6 to August 17). — By Professor A. T. DELURY: Theory of functions of a complex variable, seven and one-half hours. — By Professor G. H. LING: Theory of groups of finite order, seven and one-half hours.— By Professor H. S. WHITE: Curves and surfaces of the third order, seven and one-half hours.

UNIVERSITY OF ILLINOIS. — By Professor S. W. SHATTUCK: Differential equations, three hours, first semester. — By Professor E. J. TOWNSEND: Theory of functions of a complex variable, three hours. — By Professor G. A. MILLER: Higher algebra, three hours, first semester; Theory of groups, three hours. — By Professor ——— : Synoptic course, three hours; Differential geometry, three hours. — By Professor H. L. RIETZ: Actuarial theory, three hours, first semester; Theory of statistics, three hours. — By Professor J. W. ———: Elliptic functions, three hours. — By Professor C. H. SISAM: Algebraic surfaces, three hours. — By Dr. A. R. CRATHORNE: Advanced calculus, three hours, second semester; Theory of linear differential equations, three hours. — By Dr. R. L. BÖRGER: Projective geometry, three hours. — By Dr. G. E. WAHLIN: Partial differential equations, three hours, second semester. — By Dr. T. BUCK: Solid analytic geometry, three hours, second semester.

Summer of 1910. — By Professor G. A. MILLER: Theory of equations and determinants, five hours; Elementary theory of groups, three hours. — By Dr. E. B. LYTLE: Teachers' course,

five hours. — By Dr. G. E. Wahlin: Differential equations, five hours.

Indiana University. — By Professor S. C. Davisson: Advanced calculus (a, w, s), three hours; Fourier series (a), three hours; Fundamental concepts of mathematics (w, s), two hours. — By Professor D. A. Rothrock: Systems of geometry (a, w), three hours; Calculus of variations (s, sm), three hours; History of mathematics (w), three hours. — By Professor U. S. Hanna: Theory of numbers (a), three hours; Substitution groups and Galois theory (w, s), three hours. — By Mr. K. P. Williams: Functions defined by differential equations (a, w), two hours.

(a, w, s, sm = autumn, winter, spring, summer.)

Summer Quarter, June 22–September 3, 1910. — By Professor S. C. Davisson: Advanced calculus, five hours; Modern analytic geometry, five hours. — By Professor D. A. Rothrock: History of mathematics, three hours; Ordinary differential equations, five hours. — By Professor U. S. Hanna: Advanced differential equations, five hours. — By Mr. K. P. Williams: Fourier series, three hours.

Johns Hopkins University. — By Professor F. Morley: Higher geometry, three hours, first half year; Theory of functions, three hours, second half year. — By Professor A. Cohen: Differential equations, two hours; Calculus of variations, two hours, first half year. — By Professor A. Coble: Theory of groups, two hours; Theory of probabilities, two hours, second half year.

At the meeting of the National academy of sciences held at Washington April 19–21, Professor F. R. Moulton, of the University of Chicago, was elected to membership.

At the meeting of the American philosophical society held on April 23 the following persons were elected to membership: President R. C. Maclaurin, of the Massachusetts Institute of Technology, Professor B. O. Peirce, of Harvard University, Professor O. W. Richardson, of Princeton University, and Professor C. E. Picard, of the University of Paris.

Professor C. Carathéodory, of the technical school at Hanover, has been appointed professor of mathematics at the newly established technical school at Breslau.

PROFESSOR R. DEDEKIND, of the technical school at Brunswick, has been elected foreign member of the academy of sciences at Paris.

PROFESSOR G. FABER, of the University of Tübingen, has been appointed professor of mathematics at the technical school of Stuttgart.

AT the University of Göttingen, Dr. A. HAAR and Dr. H. WEYL have been appointed docents in mathematics.

DR. W. SCHNEE has been appointed docent in mathematics at the University of Breslau.

MR. J. H. JEANS, formerly professor of mathematics in Princeton University, has been appointed Stokes lecturer in mathematics at Cambridge University.

AT Cornell University, Professors J. I. HUTCHINSON and VIGIL SNYDER have been promoted to full professorships of mathematics.

PROFESSOR OSKAR BOLZA, of the University of Chicago, has resigned and expects to reside in Freiburg, Germany. He will however retain his connection with the University as non-resident professor. Professor E. J. WILCZYNSKI, of the University of Illinois, has been appointed associate professor of mathematics at the University of Chicago.

AT the University of Wisconsin, Professor MAX MASON has been promoted to a full professorship of mathematical physics; Professor E. B. SKINNER has been promoted to an associate professor of mathematical physics; Mr. H. L. WOLF has been promoted to an assistant professorship of mathematics; Mr. S. E. URNER has been appointed instructor in mathematics.

PROFESSOR J. W. YOUNG, of the University of Illinois has been appointed head of the department of mathematics at the University of Kansas.

AT Columbia University Dr. C. C. GROVE has been made assistant professor of mathematics. Dr. N. J. LENNES has been appointed instructor in mathematics. Mr. C. B. UPTON has been promoted to an assistant professorship of mathematics in Teachers College. Professor JAMES MACLAY has been granted a half year's leave of absence which he will spend abroad.

AT the University of Pennsylvania hereafter the chairman of each department of instruction will be elected annually by

the department. For the year 1910–1911, Professor G. E. Fisher has been chosen chairman of the Department of Mathematics in the Graduate School, and Professor I. J. Schwatt chairman of the Mathematical Department in the College.

At the Georgia School of Technology, Mr. George Rutledge and J. W. Speas have been appointed instructors in mathematics. Professor W. V. Skiles has been granted leave of absence to study at Harvard University. Mr. A. B. Morton will resume his duties this fall, after a year's leave of absence at Brown University.

At Stanford University, Mr. E. W. Ponzer, of the University of Illinois, has been appointed assistant professor of mathematics; Mr. G. F. McEwen has been appointed instructor in applied mathematics.

Mr. A. S. Hawkesworth has been appointed instructor in higher mathematics and lecturer in Semitic languages at the University of Pittsburgh.

At the University of Minnesota Dr. H. L. Slobin has been made instructor in mathematics.

Dr. Elizabeth R. Bennett has been appointed instructor in mathematics at the University of Nebraska.

Recent catalogues of second hand mathematical works: Mayer & Müller, Prinz Louis Ferdinandstrasse 2, Berlin, N. W., catalogue No. 247, containing about 4,000 titles. — List & Francke, Talstrasse 2, Leipzig, catalogue No. 419, containing 1052 titles. Süddeutsches Antiquariat, Galleriestrasse 20, Munich, catalogue 23, containing 1123 titles.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* Professor W. B. Ford, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

Berkeley (H.). Mysticism in modern mathematic. Oxford University Press, 1910. 8vo. 12 + 264 pp. 8s.

Bernoulli (J.). Ueber unendliche Reihen (1689–1704). Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, Engelmann, 1909. 8vo. 141 pp. M. 2.50

BÖGER (R.). Projektive und analytische Schulgeometrie. Ein Lehr- und Uebungsbuch für die Oberklassen. Leipzig, Göschen, 1910. M. 3.60

BÜCHER, neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet.) Mitgeteilt Winter 1909–10. Leipzig, Hinrich. 8vo. Pp. 69–91. M. 0.30

COUTURAT (L.). Internaciona matematikal lexiko en Ido, Germana, Angla, Franca e Italiana. Internationales mathematisches Lexikon in Ido, Deutsch, Englisch, Franzözisch und Italienisch. Jena, Fischer, 1910. 8vo. 4 + 36 pp. M. 1.50

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome II: Analyse. Vol. 3: Equations différentielles ordinaires. Fasc. 1: P. Painlevé, Existence de l'intégrale générale; détermination d'une intégrale particulière par ses valeurs initiales. E. Vessiot, Méthodes d'intégration élémentaires; étude des équations différentielles ordinaires au point de vue formel. Paris, Gauthier-Villars, 1910. 8vo. Pp. 1–170.

ESTANAVE (E.). Construction de modèles de surfaces applicables sur le paraboloide de révolution, surfaces définies de G. Darboux. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling. Neue Folge. Nr. 7.) Leipzig, Schilling, 1909. 8vo. 22 pp. M. 1.20

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. 1ter Teil. Die analytische Geometrie der Ebene. 7te, verbesserte Auflage. Leipzig, Teubner, 1910. M. 3.00

HARTENSTEIN (R.). Die Diskriminantenfläche der Gleichung 4ten Grades. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling. Neue Folge. Nr. 8.) Leipzig, Schilling, 1909. 8vo. 19 pp. M. 1.20

JURETZKA (E.). Die Entwickelung unstetiger Funktionen nach den Eigenfunktionen des schwingenden Stabes auf Grund der Theorie der Integralgleichungen. (Diss.) Breslau, 1909. 8vo. 52 pp.

KOWALEWSKI (G.). See BERNOULLI (J.).

LIND (B.). Ueber das letzte Fermatsche Theorem. Leipzig, Teubner, 1910. 8vo. M. 2.00

PAINLEVÉ (P.). See ENCYCLOPÉDIE.

ROTHENBERG (S.). Geschichtliche Darstellung der Entwicklung der Theorie der singulären Lösungen totaler Differentialgleichungen von der ersten Ordnung mit 2 variablen Grössen. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, 20tes Heft, 3tes Stück.) Leipzig, Teubner, 1910. 8vo. Pp. 315–404. M. 3.60

RUDIO (F.). See GANTER (H.).

VESSIOT (E.). See ENCYCLOPÉDIE.

WEHRHEIM (H.). Ueber das kombinatorische Produkt dreier Kollineationen in der Ebene. (Diss.) Giessen, 1909.

WIDDER (W.). Untersuchungen über die allgemeinste lineare Substitution mit vorgegebener p^{ter} Potenz. Leipzig, Teubner, 1909. 53 pp. M. 3.00

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

ANDOYER (H.). Cours de géométrie. 8e édition. Paris, Berlin, 1910. 12mo. 440 pp. Fr. 3.50

EDWARDS (R. W. K.). An elementary textbook of trigonometry. London, Rivers, 1910. 8vo. 266 pp. Cloth. 5s.

EMMERICH (A.). Leitfaden und Uebungsbuch der ebenen und sphärischen Trigonometrie. Neubearbeitung der 10ten Auflage von W. Wink's Lehrbuch der Geometrie II, Abteilung A und C. Weinheim, Ackermann, 1910. 8vo. 8 + 126 pp. M. 180

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria, ad uso delle scuole secondarie superiori. 4a edizione. Bologna, Zanichelli, 1910. 16mo. 607 pp. L. 4.50

GIBELLI (G.). Elementi di geometria, per le scuole secondarie. Milano, Vallardi, 1910. 16mo. 171 pp. L. 2.00

HARTL (H.). Aufgaben aus der Arithmetik und Algebra. Resultate. 3te, verbesserte Auflage. Wien, Deuticke, 1910. 8vo. 3 + 96 pp. M. 1.80

KANDUSCHER (E. A.). Studienbehelf der Stereometrie. Flächen-, Polygon-, und Kreisberechnungen. Leichtfassliches und gemeinverständliches Lehrbuch besonders für das Selbststudium geeignet. Wien, Seidel, 1909. 8vo. 16 + 79 pp. M. 3.00

KNOPS (K.) und MEYER (E.). Lehr- und Uebungsbuch für den Unterricht in der Mathematik an den höheren Mädchenschulen, Lyceen und Studienanstalten. Nach Heilermann-Diekmann's Algebra und Koppe-Diekmann's Geometrie bearbeitet. 1tes Heft für Klasse IV der höheren Mädchenschule. 2te Auflage. Essen, Baedeker, 1910. 8vo. 4 + 27 pp. M. 1.01

KUNDT (F.). Arithmetische Aufgaben mit einem Anhang von Aufgaben aus der Stereometrie für höhere Mädchenschulen und die unteren Klassen der Studienanstalten. Auf Grund der Ausführungsbestimmungen zu dem Erlasse vom 18. VIII. 1908 über die Neuordnung des höheren Mädchenschulwesens bearbeitet. 2te Auflage. Leipzig, Teubner, 1910. 8vo. 6 + 172 pp. Cloth. M. 2.00

KÜSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. 10te, neu berechnete Auflage. Leipzig, Veit, 1910. 8vo. 107 pp. Cloth. M. 2.40

MAHLERT (A.). See MÜLLER (H.).

MANDL (M.). Lehrbuch der Geometrie für die oberen Klassen der Realschulen. IV. bis VII. Klasse. Wien, 1910. K. 4.50

MASSENET (G.) et PEIRONET (J.). Compléments de mathématiques, brevet ordinaire. Compléments d'algèbre ; vecteurs ; trigonométrie rectiligne; géométrie de la sphère ; trigonométrie sphérique ; mécanique ; dessin industriel. Paris, Challamel, 1910. 8vo. 475 pp.

MATRICULATION mathematics papers : being the papers in elementary mathematics set at the matriculation examination of the University of London from June 1903 to January 1910. London, Clive, 1910. 8vo. 116 pp. 1s. 6d.

MENGOD (A. M.). Prolegómenos de matemáticas (nociones de aritmética y geometría). Malaga, 1909. 6 + 170 pp. P. 6.50

MEYER (E.). See KNOPS (K.).

MILLIS (J. F.). See STONE (J. C.).

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch für höhere Mädchenschulen und für Studienanstalten. Ergebnisse. 1ter Teil. Leipzig, Teubner, 1910. 8vo. 44 pp. M. 1.50

——. Lehr- und Uebungsbuch der Arithmetik und Algebra für Studienanstalten. Ausgabe B: Für Oberrealschul- und realgymnasiale Kurse. 2ter Teil: Lehraufgabe der oberen 3 Klassen. Mit ausgewählten Abschnitten aus der Geschichte der Schulmathematik. Leipzig, Teubner, 1910. 8vo. 7 + 237 pp. Cloth. M. 3.00

OXFORD University. Local examinations. Papers of the examination held in March, 1910, with answers to the questions set in mathematics and physics and list of delegates and examiners. London, Parker, 1910. 8vo. 2s.

PEIRONET (J.). See MASSENET (G.).

SALAZAR É IBAÑEZ (M.). Problemas matemáticos. Contestaciones al programa de esta asignatura para las oposiciones al Cuerpo de Aduanas. Madrid, Alvarez, 1910. 300 pp. P. 7.00

STONE (J. C.) and MILLIS (J. F.). Elementary geometry, plane. Boston, Sanborn, 1910. 12mo. 9 + 252 pp. Cloth. $0.80

III. APPLIED MATHEMATICS.

BAHRDT (W.). Magnetische und magnetisch-elektrische Messungen im Unterricht. (Band II, Heft 4, der Abhandlungen zur Didaktik und Philosophie der Naturwissenschaft.) Berlin, Springer, 1910. M. 2.40

BOULVIN (J.). Cours de mécanique appliquée aux machines professé à l'Ecole spéciale du génie civil de Gand. 2ème édition. Paris, Geisler, 1910. 8vo. 8 + 568 pp.

CHRISTIANSEN (C.) und MÜLLER (J. J. C.). Elemente der theoretischen Physik. Mit einem Vorwort von E. Wiedemann. 3te, verbesserte Auflage. Leipzig, Barth, 1910. 8vo. 10 + 690 pp. Cloth. M. 15.00

DÖRFFURT (B.). Graphische Tabellen für Transmissionsberechnungen. Berlin, Runge, 1910. 4 pp. M. 2.50

FLAMANT (A.). Stabilité des constructions. Résistance des matériaux. 3ème édition, revue et corrigée. Paris, Béranger, 1910. 8vo. 7 + 674 pp.

FRICKE (R.). See PERRY (J.).

GERDAY (C.). See MOULAN (P.).

HESS (L.). Lehrbuch der Baumechanik für Hoch- und Tiefbautechniker. 2te, vermehrte und verbesserte Auflage. Wien, Fromme, 1910. 8vo. 7 + 258 pp. Cloth. M. 6.00

JAEGER (M.). Graphische Integrationen in der Hydrodynamik. (Diss.) Göttingen, 1909. 8vo. 48 pp.

KNAPP (W.). Statik der Hochbaukonstruktionen. Leipzig, Scholtze, 1910. 8vo. 8 + 214 pp. M. 6.00

LIPPMANN (O.). Hilfsbuch für die Praxis des Maschinenbaues und der Mechanik, mit einem Anhang: Die Elektrotechnik und ihre Anwendung. 6te Auflage. Dresden, Lippmann, 1910. 8vo. 8 + 158 pp. Cloth. M. 3.20

LORENZ (H.). Lehrbuch der technischen Physik. 3ter Band: Technische Hydromechanik. München, Oldenbourg, 1910. 8vo. 22 + 500 pp. M. 15.00

LÖSER (B.). Hilfsbuch für die statischen Berechnungen des Hochbaues-Formeln und Tabellen für die Praxis unter besonderer Berücksichtigung des Eisenbetonbaues. 3te, vermehrte und verbesserte Auflage. Leipzig, Gilbers, 1910. 8vo. 8 + 228 pp. Cloth. M. 6.00

LUEGER (O.). Lexikon der gesamten Technik und ihrer Hilfswissenschaften. 2te, vollständig neu bearbeitete Auflage. 8ter (Schluss-) Band. Stuttgart, Deutsche Verlags-Anstalt, 1910. 8vo. 1046 pp. M. 30.00

MOULAN (P.). Cours de mécanique élémentaire à l'usage des écoles industrielles. 3ème édition, revue et notablement augmentée par C. Gerday. Paris, Béranger, 1910. 8vo. 2 + 1275 pp.

MÜLLER (J. J. C.). See CHRISTIANSEN (C.).

NOWAK (A.). Beispiele aus der Festigkeitslehre. Elementares Hilfsbuch für den Unterricht und das häusliche Studium. 3te Auflage. Altenburg, Schnuphase, 1910. 8vo. 74 pp. M. 3.00

PERRY (J.). Höhere Analysis für Ingenieure. Autorisierte deutsche Bearbeitung von R. Fricke und F. Süchting. 2te, verbesserte und erweiterte Auflage. Leipzig, Teubner, 1910.

RAMSAUER (C.). Experimentelle und theoretische Grundlagen des elastischen und mechanischen Stosses. (Hab.) Heidelberg, 1909. 8vo. 77 pp.

SALIGER (R.). Antrittsrede, zur Eröffnung der Vorlesungen über Mechanik und Statik des Hochbaues sowie über Eisenhochbau an der k. k. technischen Hochschule in Wien am 30. XI. 1909 gehalten. Wien, Lehmann, 1910. 8vo. 15 pp. M. 0.50

SÜCHTING (F.). See PERRY (J.).

WEIGEI und WERNICKE. Handbuch der Starkstromtechnik. IIter Band. Leipzig, Hochmeister, 1910. 8 + 401 pp. M. 18.00

WERNICKE. See WEIGEI.

WIEDEMANN (E.). See CHRISTIANSEN (C.).

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 2ter Band. Festigkeitslehre. 545 Aufgaben nebst Lösungen und einer Formel. sammlung. Berlin, Springer, 1910. 8vo. 8 + 348 pp. Cloth. M. 6.80

A THEOREM ON THE ANALYTIC EXTENSION OF POWER SERIES.

BY PROFESSOR WALTER B. FORD.

IN a note published in the *Journal de Mathématiques* in 1903 I considered the function $f(z)$ defined by a given power series*

$$(1) \quad g(0) + g(1)z + g(2)z^2 + \cdots + g(n)z^n + \cdots \qquad (r = rad.\ of\ conv. > 0).$$

The problem was to determine the character of $f(z)$ *outside* the circle of convergence and it was shown that if the coefficient $g(n)$ could be generalized into a function $g(w)$ of the complex variable $w = x + iy$ satisfying certain specified conditions, then $f(z)$ could be extended analytically throughout any (finite) region which did not include the positive half of the real axis, and an explicit form was given defining the function throughout such region. The conditions there imposed upon $g(w)$ were unnecessarily restrictive and, in view of certain analogous but more general theorems of Mellin, Le Roy, and Lindelöf,† it is proposed to show in the present note that my earlier results may be generalized into the following

THEOREM: *If the coefficient $g(n)$ of the power series*

$$(2) \quad \sum_{n=a}^{\infty} g(n)z^n \qquad \begin{array}{l} a = integer,\ positive,\ negative,\ or\ zero \\ r = radius\ of\ convergence > 0 \end{array}$$

may be considered as a function $g(w)$ of the complex variable $w = x + iy$ satisfying the following conditions: (a) it is single valued and analytic throughout all portions of the plane lying to the right of (or upon) the vertical line $w = a - \frac{1}{2} + iy$; and (b) it is such that

$$\lim_{y=+\infty} e^{-\epsilon y} g(x \pm iy) = 0 \qquad (x \geqq a - \tfrac{1}{2}),$$

* Subsequently generalized to functions defined by double power series. Cf. *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 260–274.

† For an exposition of this subject with bibliography see Lindelöf, Le Calcul des Residus etc., Chap V. (Paris, Gauthier-Villars, 1905). It is believed that the theorem of the present paper, while equally general with those of Lindelöf and others, has the advantage of furnishing a considerably simpler form for the function $f(z)$.

in which ϵ *represents an arbitrarily small positive quantity, then the function* $f(z)$ *of the complex variable* z *defined by* (1) *when* $|z| < r$ *may be extended analytically throughout the whole (finite)* z *plane with the exception of the positive half of the real axis, and for this region* $f(z)$ *will be defined by the equation*

$$(3) \qquad f(z) = (-1)^a \frac{(-z)^{a-\frac{1}{2}}}{2} \int_{-\infty}^{\infty} \frac{g(a-\frac{1}{2}+iy)(-z)^{iy}}{\cosh \pi y} dy$$

in which if we place $z = \rho(\cos\phi + i\sin\phi)$ *it is supposed that we take* $-2\pi < \phi < 0$ *and write* $(-z)^{iy} = e^{iy\log(-z)} = e^{iy[\log\rho + (\phi+\pi)i]}$.

For the proof of this theorem let us consider, as in the former note, the result obtained by integrating the function

$$\frac{\pi g(w)(-z)^w}{\sin \pi w}$$

about a rectangular contour C_n in the w plane. In the present instance let this rectangle be formed by the lines $w = a - \frac{1}{2} + iy$, $w = \frac{1}{2} + 2n + iy$, $w = x \pm ij$, where n is any integer such that $2n > a$ and where j is any positive quantity, arbitrarily large. We thus arrive directly by elementary results in the calculus of residues at the equation

$$\sum_{n=a}^{2n} g(n)z^n = \frac{1}{2i}\int_{C_n} \frac{g(w)(-z)^w}{\sin \pi w} dw.$$

We proceed to study the integral here appearing, supposing at first that z is real and negative.

First, along the side upon which $w = x + ij$ we have $dw = dx$ and $\sin \pi w = \sin \pi(x+ij) = \sin \pi j\,(\sin \pi x \coth \pi j + i \cos \pi x)$ so that, if we call the contribution from the side in question I, we may write

$$I = \frac{(-z)^{ij}}{2i \sinh \pi j} \int_{a-\frac{1}{2}}^{2n+\frac{1}{2}} \frac{g(x+ij)(-z)^x}{\sin \pi x \coth \pi j + i \cos \pi x} dx.$$

Whence, $\lim_{j=\infty} I = 0$, provided that

$$(4) \qquad \lim_{j=\infty} e^{-\pi j} g(x+ij) = 0 \qquad (x \geqq a - \tfrac{1}{2}).$$

Similarly, we find the same result for the contribution

arising from the side of C_n upon which $w = x - ij$, provided that

$$(5) \qquad \lim_{j=\infty} e^{-\pi j} g(x - ij) = 0 \qquad (x \geqq a - \tfrac{1}{2}).$$

Next, let us consider the side upon which $w = \frac{1}{2} + 2n + iy$. Here we have $dw = idy$, $\sin \pi w = \cos i\pi y = \cosh \pi y$, so that having taken $j = \infty$, the contribution in question becomes

$$J = \frac{(-z)^{\frac{1}{2}+2n}}{2} \int_{-\infty}^{\infty} \frac{g(\frac{1}{2} + 2n + iy)(-z)^{iy}}{\cosh \pi y} dy.$$

If we now substitute for conditions (4) and (5) the single, stronger condition

$$(6) \qquad \lim_{y=+\infty} e^{-\epsilon y} g(x \pm iy) = 0,$$

in which ϵ represents an arbitrarily small positive quantity, it appears directly that the improper integral here occurring has a meaning (z real and negative). If in particular $|z| < 1$ we shall evidently have also $\lim_{n=\infty} J = 0$.

Whence, if we now take account of the contribution arising from the remaining side $w = a - \frac{1}{2} + iy$ of C_n, noting that we here have $\sin \pi w = (-1)^{a-1} \cosh \pi y$ while the integration takes place from $y = +\infty$ to $y = -\infty$, we may write

$$(7) \quad \sum_{n=a}^{\infty} g(n) z^n = (-1)^a \frac{(-z)^{a-\frac{1}{2}}}{2} \int_{-\infty}^{\infty} \frac{g(a - \frac{1}{2} + iy)(-z)^{iy}}{\cosh \pi y} dy.$$

This relation must hold good, as we have indicated, for all values of z which are real and negative and such that $|z| < 1$. But the first member represents a function of the complex variable z which is single valued and analytic throughout the the circle of convergence of (1), while the second member, with proper conventions as regards the meaning of $(-z)^{iy}$, represents, as we shall now show, a function of z which is analytic and single valued throughout the whole z plane except for the positive half of the real axis.

Thus let us place $z = \rho(\cos \phi + i \sin \phi)$ and agree to write

$$\log(-z) = \log \rho + i(\phi + \pi).$$

Then

$$(-z)^{iy} = e^{iy \log(-z)} = e^{iy[\log \rho + i(\phi + \pi)]} = e^{iy \log \rho} e^{-(\phi + \pi) y}.$$

Moreover, for all values of z within a region T which does not cut or touch the positive half of the real axis we shall have $-\pi < \phi + \pi < \pi$, provided we agree to choose ϕ at every point so that $-2\pi < \phi < 0$. It follows, upon introducing (6), that when the above agreements are made we may always choose ϵ so small that the improper integral in (7) will converge *uniformly* for all values of z in T. Whence, the same integral and hence also the second member of (7) will have the analytic properties indicated above.*

Thus we reach in summary the theorem stated at the beginning.

It may be observed that in case the function $g(w)$ satisfies the conditions demanded except that it has a finite number of singularities in the region of the w plane lying to the right of the line $w = a - \frac{1}{2} + iy$ the theorem continues true provided we subtract from the second member of (3) the sum of the residues of the function

$$\frac{\pi g(w)(-z)^w}{\sin \pi w}$$

corresponding to such singularities.

UNIVERSITY OF MICHIGAN,
May, 1910.

EXTENSIONS OF TWO THEOREMS DUE TO CAUCHY

BY PROFESSOR G. A. MILLER.

(Read before the Chicago Section of the American Mathematical Society, April 9, 1910.)

THE last one of the noted series of papers on substitution groups published by Cauchy during 1845–6 in the Paris *Comptes Rendus* is devoted to a simplification of his earlier proof of an important theorem which may be stated as follows: If the symmetric group G of degree n involves at least one substitution which transforms one of its subgroups H_1 into a group having only identity in common with the subgroup H_2, the total number of such substitutions in G is divisible by the product of the orders of H_1 and H_2. The proof given by Cauchy is

*Cf. Osgood, Encyklopädie, II, p. 21.

very simple and applies equally when G in any group involving H_1 and H_2, as has been observed by Jordan * and others.

The object of the present note is to extend this theorem by modifying only very slightly the method of proof employed by Cauchy and to indicate how easily Sylow's theorem may be obtained from this extension. By doing this we hope to give one of the simplest proofs of Sylow's theorem and to exhibit, at the same time, how close Cauchy, Jordan, and others were to this fundamental theorem a number of years before it was announced by Sylow. While the historical setting is a prominent element of the present note, the subject matter appears sufficiently fundamental to justify various forms of presentation and emphasis on slight extensions.

Let $H_1 = 1, s_2, s_3, \cdots, s_{h_1}$ and $H_2 = 1, t_2, t_3, \cdots, t_{h_2}$ be two subgroups of any group G and suppose that H_2 and $s^{-1}H_1s$, s being any operator of G, have exactly ρ operators in common. These ρ common operators form a common subgroup of H_1 and H_2. In the following rectangular array of h_1h_2 operators

$$\begin{array}{ccccc} s & st_2 & st_3 & \cdots & st_{h_2} \\ s_2s & s_2st_2 & s_2st_3 & \cdots & s_2st_{h_2} \\ s_3s & s_3st_2 & s_3st_3 & \cdots & s_3st_{h_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{h_1}s & s_{h_1}st_2 & s_{h_1}st_3 & \cdots & s_{h_1}st_{h_2} \end{array}$$

the operators of the same column transform H_1 into the same group, and each of the operators of this array transforms H_1 into a group which has exactly ρ operators in common with H_2. The supposition that two operators of the array are equal implies an equation of the form

$$s_\alpha st_\beta = s_\gamma st_\delta, \quad \text{or} \quad s^{-1}\cdot s_\gamma^{-1}s_\alpha \cdot s = t_\delta t_\beta^{-1}.$$

If we suppose α and β fixed, γ can be chosen in exactly ρ ways so as to satisfy the latter equation. That is, there are exactly ρ operators in the given array which are equal to any given operator of the array. In other words, this array involves exactly $h_1h_2 \div \rho$ distinct operators.

If s' is any operator of G which transforms H_1 into a group having exactly ρ operators in common with H_2 but is not in-

* Traité des substitutions, 1870, p. 26.

cluded in the given array, we may form another array which can be obtained by replacing s by s' in the one given above. This new array must again involve exactly h_1h_2/ρ distinct operators and none of these can be equal to one of the preceding array since the equation

$$s_\alpha s' t_\beta = s_\gamma s t_\delta$$

implies that s' is in the former array. As this process may be repeated until all the operators of G which transform H_1 into a group having exactly ρ operators in common with H_2 have been exhausted, we have proved the following theorem :

*In any group G the number of distinct operators which transform any subgroup H_1 into a group having exactly ρ operators in common with a subgroup H_2 is a multiple of h_1h_2/ρ, h_1 and h_2 being the orders of H_1 and H_2 respectively.**

For the special case $\rho = 1$ this theorem was proved by Cauchy in the article noted above; since Cauchy used almost the same method as we employed, the present theorem should be regarded as merely a slight extension of the one given by him. Suppose now that G contains a Sylow subgroup K_1 of order p^α and that K_2 is any other subgroup of G, the order of K_2 being divisible by p^β but not by $p^{\beta+1}$. We proceed to prove that it follows from the given theorem that K_2 must involve a Sylow subgroup of order p^β. If K_2 did not contain a subgroup of order p^β, it would involve a subgroup of order $p^{\beta'}$, $\beta' < \beta$, but no subgroup of order $p^{\beta'+1}$, β' being properly chosen. In this case the number of operators of G which transform K_1 into a group having no more than $p^{\beta'}$ operators in common with K_2 would be divisible by $p^{\alpha+\beta-\beta'}$. In other words, the order of G would be divisible by $p^{\alpha+\beta-\beta'}$. As this is contrary to the hypothesis that K_1 is a Sylow subgroup of G, the given theorem implies the corollary :

If a group G involves a Sylow subgroup of order p^m, each of the subgroups of G whose order is divisible by p contains at least one Sylow subgroup whose order is a power of p.

From the preceding paragraph it results that the given theorem implies that if a group G involves at least one Sylow sub-

* This theorem evidently remains true when H_1 coincides with H_2 as well as when either one of these subgroups coincides with the entire group G. In the latter case it reduces to the theorem that the order of every subgroup divides the order of the group. This special case is sometimes called Lagrange's theorem.

group for every prime divisor of its order every subgroup of G has the same property. In particular, if the symmetric group of degree n involves Sylow subgroups for every prime which divides its order, then every substitution group of degree n (and hence every group of finite order) must involve at least one Sylow subgroup for every prime which divides its order. It is very easy to prove, as Cauchy observed, that every symmetric group of degree n has the given property, and hence the theorem which was proved above as a slight extension of one due to Cauchy implies that every group of finite order involves at least one Sylow subgroup for every prime divisor of its order. As this is the main element in Sylow's theorem it is clear that Cauchy used a method which required only slight changes to yield an easy proof of the fundamental theorem known as Sylow's theorem. It would evidently be necessary only to prove that every symmetric group whose degree is a power of p involves Sylow subgroups of order p^m in order to establish the existence of Sylow subgroups in every group of finite order by means of the theorem proved above.

The preceding remarks may also serve to exhibit additional reasons for regarding Sylow's theorem as merely an extension of Cauchy's fundamental theorem, which established the fact that every group whose order is divisible by the prime p involves operators of order p. In fact, if Cauchy had used a general value of ρ instead of $\rho = 1$ in the theorem proved above, he would have arrived at Sylow's theorem by the same steps as those which led him to his fundamental theorem. The oversight of this slight increase in generality retarded Sylow's theorem nearly thirty years and made Jordan's Traité des Substitutions much more difficult reading.

EXISTENCE THEOREMS FOR CERTAIN UNSYMMETRIC KERNELS.

BY MRS. ANNA J. PELL.

In this paper is given a brief account of the existence and expansion theorems for certain integral equations with unsymmetric kernels. Full details of the method involved and a discussion of a less general integral equation are contained in an article, "Biorthogonal systems of functions with applica-

tions to the theory of integral equations," which is at present in the hands of editors. Marty * has recently treated by another method an integral equation corresponding to a special case of the functional transformation $T\,[(1), (2), (3)]$, viz.,

$$Tf(s) = \int_a^b K(s, t) f(t) dt,$$

where $K(s, t)$ is a definite symmetric kernel.

We denote by $Tf(s)$ a linear functional transformation, which transforms every continuous function into a continuous function and which has the three following properties:

$$(1) \qquad \int_a^b [Tf(s)]^2 ds \leqq M \int_a^b [f(s)]^2 ds,$$

where M is a given positive quantity;

$$(2) \qquad \int_a^b f(s) Tf(s) ds \geqq 0,$$

the equality sign holding only for $f(s) = 0$, or for $f(s) = p(s)$, where $p(s)$ is a continuous function such that $Tp(s) = 0$;

$$(3) \qquad \int_a^b f_1(s) Tf_2(s) ds = \int_a^b f_2(s) Tf_1(s) ds.$$

The transformed function of a continuous kernel $K(s, t)$ with respect to the variable s we designate by $T_s K(s, t)$, and assume that it is continuous in s and t.

Let $L(s, t)$ be an unsymmetric kernel satisfying the condition that

$$(4) \qquad M(s, t) = T_s L(s, t)$$

is a *symmetric* kernel. By means of a biorthogonal system $(u_i(s), v_i(s))$ complete as to u and such that

$$v_i(s) = Tu_i(s)$$

the integral equations

$$u(s) + \mu p(s) = \lambda \int_a^b L(s, t) u(t) dt,$$

* *Comptes Rendus*, Feb. 28 and April 25, 1910.

$$(5) \qquad \int_a^b u(s)p(s)ds = c, \qquad v(s) = \lambda \int_a^b L(t, s)v(t)dt$$

are reduced to a system of linear equations in infinitely many variables

$$x_i = \lambda \sum_{k=1}^{\infty} x_k \int_a^b \int_a^b M(s, t)u_i(s)u_k(t)dsdt,$$

where the system of coefficients is symmetric and continuous. We obtain the following theorems:

THEOREM 1. If an unsymmetric kernel $L(s, t)$ satisfies the condition (4) and if $M(s, t) \not\equiv 0$, there exists at least one characteristic number λ, which is real and of finite multiplicity, and the characteristic functions $u_i(s)$, $v_i(s)$ form a biorthogonal system belonging to the type T.

THEOREM 2. Any continuous function $f(s)$ expressible in the form

$$f(s) = \int_a^b L(t, s)Tf_1(t)dt,$$

where $f_1(s)$ is an arbitrary continuous function, can be developed into the uniformly convergent series

$$f(s) = \sum_i v_i(s) \int_a^b u_i(s)f(s)ds.$$

THEOREM 3. If an unsymmetric kernel $L(s, t)$ has an infinite number of real characteristic numbers, and if the system of corresponding characteristic functions $(u_i(s), v_i(s))$ is complete as to u, then the kernel

$$M(s, t) = T_s L(s, t)$$

is symmetric, and T is the functional transformation belonging to the biorthogonal system (u_i, v_i) and therefore has the properties (1), (2), and (3).

CHICAGO,
May, 1910.

MULTIPLY PERIODIC FUNCTIONS.

An Introduction to the Theory of Multiply Periodic Functions. By H. F. BAKER, Sc.D., F.R.S., Fellow of St. John's College and Lecturer in Mathematics in the University of Cambridge. Cambridge University Press, 1907. Royal 8vo. xv + 335 pp.

THIS is a highly interesting and suggestive contribution to a field which has engaged the attention of numerous mathematicians since the time of Abel. Except for the first chapter, the present work has little in common with other treatises relating to the same subject, while a considerable portion of the material is drawn from the author's own investigations.

The book is divided into two parts, the first dealing with hyperelliptic functions of two variables, the second with periodic functions of n variables with reference to the fundamental problem of their connection with the theory of algebraic functions and their expression in terms of the Riemann theta functions.

"The first part is centered round some remarkable differential equations satisfied by the functions, which appear to be equally illuminative both of the analytic and the geometric aspects of the theory; it was, in fact, to explain this that the book was originally entered upon." Chapter I is introductory and is chiefly concerned with a deduction of the fundamental formulas connected with a Riemann surface of two sheets and six branch points. It contains a brief and condensed account of the hyperelliptic integrals of the first, second, and third kinds, and their behavior on the surface. After developing the properties of the theta functions, a notable departure is made from the usual treatment. Following closely the analogy of the Weierstrassian theory of elliptic functions, a single theta function is retained out of the sixteen with half-integer characteristics. This is multiplied by an exponential factor and the product regarded as a function of the unnormalized integrals u_1, u_2 of the first kind and of their homogeneous table of moduli. The function so obtained is denoted by $\vartheta(u_1, u_2)$ and later on by σ. The two first derivatives of ϑ with respect to u_i are the ζ-functions, and the three second derivatives with changed signs are the $\wp$-functions. All of these have properties strictly analogous to the corresponding functions in the elliptic case. The $\wp$-functions are expressible in a simple manner in terms of two positions on

the Riemann surface. The author hopes "that the treatment here followed, which reduces the theory in a very practical way to that of one theta function and three periodic functions, may serve the purpose of encouraging a wider use of these functions in other branches of mathematics."

This would seem on the face of it to be a considerable simplification since the sixteen Riemann theta functions, connected by numerous relations, have been replaced by a single function together with the properties deducible from it and its derivatives. The advantage is not acquired, however, without sacrifice. For example, the three $\wp$-functions satisfy an algebraic relation of degree four, the equation of the Kummer surface. But in order to prove the existence of the sixteen singular tangent planes it is necessary to expand the original determinant equation of the surface and show that it can be put in the necessary algebraic form. This illustration is somewhat typical of much of the long discussion of the Kummer and Weddle surfaces, that is, considerably more algebraic manipulation (although of an elementary character) is required and not so easy an oversight of the geometrical properties is obtained as in using the sixteen theta functions and their properties. Moreover, in the field of geometric applications, to which the author devotes so much space, it does not seem possible to treat the important class of hyperelliptic surfaces in a general and comprehensive way, as Humbert* has so beautifully and clearly done, without using the Riemann theta functions and the relations among them. The author's choice is, however, deliberate, and he expresses regret that his desire to keep the work as elementary and as self-contained as possible leads to the exclusion of important methods and material.

The first chapter strikes the reviewer as somewhat less carefully worked out than the rest of the book. There seems to be an occasional uncertainty and inconsistency in the choice of notation which is not conducive to ease and economy of effort in reading the book. On page 25, for example, we start in with theta characteristic numbers q', q and period characteristics p', p. In the course of the deduction a previous formula is referred to in which the roles of the p and q numbers are interchanged, and in writing down the final result this interchange is retained, so that at the end the letters do not mean the same that they

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* Théorie générale des surfaces hyperelliptiques, *Journal de Mathématiques*, 1893.

did at the start. On page 37, λ_1 and λ_2 are used in two different senses. In the second usage they represent the limiting values of the two integrals of the second kind. As these two integrals already contain parameters previously denoted by λ_1, λ_2, a different notation would have been more suitable and more conducive to clearness. As an example of lack of symmetry in the notation we might refer to the departure from customary usage on page 13. The four cross-cuts on the Riemann surface are denoted by A_1, A_2, A_3, A_4 while the periods of the integrals of the first and second kinds at these cuts are distinguished by subscripts 1, 2 with the addition of an accent for the second pair. We do not see that the lack of association of ideas produced by an unsymmetrical notation is compensated for by any new advantage.

The desire for compactness and condensation frequently leads the author to crowd too many things into one statement, as for example the long sentence which covers nearly the whole of page 21; and the still longer one beginning on page 25, ending exactly one page later, and including a number of different theta relations. In this last case the attempt to include so much in one sentence forces the last formula of page 25 out of its logical connection, since it is not a deduction from the one immediately preceding it, but is a special case of the formula given in the eighth line above.

Chapter II is in some sense a converse to part of the preceding chapter. Namely, starting with Kummer's quartic relation previously derived, it is shown how the variables x, y, z of this equation may be expressed as hyperelliptic functions of two variables w_1, w_2. Not only does this lead to the identification of these variables with the $\wp$-functions, but of especial significance is the culmination of the developments in a set of five partial differential equations whose general integral is

$$e^{a_1 w_1 + a_2 w_2 + b} \vartheta(w_1 + c_1, w_2 + c_2),$$

involving five arbitrary constants. A particular integral, denoted by σ, gives x, y, z as the second logarithmic derivatives with changed signs. It would at first glance seem quite out of the question to use these equations in order to obtain expansions of the σ-functions. The successful accomplishment of this step is one of the most brilliant achievements of the book. This is done by multiplying each equation by an arbitrary

parameter and adding together. The result is then written in a condensed form by the aid of the Clebsch-Aronhold symbolic algebra. By means of this composite differential equation a recurring formula is obtained for the terms in the expansion of the σ-functions, of which six odd and ten even functions are introduced. "These expansions in their turn enable us to prove succinctly various relations involving the $\wp$-functions." The processes are quite analogous to those which are well known in the case of the elliptic σ-functions.

The remainder of Part I is devoted to a detailed study of the Kummer and Weddle surfaces, the geometrical relations developed being largely restricted to those which illustrate properties of the $\wp$-functions.

Part II is confined to a general investigation of periodic functions of n variables with the object of reducing the theory to that of algebraic functions. The method is admirable for its logical simplicity and directness. There is but one other book,* as far as we know, that gives this matter a systematic treatment, and that too from a transcendental point of view. The method of the present work is algebraic. Starting in Chapter VI with some introductory theorems on power series in several variables, we are led in Chapter VII to a study of periodic functions in the neighborhood of a given point. Particular consideration is given to their behavior when the values of the arguments u_i are restricted so as to depend on a single complex variable x as follows: Let $(a_1^{(r)}, \cdots, a_n^{(r)})$ $(r = 1, \cdots, n)$ be n sets of values of the arguments $u_1, \cdots, u_n$ at which the periodic functions $\phi(u_1, \cdots, u_n)$ is regular; define n functions $\phi_r(u)$ by means of the equations

$$\phi_r(u) = \phi(u_1 + a_1^{(r)}, \cdots, u_n + a_n^{(r)}) - \phi(a_1^{(r)}, \cdots, a_n^{(r)}) \qquad (r = 1, \cdots, n);$$

then form the n equations

$$\phi_r(u) = c_{r1}x + \cdots + c_{rn}x^n,$$

of which the determinant $|c_{rs}|$ is different from zero. Regarding the values of the n variables u_i as corresponding to the points of a real space S of $2n$ dimensions, the equations last written determine a locus C (or *construct*) of two degrees of freedom in S on which the functions ϕ_r are periodic. Any

* A. Krazer, Lehrbuch der Thetafunktionen, Leipzig, 1903, Chapter XI.

given branch Γ of C may be divided up into fundamental regions such that a particular value of x occurs the same number of times in each region. The values of the u_i at points in different regions corresponding to the same value of x differ by the addition of a set of periods. The derivative $y = du/dx$ of the function $u = \Sigma\lambda_i u_i$ is shown, from its behavior in Γ, to satisfy an algebraic equation $f(y, x, \lambda_1, \cdots, \lambda_n) = 0$ with coefficients which are rational in x, the degree in y being the number of values this derivative takes for any given value of x. To each value of x are thus associated values of $u_1, \cdots, u_n$ which are determined by the equations

$$u_s = -\int \frac{\partial f/\partial\lambda_s}{\partial f/\partial y}\, dx,$$

these integrals being of the first kind. It is in this way shown that the arguments of any multiply periodic function ϕ can be expressed as abelian integrals of the first kind associated with a Riemann surface, and that the periods of ϕ are the moduli of periodicity of these integrals on the surface. The integrals u_s are in general defective integrals on a Riemann surface of genus $p > n$. A long chapter, VIII, is then given to the study of defective integrals, including a detailed treatment of several simple cases.

Chapter IX is devoted to establishing the theorem which is one of the main objects of Part II, viz., the most general single-valued multiply periodic meromorphic function is expressible by theta functions whose arguments are written in the Jacobi form as linear functions of n integrals of the first kind.

The tenth and last chapter is a discussion of the number and sum of zeros of a set of Jacobian functions, that is, functions whose second logarithmic derivatives are periodic. As the Jacobian functions are expressible in terms of theta functions, we have (in a generalized form) Poincaré's theorem concerning the number of zeros common to a system of theta functions.

Several appendices are added for the elaboration of certain topics associated with the subject in hand. Four of these relate to the algebra of matrices, of which free use is made throughout the book; one note gives a proof of Abel's theorem and its converse; and a final note considers some examples of algebraic curves on the Kummer surface having defective integrals.

The book has one noticeable feature which will commend

itself to all readers, namely, a separate caption for each page intended to indicate briefly the contents of that page. This is especially useful as the author seldom sums up results in a way readily to catch the reader's attention.

J. I. HUTCHINSON.

BÔCHER'S HIGHER ALGEBRA.

Introduction to Higher Algebra. By MAXIME BÔCHER, Professor of Mathematics in Harvard University; prepared for publication with the cooperation of E. P. R. DUVAL, Instructor in Mathematics in the University of Wisconsin. New York, Macmillan, 1907. xi + 321 pp.

Einführung in die höhere Algebra. Von MAXIME BÔCHER, Deutsch von HANS BECK, mit einem Geleitwort von E. STUDY. Leipzig, Teubner, 1910. xii + 348 pp.

THE term "higher algebra" has been so often used in America to denote a very low type of merely formal algebra and to include subjects like infinite series, which are not properly algebraic at all, that it is refreshing to find a book like this one of Professor Bôcher's, which really corresponds to its title. It does so, not only by reason of the purely algebraic character of its material, but also because this material is worked up in a strictly logical as well as systematic manner.

The amount of available algebraic material is so enormous, and it branches out in so many different directions, that some selection is inevitable; even the extensive two-volume works of Weber and Netto are confined to certain special lines. The volume under review aims to furnish the reader with an *introduction* to the whole field, to lay a broad and deep foundation for further study, and in particular, to give an adequate algebraic preparation for the study of modern analytic geometry. This aim has been accomplished with remarkable success.

There is one special topic, however, to which the author gives more than an introduction, and that is the theory of elementary divisors (Elementarteiler). In the last three chapters he not only introduces elementary divisors in a most expeditious and satisfactory manner, but carries their theory through to a fair degree of completeness, so far as the more important applications are concerned.

It has for a long time seemed to me that the theory of elementary divisors was destined to assume a much more prominent position in the science of mathematics than has hitherto been given it. For in all the applications of linear substitutions and quadratic forms, whether to geometry, algebra, or the theory of numbers, there is a whole class of problems whose complete solution is essentially dependent on elementary divisors. This belief is strengthened by observing that the subject has recently found a place in several text-books, including Muth's Elementarteiler, Bertini's Geometria Proiettiva degli Iperspazi, Bromwich's Quadratic Forms, Kowalewski's Determinantentheorie, as well as Bôcher's Algebra.

In the early chapters the author, after deriving some of the elementary properties of polynomials and determinants, considers the theory of linear dependence and the solution of systems of linear equations, as based on the idea of the rank of a matrix. The great progress that has been made in recent years in the theory of linear equations is vividly illustrated by the striking contrast between the simplicity and completeness of Bôcher's treatment in Chapter IV and the complexity and incompleteness of Chrystal's treatment in the sixteenth chapter of the first volume of his Algebra, published in 1886.

The author then defines a matrix, namely, a square array of n^2 ordinary numbers, as a single complex quantity (hypercomplex number), and develops the algebra of matrices in such a way as to be able to apply it to linear transformations, collineations, bilinear forms, and quadratic forms. He very wisely makes these subjects concrete and tangible by constantly keeping their geometric significance before the reader.

The subject of invariants is considered from a sufficiently broad standpoint to be applicable not merely to the classical theory of the invariants of n-ary forms, but to all the mathematical theories in which invariants occur; and surely that includes a very extensive category. The reduction of a quadratic form to a sum of squares, the law of inertia for real quadratic forms, the properties of a system of a quadratic form and one or more linear forms, and the simultaneous reduction, in two special cases, of a pair of quadratic forms to sums of squares, are taken up in order. The general problem of the simultaneous reduction of a pair of quadratic forms to a normal form is postponed to the last chapter, where it becomes solvable by means of elementary divisors.

The precise nature of the reducibility of a polynomial in an arbitrary domain of rationality and of the greatest common divisor of two polynomials is carefully explained, so as to afford a firm basis for the treatment of elimination, and of resultants and discriminants. Finally, after the theory of elementary divisors has been developed, it is applied to the important problems of the classification of collineations and the classification of pairs of quadratic forms.

It is to be noticed that Galois's theory of equations and the theory of permutation groups, which necessarily accompanies it, are not included in the scope of the work, although the group concept is introduced in connection with linear transformations.

A very unusual feature is the way in which the material is systematized and unified; the connections between different lines of thought are pointed out; the origin, significance, and application of every new idea are carefully indicated. This makes the work an ideal text-book, as I have found by actual trial in the class room.

Naturally, no two persons would quite agree in their choice of the tools to be employed. Personally, it seems to me that modular systems might well have been utilized at certain points, somewhat as in Pund's algebra.

Throughout the literature of linear transformations there is a very common confusion arising from the failure to distinguish in language between the two transformations $x_i = \Sigma a_{ij} x'_j$ and $x'_i = \Sigma a_{ij} x_j$, either being referred to indiscriminately as the linear transformation of matrix a. In Bôcher's book the context usually indicates which is meant, but clearness would be gained by a more explicit statement.

There are a few trifling errors and misprints, but they will cause the reader no inconvenience, and most of them have been eliminated in the German translation. There is a good index in the original and a still better one in the German edition.

Although in a rapidly growing science like mathematics the best possible text-book must necessarily be restricted to a brief period of usefulness, yet it seems evident that this one will remain a classic for a considerable time to come.

The very existence of a German edition is a distinct compliment, not only to Professor Bôcher, but to American scholarship as well. If we are to judge by the giant strides that mathematical science is now making in this country, similar compliments will become more frequent in the future.

ARTHUR RANUM.

NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Geometry. By JULIAN LOWELL COOLIDGE. Oxford, Clarendon Press, 1909. 8vo. 291 pp.

FOR some time there has been felt in our universities a lack of English texts in the branches of higher mathematics, while in the lower branches we have been literally flooded with them. If our books in the higher branches were to be merely good translations of the best that the German, French, and Italian have to offer, something is surely gained, but when the English texts bear all the ear marks of elegance of form, clearness and originality of presentation, and when they embody within them the spirit of research, they are not only worthy of the highest praise, but they should be received with open arms. It is therefore a great pleasure to note how mathematical literature in English has been enriched within the last year by two treatises in two such far-reaching and important subjects as the differential geometry of curves and surfaces * and non-euclidean geometry. Let us hope that the good work thus begun will continue until we shall have a mathematical literature of our own which will stand comparison with that of other nations.

The book under review, the first real treatise in non-euclidean geometry written in English, is at the same time a most noteworthy addition to mathematical literature in general. It is one of the books that come under the desirable category above described. It contains some original work, part of it hitherto unpublished. As far as the general makeup of the book is concerned, we should note the good table of contents and the excellent index. The axioms are printed in heavy type and the theorems are numbered, attention being called to these by italicizing the word *Theorem,* but not the body of the theorem. A large number of theorems are stated without proof — some because they follow easily from the preceding ones, and others because the author, as he expressly states, wishes to leave them as exercises for the reader. The discussion is somewhat too condensed in parts; it would add a great deal to the value of the book if some portions were worked out

* A Treatise on the Differential Geometry of Curves and Surfaces. By L. P. Eisenhart, 1909.

in more detail — but this briefness is no doubt due to the large amount of ground covered in so little space. The reviewer would also like to see more words of explanation introduced in the proofs, and would appreciate a judicious use of paragraph markings and heavy type headings to indicate a passage from one set of related theorems to another set, thus relieving a monotonous appearance of the page and lending interest in the reading. There are copious foot-notes, giving detailed references and interesting comments on these references and on the text. There are several minor typographical errors but these offer no hindrance in the reading.

To understand clearly the author's purpose, let us quote from the preface. "Recent books dealing with non-euclidean geometry fall naturally into two classes. In the one we find the works of Killing, Liebmann, and Manning, who wish to build up certain clearly conceived geometrical systems and are careless of the details of the foundations on which all is to rest. In the other category are Hilbert, Vahlen, Veronese, and the authors of a goodly number of articles on the foundations of geometry. These writers deal at length with the consistency, significance, and logical independence of their assumptions, but do not go very far towards raising a superstructure on any one of the foundations suggested. The present work is, in a measure, an attempt to unite the two tendencies." The author has clearly and successfully done what he set himself to do. He has given a rigorous treatment of the foundations and having built strongly and wisely, he has raised a firm superstructure upon these. He has attacked the problem of non-euclidean geometry by its three approaches. (1) The elementary geometry of point, line, and distance (this development is the most complete and includes the first 17 chapters); (2) projective geometry and the theory of transformation groups (Chapter XVIII); (3) differential geometry, with the concept of distance element, extremal, and space constant (Chapter XIX). The only limitations as to subject matter are that the work does not go beyond three dimensions, thus gaining in clearness at the expense of generality, and that it treats only of the classical (i. e., archimedian and desarguian) systems.

Chapter I, "Foundations for a metrical geometry in a limited region," formulates the entities, definitions, and axioms upon which the geometry is founded. The author has found it best suited to his purpose to set up a system of axioms of his

own, which is not however as condensed as some hitherto published. For the first fundamental undefinable, the *point* (as is usual) is taken, and for the second, from the mass of objects such as segment and motion, segment and order, motion, line and separation, and distance, following Peano and Levy, *distance* is chosen. Thus the existence of two classes of objects, points and distances, is posited. Then follow 17 axioms with their discussion and easily deducible theorems, and upon these the three-dimensional type of space is constructed. They include the positing of at least two points and a unique object called their distance, the usual congruent relations, the relations of greater than and less than, then Axiom XI, "If A and C be any two points, there exists such a point B distinct from either such that $\overline{AB} \equiv \overline{AC} + \overline{CB}$," which involves the existence of an infinite number of points and removes the possibility of a maximum distance; the axioms establishing a serial order among the points of a segment and its extensions, the extension beyond the geometry of a single line and that beyond the geometry of a single plane.

Chapter II deals with "Congruent transformations." Having taken *distance* as undefinable and distance being a magnitude, the basis for a metrical geometry has been laid. To complete the metrical system, two more assumptions are added. The first is the axiom of continuity — this leads to several theorems, the archimedian axiom, the introduction of the concept of number as the numerical measure of two distances, and its extension to the irrational number. Having defined a congruent transformation between two sets of points (P) and (Q), it is by means of a second axiom which allows the enlarging of a congruent transformation to include additional points, and upon the definition of angle and its numerical measure, that all the properties of congruent figures and the relation of perpendicularity in the plane and in space are developed.

In the last chapter, dealing with the comparison of distances and angles and their numerical measures, the question of the sum of the angles of a triangle was not considered. It is the purpose of Chapter III, "The three hypotheses," to fill in this gap. Here we meet the theorems concerning the continuous change of distances and angles. Thus: "If in any triangle, one side and an adjacent angle remain fixed, while the other side, including this angle, may be diminished at will, then the external angle opposite to the fixed side will take and retain a

value differing from that of the fixed angle by less than any assigned value." It follows that in such a triangle the sum of the angles can be made to differ infinitesimally from a straight angle. Then follows the definition of quadrilateral. Then the theorems: If there exist a single rectangle, every isosceles birectangular quadrilateral is a rectangle; if there exist a single right triangle (any triangle) the sum of whose angles is congruent to, less than, or greater than a straight angle, the same is true of every right triangle (any triangle). According to the assumptions, then, of the existence of a single triangle the sum of whose angles is congruent to, less than, greater than a straight angle, we have the parabolic (euclidean), hyperbolic (lobachevskian), elliptic (riemannian) geometries. The chapter closes by showing that the euclidean hypothesis holds in the infinitesimal domain. The development is complete throughout and very satisfactory.

Having now completed the foundations in one direction, the materials out of which the superstructure is to be erected are gotten ready in this and the succeeding chapters. The development of the trigonometric formulas (Chapter IV) depends upon the proofs of the existence of two limits and of the continuity of two functions, of a distance and of an angle respectively. These are (1) the limit of the fraction $\mathrm{M}\overline{CD}/\mathrm{M}AB$ ($\mathrm{M}\overline{CD}$ means the measure of $\overline{CD}$ in terms of some convenient unit) in an isosceles birectangular triangle $ABCD$ whose right angles are at A and B, where $\overline{AB}$ becomes infinitesimally small while AD remains constant; and the continuity of the resulting function of $\mathrm{M}\overline{AD}$; this function $\phi(x)$ is shown to have the property $\phi(x+y)+\phi(x-y)=2\phi(x)\phi(y)$, whose solution is easily seen to be

$$\phi(x)=\cos\frac{x}{k}=1-\frac{x^2}{k^2\cdot 2\,!}+\frac{x^4}{k^4\cdot 4\,!}\cdots$$

This constant k is later shown to be the radius of a euclidean sphere upon which the non-euclidean plane may be developed, and therefore $1/k^2$ is called the *measure of curvature* of space. This again leads to the distinction between the three geometries, $1/k^2 \lesseqgtr 0$, according as we have the hyperbolic, parabolic, or elliptic case. (2) The limit of the fraction $\overline{AB}/\overline{AC}$ in the right triangle ABC, right angled at B, where AB becomes infinitesimal while $\angle BAC$ is constant. The resulting function

$f(\theta)$ of $\angle BAC$ is proved to be continuous, and it obeys the law $f(\theta+\phi)+f(\theta-\phi)=2f(\theta)f(\phi)$, the same law as $\phi(x)$ above does. Hence $f(\theta)=\cos\theta/l=\cos\theta$ if l is so chosen that the measure of right angle is $\pi/2$. It also follows that $\lim \overline{BC}/\overline{AC}=\sin\theta$. The development of the usual trigonometric formulas (corresponding to the spherical trigonometric formulas upon a sphere of radius k) now follows, and each is shown to be universally true, and reduces to that of the euclidean plane if we put $1/k^2=0$. The discussion is shortened by the use of the consideration that we have euclidean geometry in the infinitesimal domain. The entire chapter is a bit of rigorous work to be highly commended. Chapter V, "Analytic formulæ," opens by introducing the idea of directed distances and angles and their measurement in the plane and in space. The usual coordinate systems are set up, followed by the passage to homogeneous coordinates, the formulas for the distance between two points and the distance from a point to a plane. The element of arc length is computed and by comparison with the usual distance formula, it is shown that the non-euclidean plane may be developed upon a surface of constant curvature $1/k^2$ in euclidean space. The theorem is proven that every congruent transformation of space is represented by an orthogonal substitution in the homogeneous variables $x_0 : x_1 : x_2 : x_3$.

To complete the foundations and render them one homogeneous whole, and do away with a disadvantage under which he has been laboring from the start, the author now sets himself the task of showing (1) that the assumptions made at the outset are consistent, and (2) what degree of precision might be given to Axiom XI, where it was assumed that any segment might be extended beyond either extremity. These points are respectively dealt with in Chapter VI, "Consistency a significance of the axioms," and Chapter VII, "The geometric and analytic extension of space." That the axioms are sufficient has been shown by the possibility of expressing distances and angles analytically. The axioms are shown to be compatible by setting up actual systems of objects in the plane and in space obeying them and the three hypotheses. Their mutual independence is finally examined in the usual way. The author now passes to the extension of Axiom XI. It is easily shown that under the parabolic and hyperbolic hypotheses, any segment may be extended beyond either extremity by any desired

amount. This is not so under the elliptic hypothesis. To get the desired result, the author sets up six axioms, I′–VI′, of which the first two are: there exists a class of objects containing at least two members, called points; and every point belongs to a subclass (called a consistent region) obeying Axioms I–XIX. The others state properties of such consistent regions, e. g., any two consistent regions, having a common point, have a common consistent region. By means of overlapping consistent regions, we may by a process of analytic extension, reach a set of coordinates for every point in space. A point has a unique set of coordinates. But while under the parabolic and hyperbolic hypothesis there is but one point for each set of coordinates, under the elliptic hypothesis there are two possibilities: (1) there is but one point for each set of coordinates (elliptic space); (2) there are two points (called equivalent points) at a distance $k\pi$ which have the same set of coordinates (spherical space). From the additional axioms it also follows that there must exist, under the elliptic hypothesis, a point having any chosen set of homogeneous coordinates not all zero. This is not so in the other two cases, and the author brings these two up to an equality with the first by the extension of the concept point so as to include *ideal* elements. It is also shown how to find figures corresponding to imaginary coordinate values. The extension is completed by extending the concepts distance and angle to fit the extended space. The remainder of the chapter shows how this extension may be made by the use of the *absolute* and the *cross-ratio,* in the highly interesting procedure of Cayley's Sixth memoir upon quantics.

Chapter VIII is a discussion of "The groups of congruent transformations." Having defined a congruent transformation as any collineation of non-euclidean space that keeps the absolute invariant, and having shown that every such collineation is a congruent transformation, the author now sets up the groups and subgroups of congruent transformations, by studying the collineations of space that carry the absolute into itself, and viewing the matter both from an analytic and geometric standpoint.

Chapter IX deals with "Point, line, and plane treated analytically." Having now completed the foundations and gotten ready the machinery for treating space as a perfect analytic continuum, the author launches forth to build up his superstructure. And having built strongly and surely, the task is not a difficult one.

The object of this chapter is to express the fundamental metrical properties of the point, line, and plane in terms of the invariants of the congruent group. The absolute is the basis of all the work. The theorems are proven, using the nomenclature of the point geometry, but they are stated, in their full duality, in parallel columns. The work moves along rapidly. It includes theorems on the centers of gravity of two or more points, loci of second order, the parallel angle, the desmic configuration, and paratactic lines.

This last chapter included an introduction of the non-euclidean line geometry, using the line as element. Chapter X, "The higher line geometry," is a highly interesting continuation of that work, using a pair of lines, invariantly connected, as element. This element is the *cross* — the proper cross is a pair of real lines mutually absolute polar, neither of which is tangent to the absolute, and determining a pencil of coaxial complexes. The Plücker coordinates of the pencil are used as the coordinates of the cross, which consists of the directrices of the common congruence. In the case where one pair of lines is tangent to the absolute, all the complexes of the pencil are special, and they may be determined by any pair of a pencil of tangents; this pencil of tangents is called an improper cross. The geometry of the cross in hyperbolic space throws light upon the geometry of the point in the complex elliptic plane, for there exists a one-to-one correspondence between the two assemblages. The geometry of the cross in elliptic space, on the other hand, finds its counterpart in the geometry of pairs of points, one in each of two real planes. The simplest configurations of crosses, such as the chain and synectic congruence, are then studied and carried over to their corresponding analogues mentioned above. The Clifford surface is briefly touched here. The development of the cross-geometry in elliptic space is the author's own.

The following three chapters: Chapter XI, "The circle and the sphere," Chapter XII, "Conic sections," and Chapter XIII, "Quadric surfaces," are analytic studies, including complete classifications, of the metrical properties of these concepts and their configurations in non-euclidean space from the dual standpoint as loci and as envelopes. The first of these chapters ends with an elegant transformation from euclidean to non-euclidean space, showing that the Darboux-Dupin theorem (in any triply orthogonal system of surfaces, the intersection lines are lines of

curvature) must hold in hyperbolic space; the last one terminates with the introduction of elliptic coordinates, and a pretty bit of work whereby Staude's ring construction of the ellipsoid is extended to non-euclidean space.

Chapter XIV discusses "Areas and volumes." The first function developed is the *sine amplitude* of a triangle in terms of the sides and angles, which is closely analogous to the corresponding area of a euclidean triangle; indeed if we put $1/k^2 = 0$, then $k^2 \sin (ABC) = 2$ area $\triangle ABC$. The area of a bounded surface is now defined to be the limit of a sum of infinitesimal quadrilaterals (triangles) covering the surface; this corresponds to the definite integral, and the area of a triangle is shown to be equal to the quotient of the excess $(A + B + C - \pi)$ by the measure of curvature of space. The sine amplitude also appears in considering volumes of tetrahedrons; again if $1/k^2 = 0$, then $k^3 \sin (ABCD) = 6$ volume tetrahedron $ABCD$. Following Schläfli, a formula for the volume of a tetrahedron as a definite integral is set up. The volumes of a cone of revolution, of a sphere, and the total volume of elliptic and spherical space, are also found.

In Chapters XV, "Introduction to differential geometry," and XVI, "Differential line geometry," the differential geometry of non-euclidean space is dealt with. The developments in the first of these follow the general scheme worked out for the euclidean case in Bianchi,* but the method is different from the latter's development of the non-euclidean case.† Starting with the striking theorem: "the square of the curvature of a curve is the square of its curvature treated as a curve in a four-dimensional euclidean space, minus the measure of curvature of the non-euclidean space," and finishing with the theorems on minimal surfaces, the reviewer has found this chapter as well as the succeeding one, some of the pleasantest reading and most interesting in the entire book. In the latter chapter, the line geometry as developed in Chapters X and XV receives its final treatment. It is particularly a study of the line congruence—the general properties concerning the limiting and focal points and planes, the normal and isotropic congruences. The theorems are stated in all their duality, including some of the author's own work on isotropic congruences. The chapter ends by

* Bianchi-Lukat, Vorlesungen über Differentialgeometrie, chapters I, III, IV, VI.

† Ibid., chapters XXI, XXII.

pointing out the remarkable correspondence existing between the theory of rays in hyperbolic and elliptic space.

In Chapter XVII, "Multiply connected spaces," the author returns to complete the problem of Chapter VII. There, the proof that to each point there corresponded but a single set of homogeneous coordinates depended upon Axiom VI′, which required that "a congruent transformation of any consistent region may be enlarged in a single way to be a congruent transformation of every point." Casting this one assumption aside and keeping all the others, is it possible to have a space where each point shall correspond to several sets of coordinate values? Such spaces are found obeying Axioms I′–V′, and in the sense of analysis situs these are multiply connected. As for the congruent transformations which lead to the group of identical transformations of a multiply connected space, examples are found for the euclidean plane and space, and some little known examples for the hyperbolic case; there are no multiply connected elliptic planes, but there are such three-dimensional spaces.

Chapter XVIII, "The projective basis of non-euclidean geometry," tends to base the metrical non-euclidean geometry upon projective considerations — and truly so, for the former depended upon the cross-ratio which is a projective concept. The author thus starts anew, sets up a system of eleven axioms for projective geometry, taking *point, line*, and *separation* as fundamentals, points out the invariance of the cross-ratio, sets up the coordinate system for the point in the line and plane by means of this, also the equations of line, plane, and quadric. There are now added five axioms regarding the laws obeyed by an assemblage of transformations called congruent transformations; and a quadric cone (absolute) is found which is invariant under these; two other axioms now serve to define distance by means of a cross-ratio and the previous distance formula is finally reached; hence we have the conclusion that this set of eighteen axioms are compatible with the hyperbolic, elliptic, or parabolic hypothesis, and with these only.

In the concluding Chapter XIX, "The differential basis for euclidean and non-euclidean geometry," the problem is finally attacked from the differential viewpoint. It was shown that a non-euclidean plane was a surface of Gaussian curvature $1/k^2$; further that the sum of the distances from a point to any other two, not collinear with it, was greater than the distance of these latter; thus the straight line ought to be

looked upon as a geodesic, and a plane may be generated by a pencil of geodesics through a point. The problem then is: to determine the nature of a three-dimensional point manifold which possesses the property that every surface generated by a pencil of geodesics has constant Gaussian curvature. The concepts *point* and *correspondence of point and coordinate set* are taken as fundamental. Three axioms are set up — one positing a restricted region, the second setting up the differential distance formula in such a region, and the third positing congruent transformations between arcs of geodesics. By the introduction of curvature of space, the previous distance formulas are again arrived at, and we draw the conclusion that these three axioms are compatible with the euclidean, hyperbolic, and elliptic hypothesis, and with these alone.

The chapter terminates with a brief summary of the entire work, in which the author discusses the pros and cons of the three methods of attack of the problem of non-euclidean geometry, viz., the first method developed in the first seventeen chapters, the projective method of Chapter XVIII, and the differential method of Chapter XIX. The last of these is no doubt the quickest and most direct, but this directness has been gained at the high price of "assuming at the outset that space is something whose elements depend in a definite manner on three independent parameters," instead of the more abstract view, which looks upon space as a set of objects which can be arranged in multiple series. The author concludes, and we join with him in this, that there is no answer to the question which method is the best, but that the choice is a matter of personal æsthetic preference.

In conclusion, the reviewer might again call attention to the elegant and vigorous style, to the very thorough and consistent treatment of the subject, and to the logical building up of the material step by step into one complete and harmonious whole. We might close with the author's words: "Let us not forget that, in a large measure, we study pure mathematics to satisfy an æsthetic need."

JOSEPH LIPKE.

SPECIAL PLANE CURVES.

Spezielle ebene Kurven. Von Dr. HEINRICH WIELEITNER. Leipzig, Göschen (Sammlung Schubert LVI), 1908. 8vo. xvi + 409 pp.

THE high standard which the author's book on Algebraische Kurven set is maintained in this admirable treatise both in subject matter and presentation. The fact that it has 189 figures and 282 sections in its text, each of which contains a discussion of several curves, gives an idea of the comprehensiveness of the treatment.

The arrangement of the book is unique and the reader is not bored as though he were reading a curve catalogue, but has his interest continually quickened by the clever manner in which one group of curves leads up to another and the curves of a group are related. Of the two most noteworthy books on this subject, Gino Loria arranges his "Spezielle algebraische und transzendente ebene Kurven" with their historical significance as a background; while F. Gomes Teixeira wrote his "Tratado de las curvas especiales notables" more as an encyclopædia. Wieleitner however takes up the subject from the standpoint of the mode of generation of the curves, without regard to their order or transcendency. The distinctive and most interesting feature of the book is that the author avoids the chaos of separate headings and at the same time makes a connected whole by putting in each family of curves those which arise from the original curves of the group by certain derivations such as inversion, pedal construction, polar reciprocation, evolute formation, and others. Thus certain curves are studied from many points of view and great numbers of the more noted curves are found to possess relations to each other which are fascinating to the reader.

The book contains five chapters headed respectively Cissoids, Conchoids, Other curves with simple kinematic generation, Roulettes, and The method of change of coordinates. We will mention briefly the contents of the first three chapters in order to speak more fully of the important things covered by the last two.

The generalized cissoids are defined as those curves formed by drawing through any point O a straight line G which cuts two arbitrary curves Γ and Γ' at P and P' respectively; the

locus of a point Q on G chosen so that $OQ = OP' - OP$ is the generalized cissoid. The author immediately gets its general equation, a formula for its order, and some general theorems on the family. Then comes a thorough treatment of the case where Γ and Γ' are straight lines and circles; also of the pedal curves of central conics (such as the familiar lemniscates), which are found to be intimately connected with cissoids, and so the discussion is led to properties of the family of quartics having three inflexion nodes. The chapter concludes with a connected analysis of the lines of Perseus, pedal curves of the parabola, non-circular rational cubics as cissoids, and two other types of rational cubics, namely the normal curve of the parabola and the curves generated from a circle and straight line by Maclaurin's transformation.

Conchoids, or those curves obtained by lengthening or shortening the radii vectores of any curve by a constant amount, which are really special cissoids, have the importance given to them justified by the so-called mechanical conchoidal construction, which is this: Given two planes Δ and Δ' one above the other; in Δ a fixed point O and a fixed curve Γ, in Δ' a straight line G and on it two fixed points P and Q. If Δ' is displaced so that G always passes through O and P moves on G, then Q describes in Δ a conchoid of Γ; at the same time points outside G and in Δ' describe what are called oblique conchoids, and the motion of Δ' with respect to Δ is called conchoidal. The fundamental notions of kinematic geometry are taken up and general theorems on such motions as the above, and in particular where Δ and Δ' coincide, are derived. The author then makes a complete study of conchoids of straight lines, and as their general equation contains a parameter angle, many special cases are at once evident, which in consonance with his general method of attack he immediately defines as various sorts of loci. This is followed by consideration of a family of rational quartics which have double points at infinity, such as the trisecant and cocked hat. Then come conchoids of circles and conics; the former turn out to be pedal curves of a circle, and their inverses and evolutes are of interest. The chapter ends with a discussion of Cartesian ovals as those curves which have the same power with reference to a circle conchoid, and with the determination of these ovals as loci of points satisfying certain conditions.

Chapter III begins with the definition of a motion where-

by two points P and Q of the plane Δ' are made to slide on two straight lines G and Γ of Δ, i. e., a constant length PQ is forced to move with its end-points on two given straight lines in the same plane. The questions then are: What curve does any point of Δ describe? What envelope has any line of Δ'? In answering these questions one gets the line equation of the regular astroids and of their parallel curves the oblique astroids. Wieleitner introduces here for the first time one of his sharpest tools, namely intrinsic or natural coordinates, in which the astroids have simple forms, e. g., the regular astroids have the ordinary rectangular elliptic form. These forms yield many properties, and by projection and the formation of their evolutes a number of curves are related. The cardioid and its dependent curves are then considered in detail and by means of its natural equation $9R^2 + s^2 = (8r)^2$ a chain of curves, such as cardioids, Tschirnhausen's cubics, parabolas, and Cayley sextics, is woven by simple transformations. The final sections of the chapter are given to Steiner's curve (tricuspid hypocycloid) and the "Koppelcurve des Kurbelgetriebes."

Chapter IV, which occupies 143 pages, is devoted to roulettes and special cyclic curves and is probably the most noteworthy of the book. It begins with the basal principles of natural geometry, i. e., curves are given by equations $R = f(s)$, and a discussion of the general problem of the locus of a point in the plane of a curve Λ which rolls on another curve L. By means of natural coordinates there are derived the general equation of roulettes and general formulas, the most important of which is the Savary formula, connecting the coordinates of the two base curves.

After this general discussion the author gets at once to the cycloidal curves where the coordinates R of L and R_λ of Λ are constant and equal to $\bar{R}$ and r respectively. These cycloidals have the equation $s^2/a^2 + R^2/b^2 = 1$ (epicycloids) or $s^2/a^2 - R^2/b^2 = 1$ (hypocycloids). If $r = \infty$ their equation becomes $R^2 = 2\bar{R}s$ which is the evolute of a circle; and if $\bar{R} = \infty$, $s^2 - R^2 = (4r)^2$ which is the ordinary cycloid. An interesting set of theorems follow connecting these curves and their evolutes, and also the cycloidals and the pedal curves of conics. By writing the general cycloidals in the form

$$(s + \beta/\alpha)^2 - R^2/\alpha = \Delta/\alpha^2 \quad [\Delta = \beta^2 - \alpha\gamma],$$

classification is immediate: if $\alpha < 0$, we have cycloidals with real

base curves; $\alpha > 0$, pseudo-cycloidals; $\alpha = 0$, circle evolute. If $\Delta > 0$ and $-1 < \alpha < 0$, epicycloids; $\alpha = -1$, ordinary cycloids; $\alpha < -1$, hypocycloids; etc. If $\Delta = 0$ we get $R = ks$, which are logarithmic spirals; an interesting discussion of them follows.

So far in this chapter curves of the form $R^2 = \alpha s^2 + 2\beta s + \gamma$ have been considered, which naturally suggests an analogy to conics. This is carried out neatly by the introduction of the Mannheim curve defined thus: "A curve Γ rolls on a straight line, the center of curvature of its point of contact describes a curve Γ' which is the Mannheim of Γ." So the Mannheim of $R = f(s)$ is $y = f(x)$, i. e., of the cycloidals $s^2/a^2 + R^2/b^2 = 1$ it is the ellipse $x^2/a^2 + y^2/b^2 = 1$ ($a > b$ epicycloids, $a < b$ hypocycloids, $a = b$ cycloids). Similarly the Mannheim of the circle evolute $R^2 = 2ps$ is the parabola $y^2 = 2px$, those of the paracycloids are hyperbolas, and those of the logarithmic spiral two straight lines.

This brings the author to the next important family of trochoidal curves, i. e., the curves traced by arbitrary points in the plane of two circles which roll upon each other. Their general equation in parametric form is derived and epi- and hypotrochoidals are distinguished. By transforming one set of trochoidals (stellar) to polar coordinates he finds them to be the ordinary rose curves and then follows an interesting section connecting these various curves, e. g., "the pedal curve of a cycloidal is a stellar trochoidal," and "the rose curve of modulus 2 is the pedal curve of an epicycloid of the same modulus."

If the radius of one of the rolling circles becomes infinite, a set of curves known as trochoids arise and these turn out to be projections of an ordinary helix. Thus by projecting a helix from infinity we get a sine curve; from a point on the axis of the helix a hyperbolic spiral. If the radius of the other circle becomes infinite we get an ordinary circle evolute, and these helices, evolutes, spirals, etc., are connected by a chain of theorems, e. g., if we call circle evolutes, Archimedes spirals, and hyperbolic spirals, 1, 2, and 3 respectively, we can state: "2 is the inverse of 3 and the pedal of 1; 1 is the polar reciprocal of 3, and by getting the curve (tractrix complicata) of whose existence we are assured which is the inverse of 1 and the pedal of 3, the series is complete." By projecting the helix from the surface of the cylinder, Wieleitner gets a new curve, the kachleoide, which has interesting properties as a locus and is intimately connected with the hyperbolic spiral and the quadratrix of Dinostratus.

The last part of the chapter is given up to roulettes of various kinds and begins with higher circle evolutes, i. e., evolutes of evolutes. The equation of the nth evolute of a circle can be written

$$R^{n+1} = \frac{(n+1)^n}{\underline{|n}} s^n;$$

its pedal curves are algebraic spirals and its Mannheim curves are simple. Then come such curves as Sturm's spiral, Tschirnhausen's cubic, Galileo's spiral (pedal curve of the second circle evolute), Fermat's spiral, parabolic spiral, and lituus; roll curves generated by conics rolling on a straight line, the rectification of some of which leads to elliptic integrals and whose natural equations are expressed in infinite form. One of the theorems, part of which is familiar, may be of interest as exemplifying the general treatment: "If a logarithmic spiral rolls on a straight line G, its 'Auge' describes a straight line; the focus of a parabola rolling on G describes a catenary and its directrix envelopes a catenary; the cusp of a cardioid rolling on G describes an astroid with two cusps on G; if a Ribaucour curve of index n rolls on G its directrix envelopes a Ribaucour curve of index $(n-1)/(n+3)$ having G as a directrix; if a cycloid rolls on G, its directrix envelopes an astroid and if a catenary rolls on G, its directrix passes through a fixed point." The chapter ends with some general theorems on roll curves.

Chapter V is rather novel and is on the methods of transformation of coordinates, i. e., the results to be obtained by mapping one plane on another. We have already mentioned that to each curve $f(s, R)=0$ corresponds the Mannheim $f(x, y)=0$ and the question now is what properties of curves do we get if we reverse the process, i. e., set $x = s$ and $y = R$, or ask what curve K belongs to $K'(F(x, y) = 0)$ if K' is to be the Mannheim of K? Wieleitner also considers the transformations $x = r\theta$, $y = \rho$ and $R = \rho - r$, $s = r\theta$ and calls $f(\rho - r, r\theta)$ the general Mannheim, and then seeks, for example, those curves whose general Mannheims are conchoids.

The sections of especial interest in the remainder of the chapter are on pseudo-spirals ($R = a^{1-n}s^n$), W curves, radials, and arcuids. The treatment of W curves is of especial value. Starting with $y = Kx^{p/q}$, the author discusses the singularities, etc., of these binomial curves and shows them to be polar reciprocal to themselves with respect to certain conics. Finally he comes

to the case where p/q is irrational and thus gets a set of so-called interscendental curves. The W curves are now defined as projections of these interscendental curves and have the equation $x_1^{\alpha_2-\alpha_3}x_2^{\alpha_3-\alpha_1}x_3^{\alpha_1-\alpha_2} = K$ where α_1, α_2, and α_3 are logarithms. They are the path curves of a certain projective group and are the basis of many beautiful theorems, such as: "If the above conic has the fundamental coordinate triangle for its polar triangle, the reciprocals of the W curves with respect to it are W curves of the same system. If the conic touches the W curve, the latter's polar reciprocal is identical with itself." "The cross ratio of the point of contact of a tangent T of a W curve and the three points of intersection of T and the sides of the fundamental triangle is constant for the entire curve and for each curve of the system." As the logarithmic spirals are W curves, we get many theorems about them, their pedal curves and evolutes.

Radials ($\rho = f(\theta)$ is the radial of $R = f(\tau)$) are treated in general and in detail and such theorems as the following hold: "The radial of a curve W is the cissoid of the pedal curve of W and the pedal curve of the second evolute of W with respect to an arbitrary point." Lamé's curves $(x/a)^m + (y/b)^m = 1$ are well classified and the chapter ends with arcuids, i. e., curves given by $s = \phi(\tau)$.

In summing up we would say that the large amount of new results in the book and the attractive setting of the old material show such mastery of the subject and render the book such a standard that it is hard to see how it can be improved upon. The type adopted, the arrangement of the sections, the completeness of the index, and the distinctness and accuracy of the figures are a delight to the reader. The only fault, and one that is sure to be found with any such book, is that very often curves are dragged into relationship with others and are made the subjects of many theorems without any apparent consent on their part.

E. GORDON BILL.

SHORTER NOTICES.

Die Elemente der Mathematik, Band II: *Geometrie*. Von E. BOREL. Von Verfasser genehmigte deutsche Ausgabe, besorgt von P. STÄCKEL. Leipzig, Teubner, 1909. 8vo. xii + 324 pp.

THE work here considered is the second volume of the German edition of Borel's Elements of Mathematics. It is intended as a text-book of geometry for use in the secondary schools of Germany. While the language in which it is written precludes its use as a text-book in this country, it contains so many modifications, both in subject matter and in method of presentation, of the customary course in elementary geometry, that it will be read with interest and the results obtained from the use of it will be watched carefully by all who are interested in the reform of the secondary school course in geometry in the United States.

The spirit which animates the book is stated in the preface in the following way: "The conviction that the instruction in elementary geometry ought to be revised is daily gaining ground, although certain persons strongly oppose it. These people fear that the logical structure which the Elements of Euclid has for its foundation will be torn down. Would it not be better, they say, to improve and extend this structure, as in the past, rather than to tear it down and attempt the dubious experiment of building a new one in its place? I can not participate in this view of the matter and believe, on the contrary, that, in a few decades at the latest, instruction in geometry will be based on a new principle. This new principle was arrived at only in the course of the nineteenth century by the efforts of eminent mathematicians. It consists in the realization that *elementary geometry is equivalent to the investigation of the group of motions.*"

As an illustration of the differences of this text from the customary books based on Euclid's Elements may be cited the discussion of incommensurable quantities and the proofs by limits to which they give rise. The existence of incommensurable lengths, for example, appears as a difficulty which would naturally be encountered in measurement. The diffi-

culties in the demonstrations to which these quantities give rise are surmounted by a device which again is suggested by measurement and which exhibits all the rigor that the student is likely to be able to appreciate.

It will hardly be questioned that this text will appeal more strongly to the students' interest than Euclid, nor that the material is better selected with reference to the students' capacity to receive it, nor that the student can, by the expenditure of a given amount of energy, obtain a greater amount of mathematical information from this text than from Euclid's Elements. It still remains in doubt, however, whether the student will obtain the same thorough training in rigorous, careful reasoning in this course as under the present discipline.

C. H. SISAM.

The Foundations of Mathematics. A Contribution to the Philosophy of Geometry. By Dr. PAUL CARUS. Chicago, The Open Court Publishing Co., 1908. 141 pp.

THIS book is, mathematically speaking, a more or less popular treatise, which would appear to have for its primary object an effort to show that geometry can be obtained a priori, by abstraction, from the notion of motility, and can be constructed from this alone by making use of the principles of reasoning, *all axioms being unnecessary.*

The book opens with a historical sketch, which is fairly accurate, mentioning particularly the work of Euclid, Gauss, Riemann, Lobachevsky, Bolyai, Cayley, Klein, and Grassmann. The author then introduces chapters on "The philosophical basis of mathematics" and "mathematics and metageometry" in which his philosophical theories are presented. Briefly expressed, his doctrine seems to be about as follows: "Space is the possibility of motion, and by ideally moving about in all possible directions, the number of which is inexhaustible, we construct our notion of pure space. If we speak of space we mean this construction of our mobility. It is an a priori construction and is as unique as logic or arithmetic. There is but one space, and all spaces are but portions of this construction." Mathematical space is a priori, in the Kantian sense, not however ready made in the mind, but the product of much toil and careful thought. Mathematical space is an ideal construction, hence all mathematical problems must be settled by a priori operations of pure thought, and can not be decided by external

experiment or by reference to a posteriori information. Space being obtained by abstraction, is unique, and has definite properties, and requires no *axioms* for its development. The theory of parallels is only a side issue of the implications of the straight line. The author leads the reader to expect the conclusion that Euclid alone is valid, yet he says later (page 121), "The result of our argument is quite conservative. It reestablishes the apriority of mathematical space, yet in doing so it justifies the method of metaphysicians in their constructions of the several non-euclidean systems."

There is much vagueness and apparent contradiction in the book. The abstraction process, except in so far as it is purely intuitional, would seem, if definite at all, to be nothing more than an arbitrary process, and hence equivalent to a set of axioms. The author is not concerned with any question of betweenness, or of continuity, except as involved in notions of homogeneity, evenness, his interest being almost entirely in the parallel axiom and its implications.

The book concludes with an epilogue in which the analogy between mathematics and religion is discussed, although the precise analogy is not quite clear.

F. W. OWENS.

Mechanics. By JOHN COX. Cambridge University Press (Cambridge Physical Series), 1904. Demy 8vo. xiv + 332 pp.

THIS book ought to have a far reaching influence on the teaching of elementary mechanics. It contains really good illustrative examples, concrete, practical, and instructive, and at the same time, gives clear and accurate statements of the fundamental principles. It is not overloaded with theory more general than ordinary applications require. Further, principles are expressed in words rather than by formulas. In simple examples it is clumsy to use a general formula, in complicated examples verbal expression often clears the view, in all examples the mere substitution of numerical values in a formula is poor practice.

Two paragraphs from the author's preface are worth quoting. "Some years ago I stumbled on the first German edition of Professor Mach's Die Mechanik in ihrer Entwicklung. . . . Since then my teaching has been based more and more on the lines laid down by Mach, and as I have found it impossible to

induce ordinary students to read the original even when translated, I recurred to the idea of writing a text book which should yet be based on Mach's method."

"Until Mechanics is clad in its historical flesh and blood, it will remain the dull and tiresome subject that has convinced so many generations of students that an abysmal gulf separates theory from practice."

The book is divided into four sections: historical, mathematical formulation, various applications, rigid dynamics.

The first section deals successively with Archimedes' treatment of the theory of the lever in the third century B. C., and applications including centers of gravity and the balance; Stevin's solution of the problem of the inclined plane in the sixteenth century and extensions to the parallelogram of forces and virtual work; Galileo's foundations for dynamics, their application to uniform motion by Huyghens, and Newton's deductions from the theory of universal gravitation, all in the seventeenth century.

The second section begins with a chapter on kinematics, followed by the laws of motion and their verification by Atwood's machine, energy, the usual treatment of the equilibrium of forces in a plane, and friction.

The various applications in the third section include brachistochrones, projectiles, simple harmonic motions, with Fresnel's rule for compounding them, the simple pendulum, an outline of Newton's Principia, and impact.

The elementary theory of rigid dynamics contains chapters on Huyghens's treatment of the compound pendulum, D'Alembert's principle, moments of inertia and their experimental verification, determination of gravity by Kater's pendulum and the Cavendish experiment.

It is by following the historical order of development, by bringing the reader into close contact with original masterpieces, and by the concrete illustration of principles in practice that the author has produced a book so valuable that any criticism seems ungenerous.

But the book is the beginning of a new era, not the last word in an old one. There are a few errors in the answers to examples, but this is no disadvantage for classwork. A freer use of the concepts and notation of the calculus would be more in accord with present ideas, and amongst other things, would avoid the unsatisfactory treatment of motion down a smooth

curve as the limit of motion down a series of inclined planes without any consideration of the impacts involved.

In the section devoted to mathematical formulation no mention is made of centers of gravity or of the vector law of addition. Varignon's theorem of moments should have a proof more modern than his own geometrical one. The treatment of forces in one plane acting on a rigid body would be more satisfying if the logical steps were made clearer. The following definition of equilibrium is tacitly assumed : A system of forces in one plane acting on a rigid body is in equilibrium if the sum of their moments about every point in that plane is zero.

Lastly, how about the definitions of mass and force? Of all sources of error these are the most persistent. There are two logical systems, to be found respectively in Mach's Science of Mechanics and Maxwell's Matter and Motion. In the former, which is purely dynamical, first mass and then force are defined by combining Newton's second and third laws. "All those bodies are of equal mass, which, mutually acting on each other, produce in each other equal and opposite accelerations." In the latter, equal forces are defined by their statical effect and Newton's first law and then Newton's second law furnishes a definition of mass.

"If a body moves with constant velocity in a straight line, the external forces, if any, which act upon it balance each other, or are in equilibrium." (Article 42.)

"We shall assume that it is possible to cause the force with which one body acts on another to be of the same intensity on different occasions. . . . We know that a thread of caoutchouc when stretched beyond a certain length exerts a tension which increases the more the thread is elongated. . . . When the same thread is drawn out to the same length it will, if its properties remain constant, exert the same tension." (Article 45.)

"Hence any two bodies are of equal mass if equal forces applied to these bodies produce, in equal times, equal changes of velocity. This is the only definition of equal masses which can be admitted in dynamics, and it is applicable to all material bodies, whatever they may be made of." (Article 46.)

The author is surely right in throwing over Mach's definitions and adopting those of Maxwell. In Matter and Motion, however, there are passages which tend to obscure the scheme

outlined in the three extracts quoted above, and in the author's article dealing with mass and force (pages 116–118) there is a corresponding haziness of outline. It is distinctly unfortunate that the only definition of force given in the book is that of Newton, "Force is anything which changes or tends to change a body's state of rest or of uniform motion in a straight line." That this definition almost immediately precedes Maxwell's definition of equal masses quoted above, makes the omission still more serious. Still, the work should rank as the most important contribution of late years to the teaching of elementary mechanics.

W. H. JACKSON.

Theorie der Elektrizität. Von M. ABRAHAM. Zweiter Band: *Elektromagnetische Theorie der Strahlung.* Zweite Auflage. Leipzig, Teubner, 1908. xii + 404 pp.

THE earlier edition of Abraham's Theory of Electricity was reviewed in these pages,* and no very extensive mention of the present edition seems needed. Although the number of pages in 1908 remain identical with that of 1905, and neither the titles nor the numbers of the sections are disturbed until the last quarter of the book, there have been introduced into the new edition some considerable and important alterations and improvements aimed to keep the work up to date. A saving here and there of about 18 pages up to the point where the author discusses phenomena in moving bodies and a thorough rewriting of the theory of moving bodies enables him to give to this subject the careful and critical discussion which its rapid advance and numerous controversies of the last few years necessitate. As points of especial interest may be mentioned the presentation of the equations of Lorentz, Cohn, and Minkowski with a discussion of their individual characteristics, the development of the dynamics of the Hohlraum, the treatment of local time, and the investigation of the principle of relativity. These matters are all still under critical discussion in the scientific world and are perhaps likely to remain in the spotlight for some time. The account here given by Abraham seems particularly valuable in that it enables the reader to get a good knowledge of the subjects from a single author and a single reference.

The principle of relativity is an interesting hypothesis and

* BULLETIN, 2d ser., volume 14 (1908), pp. 230–237.

has probably taken a grip upon pure mathematicians to a considerable extent since it was mathematically formulated by Poincaré and Minkowski. It would be particularly interesting to know *just* what this principle is. Many a person seems to have a general idea as to what it is, and the general ideas seem in a general way to be very much the same; but it is doubtful if these various persons agree in the details of their ideas on the subject. For instance, if we understand correctly the point of view of Poincaré, it is an essential element of the principle of relativity that the transformations of Lorentz form a group; whereas we find Abraham stating that Cohn's theory also satisfies the principle of relativity, and Cohn's transformations do not form a group. From Lorentz's point of view of motion relative to an ether, the principal of relativity appears as a physical theorem; for some more recent writers it appears to be a metaphysical principle or at least a psychological theorem. The whole matter needs an exhaustive analysis. In his closing remarks Abraham seems to suggest that a different procedure be employed according as electrons or ponderable bodies are considered. This might avoid some difficulties at present but is not finally satisfactory from a scientific point of view.

E. B. Wilson.

CORRECTION.

In Dr. W. B. Carver's paper on "Degenerate pencils of quadrics" in the Bulletin for July, 1909, the first sentence of the third paragraph on page 486 should read: Proper configurations exist for types 1, 2, and 4, and do not exist for types 3 and 5.

NOTES.

THE following papers have been read at recent meetings of the Edinburgh mathematical society. May 13: by G. E. CRAWFORD, "Properties of quadrilaterals and five-point space figures"; by D. M. Y. SOMMERVILLE, "Note on the geometries in which straight lines are represented by circles." June 10: R. C. ARCHIBALD, "Discussion and history of certain geometrical problems of Heraclitus and Apollonius"; C. M'LEOD and W. P. MILNE, "Triangles triply in perspective."

THE eighty-second meeting of the society of German naturalists and physicians will be held at Königsberg, September 18 to 24. As usual, the Mathematiker-Vereinigung will meet in affiliation with the general society. Papers to be presented at this meeting should be submitted to the secretary, Professor A. KRAZER, Karlsruhe, Westendstrasse 57.

THE summer meeting of the British association for the advancement of science will be held at Sheffield during the week beginning August 31. Professor E. W. HOBSON is chairman of the section of mathematics and physics.

THE French association will hold its general meeting at Toulouse during the last week in August. The section of mathematics and astronomy is under the presidency of Professor E. BELOT.

A MEETING of the commissioners of the international commission of mathematical instruction will be held at Brussels during the week of August 9. While the meeting is of particular interest to Belgium and adjacent countries, some of the sessions will be public and of general interest. After the routine business, the chairman, Professor F. KLEIN, will deliver an address on the aims of the commission and give a report of the work already accomplished; Professor BOURLET will speak on the reciprocal relations between pure and applied mathematics in secondary instruction. A third report of the German sub-committee is in the press, and will be presented at the forthcoming meeting; it is by W. LIETZMANN, on the organization of mathematical instruction in the boys' high schools of Prussia. Three reports from Austria, and one from France are also in the press; the latter is to appear in the next number of the *Enseignement Mathématique*.

In the press of B. G. Teubner, Leipzig, are the following books on mathematics (compare the list on page 94 of the present volume of the BULLETIN): "E. B. Christoffels gesammelte mathematische Abhandlungen," by L. MAURER; "Logische Grundlagen der exacten Wissenschaften (Wissenschaft und Hypothese, volume 12)," by P. NATORP; "Schröders Abriss der Algebra der Logik, Band 3," by E. MÜLLER; "Determinanten," by E. NETTO; "Graphische Statik der starren Systeme," by L. HENNEBERG; "Enriques Fragen der Elementargeometrie, vol. 1," by H. FLEISCHER and H. THIEME.

THE first volume of Lectures on elliptic functions, by HARRIS HANCOCK, has just been published by Wiley and Son, New York. The present volume treats of the analysis of the subject, a second will consider applications to geometry and to mechanics, while a third will be devoted to higher algebra and the theory of numbers.

THE publishing house of Ginn and Company has in press a volume on The Hindu-Arabic Numerals, by D. E. SMITH and L. C. KARPINSKI.

THE following advanced courses in mathematics are announced for the year 1910–1911.

HARVARD UNIVERSITY.—By Professor W. E. BYERLY: Dynamics of a rigid body, three hours; trigonometric series, three hours, with Professor B. O. PEIRCE.—By Professor W. F. OSGOOD: Advanced calculus, three hours; Infinite series and products, three hours, first half year; Advanced algebra, three hours, second half year; Theory of functions, II, three hours. —By Professor M. BÔCHER: Linear differential equations, three hours.—By Professor J. L. COOLIDGE: Introduction to modern geometry and modern analysis, three hours.—Projective geometry, three hours, first half year; Non-euclidean geometry, three hours, second half year.—By Professor J. K. WHITTEMORE: Theory of functions, I, three hours; Calculus of variations, three hours, first half year; Equations of mechanics, three hours, second half year; Introduction to differential geometry of curves and surfaces, three hours, first half year. Professors Osgood and Whittemore will conduct a fortnightly seminary in the theory of functions.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Modern analytic geometry, two hours: Theory of numbers, three hours. Mathematics of insurance, two hours. — By Professor G. E. FISHER: Differential equations, three hours; Theory of functions of a complex variable, three hours; The calculus of variations, two hours. — By Professor I. J. SCHWATT: Theory of functions of a real variable, three hours: Infinite series and products, three hours. — By Professor G. H. HALLETT: Introduction to modern higher algebra, first term, three hours; The Galois theory of equations, second term, three hours; Theory of groups of a finite order, three hours. — By Professor F. H. SAFFORD: Partial differential equations, three hours. — By Dr. O. E. GLENN: Geometry of contact transformations, first term, three hours: Higher algebraical equations, second term, three hours.

NEW lectureships in mathematics have been established at the University of Glasgow; students in engineering will receive separate instruction in mathematics and allied subjects.

PROFESSOR F. HAUSDORFF, of the University of Leipzig, has been appointed associate professor of mathematics at the University of Bonn.

DR. O. PERRON, of Munich, has been appointed associate professor of mathematics at the University of Tübingen.

DR. G. W. HILL has been elected foreign member of the Brussels academy of sciences.

AT the University of Missouri, Dr. O. D. KELLOGG has been promoted to a professorship of mathematics.

AT Purdue University, Professor ERASTUS TEST will retire at the close of the present academic year. Professor JACOB WESTLUND has been promoted to a full professorship of mathematics.

AT the University of Pennsylvania, Dr. G. H. HALLETT has been promoted to a full professorship and Drs. M. J. BABB, G. G. CHAMBERS, and O. E. GLENN to assistant professorships of mathematics.

AT the University of Kansas Drs. C. H. ASHTON and J. N. VAN DER VRIES have been promoted from assistant profes-

sorships to associate professorships of mathematics. Dr. U. G. MITCHELL, of Princeton University, and Drs. ARTHUR PITCHER and M. B. WHITE, of the University of Chicago, have been appointed assistant professors of mathematics.

AT the University of Michigan Dr. W. B. FORD has been promoted to a junior professorship, and Dr. L. C. KARPINSKI to an assistant professorship of mathematics. Professor J. L. MARKLEY has been granted leave of absence for the academic year 1910–1911.

PROFESSOR D. N. LEHMER, of the University of California, has been promoted to an associate professorship of mathematics.

MR. N. C. GRIMES has been appointed professor of mathematics in the University of Arizona.

DR. L. I. HEWES has been appointed assistant professor of mathematics at Whitman College, Walla Walla, Wash.

ON the occasion of his retirement from the University of Chicago, Professor OSKAR BOLZA was presented with a loving cup and a testimonial volume signed by more than a hundred of his former students.

CATALOGUES of mathematical books: A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogue no. 102, 2200 titles.—Ottmar Schönhuth Nachf., Schwanthalerstrasse 2, Munich, catalogue no. 22, books before 1800, 116 titles in mathematics and exact sciences. — Bernard Quaritch, 11 Grafton Street, New Bond Street, London, catalogue no. 286, 247 titles in mathematics and physics. — Galloway and Porter, Cambridge, England, catalogue no. 49, 432 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BASSET (A. B.). A treatise on the geometry of surfaces. London, Bell, 1910. 8vo. 10s. 6d.

CAPELLI (A.). Istituzioni di analisi algebrica. 4a edizione, ampliata, ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. Napoli, Pellerano, 1909. 8vo. 28 + 953 pp. L. 17.00

DESCHAMPS (J.). Notes de géométrie analytique. Démonstration de quelques identités fondamentales, et premières applications. Paris, 1909. 8vo. 42 pp.

DZIOBEK (O.). Vorlesungen über Differential- und Integralrechnung. Leipzig, Teubner, 1910. 8vo. 10 + 648 pp. Cloth. M. 16.00

FRANKLAND (W. B.). Theories of parallelism: an historical critique. Cambridge, University Press, 1910. 8vo. 88 pp. 3s.

GAEDECKE (W.). Die inversen Flächen der Mittelpunktsflächen 2ter Ordnung. (Diss.) Königsberg, 1910. 8vo. 62 pp.

GERSTENMEIER (C.). Beiträge zur Theorie der linearen Differentialgleichungen mit 4 und 5 singulären Stellen. (Diss.) Erlangen, 1910. 8vo. 88 pp.

HANCOCK (H.). Lectures on the theory of elliptic functions. Volume I. Analysis. New York, Wiley, 1910. 8vo. 23 + 498 pp. Cloth. $5.00

HARTDEGEN (F.). See UNTERRICHTS-BRIEFE.

HORN (J.). Einführung in die Theorie der partiellen Differentialgleichungen. (Sammlung Schubert, LX.) Leipzig, Göschen, 1910. 8vo. 7 + 363 pp. Cloth. M. 10.00

KAHN (G.). Eine allgemeine Methode zur Untersuchung der Gestalten algebraischer Kurven. (Diss.) Göttingen, 1909. 8vo. 43 pp.

LAURENT (H.). See STURM (C.).

NOSSOW (A.). Versuch des Beweises des letzten Fermatschen Satzes. Berlin, 1910.

RICHERT (P.). Die ganze rationale Funktion vierten Grades und ihre Kurven. Fortsetzung. (Progr.) Berlin, Weidmann, 1910. 8vo. 19 pp. M. 1.00

SAINT-GERMAIN (A. DE). See STURM (C.).

SCHEFFERS (G.). Anwendung der Differential- und Integralrechnung auf Geometrie. Band I: Einführung in die Theorie der Kurven in der Ebene und im Raume. 2te verbesserte und vermehrte Auflage. Leipzig, Veit, 1910. 8vo. 10 + 482 pp. Cloth. M. 14.00

STURM (C.). Cours d'analyse de l'Ecole Polytechnique. Revu et corrigé par E. Prouhet, et augmenté de la théorie élémentaire des fonctions elliptiques, par H. Laurent. 14me édition, revue et mise au courant du nouveau programme de la licence, par A. de Saint-Germain. 2 vols. Paris, Gauthier-Villars, 1910. 8vo. 32 + 564 + 10 + 658 pp. Fr. 15.00

TEMPERLI (H.). Ueber eine spezielle Kurve dritten Grades. (Diss.) Zürich, 1909. 8vo. 47 pp.

UNTERRICHTS-BRIEFE, mathematische, für das Selbststudium Erwachsener. Mit besonderer Berücksichtigung der angewandten Mathematik. (Methode Burkhardt-Blank.) V. Integral- und Differentialrechnung (sowie deren Anwendungen) bearbeitet von F. Hartdegen. 1ter Brief. Kahla, Thüringer Verlagsanstalt, 1910. 8vo. Pp. 1721–1740. M. 0.60

WERNER (A.). Ueber Systeme von drei Pfaffschen Gleichungen im Raume von 5 Dimensionen. (Diss.) Greifswald, 1908. 8vo. 43 pp.

WIESMANN (C.). Beiträge zur Theorie der Funktionen des elliptischen Zylinders. (Diss.) Zürich, 1909. 8vo. 46 pp.

YOUNG (W. H.). The fundamental theorems of the differential calculus. Cambridge, University Press, 1910. 8vo. 2s. 6d.

II. ELEMENTARY MATHEMATICS.

BOHNERT (F.). Elementare Stereometrie. 2te, durchgesehene Auflage. (Sammlung Schubert, IV.) Leipzig, Göschen, 1910. 8vo. 7 + 183 pp. Cloth. M. 2.40

BRIGGS (W.). First stage mathematics. (With modern geometry.) London, Clive, 1910. 8vo. 312 pp. Cloth. 2s.

CATANI (S.). Problemi di matematica, dati agli esami di licenza d'istituto tecnico, con le loro risoluzioni. 2a edizione, riveduta e aumentata. Livorno, Giusti, 1910. 16mo. 145 pp. L. 1.00

DEVILLER, SCHWEIKERT und SPAHN. Algebra für Mittelschulen und höhere Mädchenschulen. Strassburg, Bull, 1910. 8vo. 6 + 110 pp. M. 1.00

DURELL (P.). Logarithmic and trigonometric tables. New York, Merrill, 1910. 8vo. 114 pp. Cloth. $0.75

—— Plane trigonometry. New York, Merrill, 1910. 8vo. 184 pp. Cloth. $1.00

Bound with tables. $1.25

ELEMENTARY practical mathematics, for technical and industrial classes. Reissue. London, Oliver, 1910. 8vo. 240 pp. Cloth. 1s. 6d.

HALL (H. S.). A school algebra. Part I. With or without answers. London, Macmillan, 1910. 8vo. 2s. 6d.

—— and STEVENS (F. H.). Key to the exercises and examples in "A school geometry," parts 5 and 6. London, Macmillan, 1910. 8vo. 3s. 6d.

JACKSON (L. L.). See YOUNG (J. W. A.).

KLEIN (F.). See LIETZMANN (W.).

KUTNEWSKY (M.). See MÜLLER (H.).

LEFEBVRE (B.). Cours d'algèbre élémentaire, à l'usage des cours moyens et des classes d'humanités. 3ème édition. Liège, Dessain, 1909. 600 pp. Fr. 5.00

LIETZMANN (W.). Stoff und Methode im mathematischen Unterricht der norddeutschen höheren Schulen auf Grund der vorhandenen Lehrbücher. Mit einen Einführungswort von F. Klein. Leipzig, Teubner, 1909. 8vo. 12 + 102 pp. M. 2.00

LINNICH (M.). Lehr- und Uebungsbuch der Mathematik. Für höhere Mädchenschulen bearbeitet. 1ter Teil. Lehrstoff der vierten und dritten Klasse. Leipzig, Freytag, 1910. M. 2.00

MAHLERT (A.). See MÜLLER (H.).

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Bearbeitet von A. Schulte-Tigges. Ausgabe B. Unterstufe. Planimetrie und Arithmetik nebst den Anfangsgründen der Trigonometrie und Stereometrie und 3 Anhängen. 2te unveränderte Auflage. Berlin, Reimer, 1910. 8vo. 9 + 204 pp. Cloth. 2.00

MÜLLER (H.) und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. 2ter Teil. Ausgabe B, für reale Anstalten und Reformschulen. 3te Auflage. Leipzig, Teubner, 1910. 8vo. 11 + 312 pp. Cloth. M. 3.00

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch für höhere Mädchenschulen. 2ter Teil: Planimetrie und Körperberechnungen. 3te Auflage. Leipzig, Teubner, 1910. 8vo. 7 + 122 pp. Cloth. M. 1.80

——. Ergebnisse zu den Aufgaben in dem mathematischen Lehr- und Uebungsbuche für Lyzeen. Leipzig, Teubner, 1910. 8vo. 38 pp. M. 1.50

——. Mathematisches Lehr- und Uebungsbuch für das Lyzeum. Fortsetzung des mathematischen Lehr- und Uebungsbuches für höhere Mädchenschulen. Teil III: Methodik des Unterrichts. Analytische Geometrie der Ebene. Unter Mitwirkung von J. Plath. Leipzig, Teubner, 1910. 8vo. 6 + 100 pp. Cloth. M. 1.80

NARDI (P.). Geometria pratica, ad uso degli alunni delle scuole tecniche e professionali. Parte I: planimetria. 2a edizione, corretta. Livorno, Giusti, 1910. 16mo. 76 pp. L. 0.70

PLATH (J.). See MÜLLER (H.).

SCHUBERT (H.). Elementare Arithmetik und Algebra. 2te Auflage. (Sammlung-Schubert, I.) Leipzig, Göschen, 1910. 8vo. 6 + 230 pp. Cloth. M. 2.80

SCHULTE-TIGGES (A.). See MEHLER (F. G.).

SCHWEIKERT. See DEVILLER.

SPAHN. See DEVILLER.

STEVENS (F. H.). See HALL (H. S.).

WHIPPLE (F. J. W.). The public school geometry. London, Dent, 1910 8vo. 162 pp. Cloth. 2s.

WILK (E.) Geometrie für höhere Mädchenschulen. 1ter Teil (für Klasse IV und III). Dresden, Bleyl, 1910. 8vo. 63 pp. M. 1.10

YOUNG (J. W. A.) and JACKSON (L. L.). A first course in elementary algebra. New York, Appleton, 1910. 12mo. 8 + 294 pp. Cloth. \$0.95

——. A second course in elementary algebra. New York, Appleton, 1910. 12mo. 215 pp. Cloth. \$0.70

III. APPLIED MATHEMATICS.

BOUASSE (H.). Cours de mécanique rationnelle et expérimentale spécialement écrit pour les physiciens et les ingénieurs. Paris, Delagrave, 1910. 8vo. 696 pp. Fr. 20.00

CHWOLSON (O. D.). Traité de physique. Ouvrage traduit sur les éditions russe et allemande par E. Davaux. Edition revue et considérablement augmentée par l'auteur, suivie de notes sur la physique théorique, par E. Cosserat et F. Cosserat. Vol. 4. 1er fascicule: Champ électrique constant. Paris, Hermann, 1910. 8vo. 7 + 430 pp. Fr. 12.00

COSSERAT (E. et F.) See CHWOLSON (O. D.).

DAVAUX (E.). See CHWOLSON (O. D.).

DUHEM (P.). Thermodynamique et chimie. Leçons élémentaires. 2e édition, entièrement refondue et considérablement augmentée. Paris, Hermann, 1910. 8vo. 12 + 580 pp. Fr. 16.00

ENZYKLOPÄDIE der mathematischen Wissenschaften. Vol. V: Physik. Redigiert von A. Sommerfeld. 2ter Teil. 3tes Heft. Leipzig, Teubner, 1910. 8vo. Pp. 395–538. M. 4.60

GROBER (M.). Untersuchungen über Resonanzkurven. (Diss.) Halle, 1910. 8vo. 33 pp.

LONEY (S. L.). A treatise on elementary dynamics. 7th edition. Cambridge, University Press, 1910. 8vo. 12 + 348 pp. $1.90

——. The elements of statics and dynamics. Part II: Dynamics. 12th edition. Cambridge, University Press, 1910. 8vo. 8 + 254 pp. $1.00

NIMFÜHR (R.). Die Luftschiffahrt. Ihre wissenschaftlichen Grundlagen und technische Entwicklung. 2te, verbesserte, und vermehrte Auflage. Leipzig, 1910. M. 12.50

SOMMERFELD (A.). See ENZYKLOPÄDIE.

SULLIVAN (J. G.). Spiral tables. Prepared for the Canadian Pacific railroad. Reprinted from the Engineering Record. New York, McGraw, 1910. 12mo. 47 pp. $1.50

NINETEENTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALLARDICE, R. E. On the Locus of the Foci of a System of Similar Conics through Three Points. Read (San Francisco) Dec. 19, 1903. *Proceedings of the Edinburgh Mathematical Society*, vol. 27, pp. 37–50 ; Sept., 1909.

AMES, L. D. On Some Theorems in the Lie Theory. Read (Southwestern Section) Nov. 27, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 7, pp. 360–363 ; April, 1910.

ASHTON, C. H. A New Elliptic Function. Read (Chicago) Dec. 31, 1909. Included in the author's Dissertation : Die Heine'schen O-Funktionen und ihre Anwendungen auf die elliptischen Funktionen, Munich, 1909.

BIRKHOFF, G. D. Singular Points of Ordinary Linear Differential Equations. Read (Chicago) April 18, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 4, pp. 436–470 ; Oct., 1909.

—— A Simplified Treatment of the Regular Singular Point. Read Feb. 26, 1910. *Transactions of the American Mathematical Society*, vol. 11, No. 2, pp. 199–202 ; April, 1910.

BLICHFELDT, H. F. Theorems on Simple Groups. Read (San Francisco) Sept. 29, 1906, and Sept. 26, 1908. *Transactions of the American Mathematical Society*, vol. 11, No. 1, pp. 1–14 ; Jan., 1910.

BOLZA, O. An Application of the Notions of General Analysis to a Problem of the Calculus of Variations. Read (Chicago) April 8, 1910. *Bulletin of the American Mathematical Society*, vol. 16, No. 8, pp. 402–407 ; May, 1910.

BUSSEY, W. H. On the Tactical Problem of Steiner. Read Feb. 24, 1906. *Bulletin of the American Mathematical Society*, vol. 16, No. 1, pp. 19–22 ; Oct., 1909.

—— Tables of Galois Fields of Order Less than 1,000. Read Sept., 13, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 4, pp. 188–206 ; Jan., 1910.

CAJORI, F. A Note on the History of the Slide Rule. Read Oct. 30, 1909. *Bibliotheca Mathematica*, ser. 3, vol. 10, No. 2, pp. 161–163 ; May, 1910.

CARMICHAEL, R. D. On the Numerical Factors of Certain Arithmetic Forms. Read Feb. 29, 1908. *American Mathematical Monthly*, vol. 16, No. 10, pp. 153–159 ; Oct., 1909.

—— On r-fold Symmetry of Plane Algebraic Curves. Read Dec. 30, 1908. *American Mathematical Monthly*, vol. 17, No. 3, pp. 56–64 ; March, 1910.

—— Note on Some Polynomials Related to Legendre's Coefficients. Read April 24, 1909. *American Mathematical Monthly*, vol. 16, Nos. 6–7, pp. 114–117 ; June–July, 1909.

—— On Certain Functional Equations. Read Apr. 24, 1909. *American Mathematical Monthly*, vol. 16, No. 11, pp. 180–183 ; Nov., 1909.

—— Note on a New Number Theory Function. Read Sept. 13, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 5, pp. 232–238; Feb., 1910.

CHESSIN, A. S. On the So-Called Gyrostatic Effect. Read Apr. 24, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 1, pp. 22–24; Oct., 1909.

COBLE, A. B. Symmetric Binary Forms and Involutions. Read Dec. 31, 1908. *American Journal of Mathematics*, vol. 31, Nos. 2 and 4, pp. 183–212 and 355–364; Apr. and Oct., 1909.

CONNER, J. R. See MORLEY, F.

CRAIG, C. F. On a Class of Hyperfuchsian Functions. Read Dec. 31, 1908. *Transactions of the American Mathematical Society*, vol. 11, No. 1, pp. 37–54; Jan., 1910.

DEDERICK, L. S. The Solution of the Equation in Two Real Variables at a Point Where Both the Partial Derivatives Vanish. Read Sept. 14, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 4, pp. 174–187; Jan., 1910.

DENTON, W. W. On the Osculating Quartic of a Plane Curve. Read (Chicago) Jan. 2, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 297–308; July, 1909.

DICKSON, L. E. Equivalence of Pairs of Bilinear or Quadratic Forms under Rational Transformation. Read (Chicago) April 9, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 347–360; July, 1909.

—— Combinants. Read (Chicago) Apr. 9, 1909. *Quarterly Journal of Pure and Applied Mathematics*, vol. 40, No. 4, pp. 349–366; July, 1909.

—— A Theory of Invariants. Read (Chicago) Apr. 9, 1909. *American Journal of Mathematics*, vol. 31, No. 4, pp. 337–354; Oct., 1909.

EISENHART, L. P. The Twelve Surfaces of Darboux and the Transformation of Moutard. Read Apr. 24, 1909. *American Journal of Mathematics*, vol. 32, No. 1, pp. 17–36; Jan., 1910.

ESCOTT, E. B. Logarithmic Series. Read (Chicago) Apr. 2, 1904, and Dec. 31, 1909. *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, No. 2, pp. 141–156; Jan., 1910.

—— The Calculation of Logarithms. Read (Chicago) Dec. 31, 1909. *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, No. 2, pp. 157–167; Jan., 1910.

EVANS, G. C. The Integral Equation of the Second Kind, of Volterra, with Singular Kernel. Read Sept. 13, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 3, pp. 130–136; Dec., 1909.

FITE, W. B. Groups of Order 3^m in Which Every Two Conjugate Operations are Permutable. Read Sept. 11, 1908. *Mathematische Annalen*, vol. 67, No. 4, pp. 498–510; Oct., 1909.

—— Irreducible Homogeneous Linear Groups in an Arbitrary Domain. Read Dec. 31, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 315–318; July, 1909.

FORD, W. B. On the Integration of the Homogeneous Linear Difference Equation of Second Order. Read (Chicago) Apr. 17, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 319–336; July, 1909.

—— On the Determination of the Asymptotic Developments of a Given Function. Read (Chicago) April 9, 1909. *Annals of Mathematics*, ser. 2, vol. 11, No. 3, pp. 115–127; Apr., 1910.

Glenn, O. E. The Theory of Degenerate Algebraical Curves and Surfaces. Read Apr. 25, and Sept. 10, 1908. *American Journal of Mathematics*, vol. 32, No. 1, pp. 75–100; Jan., 1910.

Haskins, C. N. Numerical Computation of Reaction Velocity Constants. Read Dec. 31, 1908. *Bulletin of the University of Illinois Engineering Experiment Station*, No. 30, pp. 37–46; 1909.

Hawkes, H. E. The Reduction of Families of Bilinear Forms. Read Dec. 29, 1906, and Dec. 28, 1907. *American Journal of Mathematics*, vol. 32, No. 2, pp. 101–114; Apr., 1910.

Hawkesworth, A. S. A New Theorem in the Geometry of Conics. Read Sept. 13, 1909. *American Mathematical Monthly*, vol. 17, No. 4, pp. 82–89; April, 1910.

Hewes, L. I. Necessary and Sufficient Conditions that an Ordinary Differential Equation Shall Admit a Conformal Group. Read Dec. 27, 1907, and Sept. 11, 1908. *Annals of Mathematics*, ser. 2, vol. 11, No. 2, pp. 49–59; Jan., 1910.

Hoskins, L. M. The Strain of a Gravitating, Compressible Elastic Sphere. Read (San Francisco) Sept. 25, 1909. *Transactions of the American Mathematical Society*, vol. 11, No. 2, pp. 203–248; Apr., 1910.

Hurwitz, W. A. See Richardson, R. G. D.

Hutchinson, J. I. On Linear Transformations Which Leave an Hermitian Form Invariant. Read Dec. 30, 1908. *American Journal of Mathematics*, vol. 32, No. 2, pp. 195–206; Apr., 1910.

Jackson, D. Resolution into Involutory Substitutions of the Transformations of a Non-Singular Bilinear Form into Itself. Read Sept. 13, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 4, pp. 479–484; Oct., 1909.

Jackson, W. H. The Integral Roots of Certain Inequalities. Read Apr. 24, 1909. *Annals of Mathematics*, ser. 2, vol. 11, No. 3, pp. 128–140; Apr., 1910.

—— The Solution of an Integral Equation Occurring in the Theory of Radiation. Read Dec. 30, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 9, pp. 473–475; June, 1910.

Karpinski, L. C. Jordanus Nemorarius and John of Halifax. Read Dec. 29, 1909. *American Mathematical Monthly*, vol. 17, No. 5, pp. 108–113; May, 1910.

Kasner, E. The Infinitesimal Contact Transformations of Mechanics. Read Feb. 29, 1908. *Bulletin of the American Mathematical Society*, vol. 16, No. 8, pp. 408–412; May, 1910.

—— The Theorem of Thomson and Tait and Natural Families of Trajectories. Read Sept. 13 and Dec. 29, 1909. *Transactions of the American Mathematical Society*, vol. 11, No. 2, pp. 121–140; Apr., 1910.

Lehmer, D. N. Preliminary Report on a Table of Smallest Divisors. Read (San Francisco) Apr. 25, 1903. Author's Factor Table for the First Ten Millions, Washington, 1909.

Leib, D. D. On a Complete System of Invariants of Two Triangles. Read Dec. 31, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 361–390; July, 1909.

LUNN, A. C. The Fundamental Theorems on Ordinary Differential Equations in Real Variables. Read (Chicago) Apr. 2, 1904. Included in the author's Dissertation: The Differential Equations of Dynamics, Chicago, 1909.

—— A Continuous Group Related to Von Seidel's Optical Theory. Read (Chicago) Apr. 17, 1908. *Bulletin of the American Mathematical Society*, vol. 16, No. 1, pp. 25–30; Oct., 1909.

—— The Apparent Size of a Closed Curve. Read (Chicago) Jan. 2, 1909. *American Journal of Mathematics*, vol. 32, No. 2, pp. 186–194; Apr., 1910.

LYTLE, E. B. Proper Multiple Integrals over Iterable Fields. Read Apr. 25, 1908. *Transactions of the American Mathematical Society*, vol. 11, No. 1, pp. 25–36; Jan., 1910.

MACLAGAN-WEDDERBURN, J. H. On the Direct Product in the Theory of Finite Groups. Read Sept. 11, 1908. *Annals of Mathematics*, ser. 2, vol. 10, No. 4, pp. 173–176; July, 1909.

MACMILLAN, W. D. Periodic Orbits about an Oblate Spheroid. Read (Chicago) Apr. 18, 1908, and Apr. 10, 1909. *Transactions of the American Mathematical Society*, vol. 11, No. 1, pp. 55–120; Jan., 1910.

MILLER, G. A. Groups in Which the Subgroup which Involves all the Substitutions Omitting a Given Letter is Regular. Read (San Francisco) Sept. 28, 1907. *Prace Matematyczno-Fizyczne*, vol. 19, pp. 17–19; 1908.

—— Groups Formed by Prime Residues with Respect to Modular Systems. Read Sept. 11, 1908. *Archiv der Mathematik und Physik*, ser. 3, vol. 15, No. 2, pp. 115–121; 1909.

—— Automorphisms of Order Two. Read (Chicago) Apr. 9, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 4, pp. 471–478; Oct., 1909.

—— The Groups Which May be Generated by Two Operators, s_1, s_2 Satisfying the Equation $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α and β Being Relatively Prime. Read Sept. 13, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 2, pp. 67–69; Nov., 1909.

—— Note on the Groups Generated by Two Operators Whose Squares are Invariant. Read Oct. 30, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 4, pp. 173–174; Jan., 1910.

—— Groups Generated by Two Operators Each of which is Transformed into a Power of Itself by the Square of the Other. Read (Chicago) Jan. 1, 1910. *Bulletin of the American Mathematical Society*, vol. 16, No. 9, pp. 466–473; June, 1910.

—— Extensions of Two Theorems Due to Cauchy. Read (Chicago) Apr. 9, 1910. *Bulletin of the American Mathematical Society*, vol. 16, No. 10, pp. 510–513; July, 1910.

MOORE, C. N. The Summability of the Developments in Bessel Functions, with Applications. Read Sept. 11, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 4, pp. 391–435; Oct., 1909.

MOORE, E. H. Five papers on Functional Operations, Integral Equations, and General Analysis. Read (Chicago) Dec. 29, 1905, Sept. 5–8, 1906, (Southwestern Section) Dec. 1, 1906, (Chicago) Jan. 1, 1908, and Jan. 2, 1909. Included in the New Haven Mathematical Colloquium, New Haven, 1910.

MOREHEAD, J. C., and WESTERN, A. E. Note on Fermat's Numbers. Read (Chicago) Apr. 9, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 1, pp. 1–6; Oct., 1909.

MORLEY, F., and CONNER, J. R. Plane Sections of a Weddle Surface. Read Dec. 30, 1908. *American Journal of Mathematics*, vol. 31, No. 3, pp. 263-270; July, 1909.

NOBLE, C. A. Necessary Conditions that Three or More Partial Differential Equations of the Second Order shall have Common Solutions. Read (San Francisco) Sept. 26, 1908. *Bulletin of the American Mathematical Society*, vol. 16, No. 1, pp. 10–14; Oct., 1909.

OSGOOD, W. F. On Cantor's Theorem Concerning the Coefficients of a Convergent Trigonometric Series, with Generalizations. Read Apr. 24, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 337–346; July, 1909.

OWENS, F. W. The Introduction of Ideal Elements and a New Definition of Projective n-Space. Read (Chicago) Apr. 22, 1905. *Transactions of the American Mathematical Society*, vol. 11, No. 2, pp. 141–171; Apr., 1910.

PELL, A. J. On an Integral Equation with an Adjoined Condition. Read (Chicago) Dec. 31, 1909. *Bulletin of the American Mathematical Society*. vol. 16, No. 8, pp. 412–415; May, 1910.

RANUM, A. The Group of Classes of Congruent Quadratic Integers with Respect to a Composite Ideal Modulus. Read (Chicago) Apr. 9, 1909. *Transactions of the American Mathematical Society*, vol. 11, No. 2, pp. 172–198; Apr., 1910.

REED, F. W. On Singular Points in the Approximate Development of the Perturbative Function. Read Sept. 13, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 4, pp. 435–509; Oct., 1909.

RICHARDSON, R. G. D. Das Jacobische Kriterium der Variationsrechnung und die Oscillationseigenschaften linearer Differentialgleichungen 2. Ordnung. Read Sept. 14, 1909. *Mathematische Annalen*, vol. 68, No. 2, pp. 279–304; Jan., 1910.

RICHARDSON, R. G. D., and HURWITZ, W. A. Note on Determinants Whose Terms are Certain Integrals. Read Sept 14, 1909. *Bulletin of the American Mathematical Society*, vol. 16, No. 1, pp. 14–19; Oct., 1909.

ROE, E. D., Jr. On the Extension of the Exponential Theorem. Read Apr. 24, 1909. *American Mathematical Monthly*, vol. 16, Nos. 6–7 and 10, pp. 101–106 and 159; June–July and Oct., 1909.

SCHWEITZER, A. R. Note on a System of Axioms for Geometry. Read (Chicago) Apr. 14, 1906. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 309–314; July, 1909.

—— A Theory of Geometrical Relations. Read Sept. 6, 1907 and (Chicago) Apr. 18, 1908. *American Journal of Mathematics*, vol. 31, No. 4, pp. 365–410; Oct., 1909.

SHARPE, F. R. The General Circulation of the Atmosphere. Read Sept. 3, and Dec. 29, 1906. *American Journal of Mathematics*, vol. 32, No. 1, pp. 52–64; Jan., 1910.

—— The Identical Relations of the Strain and Stress Components of an Elastic Solid. Read Sept. 10, 1908. Included in Review of Love's Theory of Elasticity, *Bulletin of the American Mathematical Society*, vol. 16, No. 2, pp. 90–92; Nov., 1909.

—— The Topography of Certain Curves Defined by a Differential Equation. Read Dec. 30, 1908. *Annals of Mathematics*, ser. 2, vol. 11, No. 3, pp. 97–102; Apr., 1910.

Sisam, C. H. On Some Loci Associated with Plane Curves. Read (Chicago) Dec. 28, 1906, and Mar. 30 and Dec. 30, 1907. *American Journal of Mathematics*, vol. 31, No. 3, pp. 253–262; July, 1910.

Slocum, S. E. The Collapse of Tubes under External Pressure. Read Apr. 25, 1908. *Engineering* (London), Jan. 8, 1909.

Snyder, V. Infinite Discontinuous Groups of Birational Transformations Which Leave Certain Surfaces Invariant. Read Feb. 27, 1909. *Transactions of the American Mathematical Society*, vol. 11, No. 1, pp. 15–24; Jan., 1910.

—— Surfaces Invariant under Infinite Discontinuous Birational Groups Defined by Line Congruences. Read Sept. 13, 1909. *American Journal of Mathematics*, vol. 32, No. 2, pp. 177–185; Apr., 1910.

Van Benschoten, A. L. The Birational Transformations of Algebraic Curves of Genus Four. Read Sept. 6, 1907. *American Journal of Mathematics*, vol. 31, No. 3, pp. 213–252; July, 1909.

Western, A. E. See Morehead, J. C.

Wilczynski, E. J. Projective Differential Geometry of Curved Surfaces (Fifth Memoir). Read Sept. 10, 1908, and (Chicago) Jan. 1, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 3, pp. 279–296; July, 1909.

Wright, J. E. The Differential Equations Satisfied by Abelian Theta Functions of Genus Three. Read Apr. 24, 1909. *American Journal of Mathematics*, vol. 31, No. 3, pp. 271–298; July, 1909.

Young, A. E. On the Problem of the Spherical Representation and the Characteristic Equations of Certain Classes of Surfaces. Read (Chicago) Apr. 18, 1908. *American Journal of Mathematics*, vol. 32, No. 1, pp. 37–51; Jan., 1910.

Young, J. W. The Geometry of Chains on a Complex Line. Read Feb. 29, 1908. *Annals of Mathematics*, ser. 2, vol. 11, No. 1, pp. 33–48; Oct., 1909.

—— On the Discontinuous ζ-Groups Defined by Rational Normal Curves in a Space of n Dimensions. Read (Chicago) Jan. 1, 1910. *Bulletin of the American Mathematical Society*, vol. 16, No. 7, pp. 363–368; Apr., 1910.

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